# Some Convergence Results of a Multidimensional Finite Volume Scheme for a Semilinear Parabolic Equation with a Time Delay

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# Aim of the presentation

The aim of this talk is to establish a finite volume scheme along with a convergence analysis for a Semi-linear Parabolic Equation with a Time Delay.

# Plan of this presentation

# Problem to be solved

## Equation

Semi-Linear Parabolic equation with a Time Delay:

$$u_t(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t,u(\boldsymbol{x},t),u(\boldsymbol{x},t-\tau)),\ (\boldsymbol{x},t) \in \Omega \times (0,T), \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is an open domain of $\mathbb{R}^d$, $f$ is given function, and $T, \tau > 0$ are given.
**The positive value $\tau$ called the delay.**

## Initial conditions

$$u(\boldsymbol{x},t) = u^0(\boldsymbol{x},t),\ \boldsymbol{x} \in \Omega,\ \ -\tau \le t \le 0. \tag{2}$$

## Homogeneous Dirichlet boundary

$$u(\boldsymbol{x},t) = 0,\ (\boldsymbol{x},t) \in \partial\Omega \times (0,T). \tag{3}$$

# What about Delay differential equations?

## Some physics

Delay differential equations occur in many applications such as ecology and biology. They have long played important roles in the literature of theoretical population dynamics, and they have been continuing to serve as useful models.

# Introduction: Finite Volume from Admissible meshes to Nonconforming meshes

Finite volume methods are numerical methods approximating different types of Partial Differential Equations (PDEs). They are based on three principle ideas:

- Subdivision of the spatial domain into subsets called Control Volumes.
- Integration of the equation to be solved over the Control Volumes.
- Approximation of the derivatives appearing after integration.

[1] We mean here the "pure" finite volume methods and not finite volume-element methods

# Introduction (suite): Finite Volume from Admissible meshes to Nonconforming meshes

Finite Volume methods passed by two steps:

**First step**

Finite Volume methods using Admissible meshes.

**Second step**

SUSHI method.

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# Introduction (suite): Finite Volume methods on admissible meshes

**Definition**

Let $\mathcal{T}$ be an Admissible Mesh in the sense of Eymard et al. (Handbook, 2000).

$K \in \mathcal{T}$ are the control volumes and $\sigma$ are the edges of the control volumes $K$.

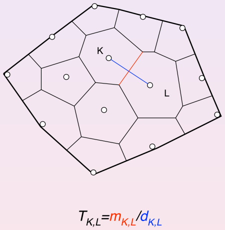


Figure : transmissivity between $K$ and $L$: $\mathcal{T}_\sigma = \mathcal{T}_{K|L} = \frac{m_{K,L}}{d_{K,L}}$

# Introduction (suite): Finite Volume methods on admissible meshes

**Main properties of Admissible mesh:**

1. Convexity of the Control Volumes.
2. The orthogonality property: the $(\boldsymbol{x}_K\boldsymbol{x}_L)$ is orthogonal to the common edge $\sigma$ between the control volumes $K$ and $L$.

# Introduction (suite): Finite Volume methods on admissible meshes

Model to be solved:

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega \quad \text{and} \quad u(\boldsymbol{x}) = 0,\ \boldsymbol{x} \in \partial\Omega. \tag{4}$$

**Principles of Finite Volume scheme**:

1. Integration on each control volume $K$ :$-\int_K \Delta u(\boldsymbol{x}) d\boldsymbol{x} = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$,
2. Integration by Parts gives :$-\int_{\partial K} \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$
3. Summing on the lines of $K$: $-\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$

# Introduction (suite): Finite Volume methods on admissible meshes

## Approximate Finite Volume Solution $u_{\mathcal{T}} = (u_K)_K$

$$-\sum_{\sigma \in \mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{d_{K|L}} (u_L - u_K) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}. \qquad (5)$$

## Matrix Form

$$\mathcal{A}^{\mathcal{T}} u_{\mathcal{T}} = f_{\mathcal{T}}.$$

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# Introduction (suite): Finite Volume methods on admissible meshes

