New optimal $L^\infty(H^1)$–error estimate of a SUSHI scheme for time fractional diffusion equations

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Problem to be soved

Time Fractional Diffusion Equation:

$$\partial_t^\alpha u(\mathbf{x},t) - \Delta u(\mathbf{x},t) = f(\mathbf{x},t), \qquad (\mathbf{x},t)\in\Omega\times(0,T), \tag{1}$$

where

- Ω is an open polyhedral bounded subset in $\mathbb{R}^d$,
- $T>0$, $0<\alpha<1$ are given.
- f is a given smooth function,
- The operator ∂_t^α is the Caputo derivative defined by:

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u_t(s)ds. \tag{2}$$

- Initial condition is given by

$$u(\mathbf{x},0)=0, \qquad \mathbf{x}\in\Omega. \tag{3}$$

- Homogeneous Dirichlet boundary conditions are given by

$$u(\mathbf{x},t)=0, \qquad (\mathbf{x},t)\in\partial\Omega\times(0,T). \tag{4}$$

Application..

Fractional differential equations have been successfully used in theory and they appear in many areas of application, see [7].

Finite volume mesh

The finite volume mesh considered is the one used in [5]. Among the properties of this mesh, we quote

- This new generic mesh is a generalization of the one introduced in [6].
- The control volumes are not necessary convex.
- No orthogonality is required.
- The discrete unknowns are located at the centers of the control volumes and at their interfaces.

Discrete Gradient

- **A discrete space.** $X_\mathcal{D}$ as the set of all $\left((v_K)_{K\in\mathcal{M}},(v_\sigma)_{\sigma\in\mathcal{E}}\right)$, where $v_K, v_\sigma\in\mathbb{R}$ for all $K\in\mathcal{M}$ and for all $\sigma\in\mathcal{E}$ such that $u_\sigma=0$.
- **A discrete Gradient.** For $u=\left((u_K)_{K\in\mathcal{M}},(u_\sigma)_{\sigma\in\mathcal{E}}\right)\in X_\mathcal{D}$, we define, for all $K\in\mathcal{M}$

$$\nabla_\mathcal{D}u(x) = \nabla_{K,\sigma}u, \ \text{ a. e. } x\in\mathcal{D}_{K,\sigma}, \tag{5}$$

where $\mathcal{D}_{K,\sigma}$ is the cone with vertex x_K and basis σ and

$$\nabla_{K,\sigma}u = \nabla_K u + \left(\frac{\sqrt{d}}{d_{K,\sigma}}\left(u_\sigma-u_K-\nabla_K u\cdot(x_\sigma-x_K)\right)\right)\mathbf{n}_{K,\sigma}, \tag{6}$$

where $\nabla_K u = \frac{1}{\mathrm{m}(K)}\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}(\sigma)\left(u_\sigma-u_K\right)\mathbf{n}_{K,\sigma}$ and d is the space dimension.

Finite volume space

We use the finite volume space considered in [4], that is $\mathcal{H}_\mathcal{D}\subset\mathbb{L}^2(\Omega)$ of functions which are constant on each control volume K of $\mathcal{M}$. We associate any $\sigma\in\mathcal{E}_{\rm int}$ with a family of real numbers $\left(\beta_\sigma^K\right)_{K\in\mathcal{M}}$ such that

$$1=\sum_{K\in\mathcal{M}}\beta_\sigma^K \quad \text{and} \quad \mathbf{x}_\sigma=\sum_{K\in\mathcal{M}}\beta_\sigma^K\mathbf{x}_K. \tag{7}$$

Then, for any $u\in\mathcal{H}_\mathcal{D}$, we set $u_\sigma=\sum_{K\in\mathcal{M}}\beta_\sigma^K u_K$, for all $\sigma\in\mathcal{E}_{\rm int}$ and $u_\sigma=0$, for all $\sigma\in\mathcal{E}_{\rm ext}$.