## Theorem

*Let $\mathcal{X}(\mathcal{T})$: functions which are constant on each control volume $K$. Let $e_{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$ be defined by $e_K = u(\boldsymbol{x}_K) - u_K$ for any $K \in \mathcal{T}$. Assume that the exact solution $u$ satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence results hold:*

1. *$H_0^1$-error estimate*

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \leq Ch\|u\|_{2,\overline{\Omega}}, \tag{6}$$

*where $\|\cdot\|_{1,\mathcal{T}}$ is the $H_1^0$-norm* $\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 = \sum_{\sigma=K|L\in\mathcal{E}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(u_L - u_K)^2.$

2. *$L^2$-error estimate:*

$$\|e_{\mathcal{T}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{7}$$

# Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

## Definition (New mesh of Eymard et *al.*, IMAJNA 2010)

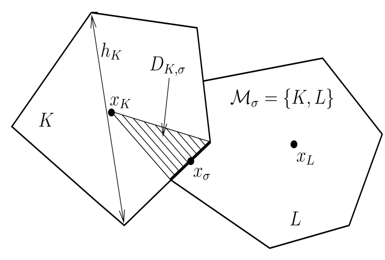


Figure : Notations for two neighbouring control volumes in $d = 2$

# Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

**Main properties of this new mesh:**

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (orthogonality property is not required): the orthogonality property is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convex, whereas the convexity property is not required in this new mesh.

# Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

## Principles of discretization for Poisson's equation:

1. Discrete unknowns: the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \{ \left( (v_K)_{K\in\mathcal{M}} \,,\, (v_\sigma)_{\sigma\in\mathcal{E}} \right) , \;\; v_K, v_\sigma \in \mathbb{R}, \;\; v_\sigma = 0, \, \forall \sigma \in \mathcal{E}_{\text{ext}} \}$$

2. Discretization of the gradient: the discretization of $\nabla$ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see Eymard et *al.* (IMAJNA, 2010):
   1. The discrete gradient $\nabla_{\mathcal{D}}$ is stable
   2. The discrete gradient $\nabla_{\mathcal{D}}$ is consistent.

# Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI

**Weak formulation for Poisson's equation:** Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega). \tag{8}$$

**SUSHI (Scheme Using stabilized Hybrid Interfaces) for Poisson's equation:** Find $u_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\int_\Omega \nabla_{\mathcal{D}} u_{\mathcal{D}}(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{9}$$

# Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI

**Theorem**

*Assume that the exact solution $u$ satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence result hold:*

1. *$H_0^1$-error estimate*

$$\|\nabla u - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{10}$$

2. *$L^2$-error estimate:*

$$\|u - \Pi_{\mathcal{M}} u_{\mathcal{D}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{11}$$

# Principles of the discretization

## Definition of a discretization in time and its parameters

- The time discretization is performed with a constrained time step-size $k$ such that $\frac{\tau}{k} \in \mathbb{N}$. We set then $k = \frac{\tau}{M}$, where $M \in \mathbb{N} \setminus \{0\}$. We denote by $\partial^1$ the discrete first time derivative given by $\partial^1 v^{j+1} = \dfrac{v^{j+1} - v^j}{k}$.
- Denote by $N$ the integer part of $\frac{T}{k}$, i.e. $N = \left[\frac{T}{k}\right]$.
- We shall denote $t_n = nk$, for $n \in [\![-M, N]\!]$.
- As particular cases $t_{-M} = -\tau$, $t_0 = 0$, and $t_N \leq T$.

## Advantages of this time discretization

The point $t = 0$ is a mesh point which is suitable since we have equation (1) defined for $t \in (0, T)$ and initial condition (2) defined for $t \in (-\tau, 0)$.

# Principles of the discretization (suite)

## Discretization in space

We use SUSHI scheme

## Formulation of scheme

The finite volume scheme can then be defined as:

- Discretization of initial condition: For any $n \in [\![-M, 0]\!]$

$$\left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n}, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\left(\Delta u^0(t_n), \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}, \qquad (12)$$

- Discretization of the delayed equation: and for any For any $n \in [\![0, N-1]\!]$, find $u_{\mathcal{D}}^n \in \mathcal{X}_{\mathcal{D},0}$ such that, for all $v \in \mathcal{X}_{\mathcal{D},0}$

$$\begin{aligned}\left(\partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} &+ \left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} \\ &= \left(f(t_{n+1}, \Pi_{\mathcal{M}} u_{\mathcal{D}}^n, \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1-M}), \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}, \qquad (13)\end{aligned}$$

where $f(t_{n+1}, \Pi_{\mathcal{M}} u_{\mathcal{D}}^n, \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1-M})$ denotes the function

$$\boldsymbol{x} \mapsto f(\boldsymbol{x}, t_{n+1}, \Pi_{\mathcal{M}} u_{\mathcal{D}}^n(\boldsymbol{x}), \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1-M}).$$

# Useful Assumption

To deal with the convergence analysis, we need the following assumption on the function $f$:

**Assumption (Assumption on $f$)**

*We assume that the function $f(\boldsymbol{x}, t, s, r)$ is Lipschitz continuous with respect to $(s, r)$ with constant $\kappa$, i.e. for all $(\boldsymbol{x}, t, s, r), (\boldsymbol{x}, t, s', r') \in \Omega \times \mathbb{R}^3$*

$$|f(\boldsymbol{x}, t, s, r) - f(\boldsymbol{x}, t, s', r')| \le \kappa \left(|s - s'| + |r - r'|\right).$$

## Statement of the convergence results

**Theorem (Error estimates)**

*We assume that u is sufficiently smooth.*

- $\mathbb{L}^\infty(H_0^1)$–*estimate. For all* $n \in [\![-M, N]\!]$

$$\|\nabla_{\mathcal{D}} u_{\mathcal{D}}^n - \nabla u(t_n)\|_{\mathbb{L}^2(\Omega)} \le C(k + h_{\mathcal{D}}). \tag{14}$$

- $W^{1,2}(\mathbb{L}^2)$–*estimate.*

$$\left( \sum_{n=-M+1}^{N} k \left\| u_t(t_n) - \Pi_{\mathcal{M}} \partial^1 u_{\mathcal{D}}^n \right\|_{\mathbb{L}^2(\Omega)}^2 \right)^{\frac{1}{2}} \le C(k + h_{\mathcal{D}}). \tag{15}$$

# Idea on the proof

The proof is manly based on two facts:

- Comparison with an optimal scheme : for any $n \in [\![ 0, N+1 ]\!]$, find $\bar{u}^n_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\nabla_{\mathcal{D}} \bar{u}^n_{\mathcal{D}}, \nabla_{\mathcal{D}}\, v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\sum_{K\in\mathcal{M}} v_K \int_K \Delta u(x, t_n)dx, \;\; \forall\, v \in \mathcal{X}_{\mathcal{D},0}. \qquad (16)$$

- A convenient a priori estimate.

# Conclusion and Perspectives

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We considered the convergence of an implicit finite volume scheme, in any space dimension, for a simple semi-linear delay parabolic equation. The order is proved to be one (both in time and space)

## First perspective

The use of Crank Nicolson method in order to improve the order in time.

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## First perspective

The use of Crank Nicolson method in order to improve the order in time.

# Perspectives

## Second perspective

Extension to the the case when the right hand side involves the exact solution and its gradient

$$u_t(\boldsymbol{x},t)-\Delta u(\boldsymbol{x},t)=f\left(\boldsymbol{x},t,u(\boldsymbol{x},t),u(\boldsymbol{x},t-\tau),\nabla u(\boldsymbol{x},t),\nabla u(\boldsymbol{x},t-\tau)\right).$$

## Third perspective

Several delays.

## Fourth perspective

Delays are not numbers but functions depending on $t$.

# Perspectives

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Extension to the the case when the right hand side involves the exact solution and its gradient

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