Time discretization and discrete temporal derivative

The discretization of $[0,T]$ is performed with a constant time step

$$k=\frac{T}{N+1},$$

where $N\in\mathbb{N}^\star$

$$t_n=nk, \forall n\in[\![0,N+1]\!].$$

The discrete temporal derivative given by

$$\partial^1 v^n = \frac{v^n-v^{n-1}}{k}.$$

Approximation of the Caputo derivative

$$\partial_t^\alpha u(t_{n+1}) = \sum_{j=0}^n k\lambda_j^{n+1}\partial^1 u(t_{j+1}) + \mathbb{T}_1^{n+1}(u), \tag{8}$$

where

$$\lambda_j^{n+1} = \frac{(n-j+1)^{1-\alpha}-(n-j)^{1-\alpha}}{k^\alpha\Gamma(2-\alpha)} \tag{9}$$

and

$$|\mathbb{T}_1^{n+1}(u)|\le Ck^{2-\alpha}. \tag{10}$$

Properties of the approximation of the Caputo derivative

$$\frac{k^{-\alpha}}{\Gamma(2-\alpha)} = \lambda_n^{n+1} > \dots > \lambda_0^{n+1} \ge \lambda_0 = \frac{T^{-\alpha}}{\Gamma(1-\alpha)}$$

and

$$\sum_{j=0}^n k\lambda_j^{n+1} \le \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}.$$

Formulation of a Finite Volume scheme

$$\sum_{j=0}^n k\lambda_j^{n+1}\left(\partial^1 u_\mathcal{D}^{j+1}, v\right)_{L^2(\Omega)} + \left(\nabla_\mathcal{D} u_\mathcal{D}^{n+1},\nabla_\mathcal{D} v\right)_{L^2(\Omega)} = \left(f(t_{n+1}),v\right)_{L^2(\Omega)}, \tag{11}$$

where $u_\mathcal{D}^0$ is defined using a discrete projection.

New $\mathbb{L}^\infty(H_0^1)$–Error estimate for the Scheme (11)

- This results improves the one of [1] which dealt with only $\mathbb{L}^\infty(L^2)$–Error estimate.
- Error estimate in the discrete norm of $\mathbb{L}^\infty(H_0^1)$

$$\max_{n=0}^{n=N+1}\|\nabla u(t_n)-\nabla_\mathcal{D}u_\mathcal{D}^n\|_{\mathbb{L}^2(\Omega)} \le C(k^{2-\alpha}+h_\mathcal{D}). \tag{12}$$

Main idea on the proof

- We use the discrete Laplace of [4].
- A well-developed discrete a priori estimate..

In Progress

- Extension to GDM [3].
- Extension to Second order time accurate schemes.

References

1. Bradji, A., Fuhrmann, J.: Convergence order of a finite volume scheme for the time-fractional diffusion equation. Numerical analysis and its applications, 33–45, Lecture Notes in Comput. Sci., 10187, Springer, Cham, 2017.

2. Bradji, A., Fuhrmann, J.: Some abstract error estimates of a finite volume scheme for a nonstationary heat equation on general nonconforming multidimensional spatial meshes. Appl. Math. 58/1, 1–38 (2013).

3. Droniou, J., Eymard, R., Gallouët, T., Guichard, C., Herbin, R.: The Gradient Discretisation Method. Mathématiques et Applications, 82, Springer Nature Switzerland AG, Switzerland, 2018.

4. Eymard, R., Gallouët, T., Herbin, R., Linke, A.: Finite volume schemes for the biharmonic problem on general meshes. Math. Comput. 81/280, 2019–2048 (2012).

5. Eymard, R., Gallouët, T., Herbin, R.: Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. IMA J. Numer. Anal. 30/4, 1009–1043 (2010).

6. Eymard R., Gallouët T. and Herbin R.: Finite Volume Methods, Handbook for Numerical Analysis, Ph. Ciarlet J.L. Lions (Eds.), North Holland, 2000, vol. VII pp. 715-1022.

7. Jin, B., Lazarov, R., Liu, Y., Zhou, Z.: The Galerkin finite element method for a multi-term time-fractional diffusion equation. J. Comput. Phys. 281, 825–843 (2015).

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