# New gradient scheme of a time fractional Fokker-Planck equation with time independent forcing and its convergence analysis

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## Original Problem: Fokker-Planck Equation

The time fractional Fokker–Planck equation can be written as follows:

$$\partial_t u(\boldsymbol{x},t) - \nabla\cdot\left({}_{\mathrm{RL}}\partial_t^{1-\alpha}\kappa_\alpha \nabla u(\boldsymbol{x},t) - \mathbf{F}(\boldsymbol{x},t)\,{}_{\mathrm{RL}}\partial_t^{1-\alpha}u(\boldsymbol{x},t)\right) = g(\boldsymbol{x},t), \quad (1)$$

- $\Omega$ is an open polyhedral bounded subset in $\mathbb{R}^d$
- $T>0$, $0<\alpha<1$, $\kappa_\alpha>0$
- $\mathbf{F}$ (the Force) and $g$ are given functions.
- The operator ${}_{\mathrm{RL}}\partial_t^{1-\alpha}u(t)$ is the Riemann–Liouville derivative defined by $\partial_t\left(\partial_t^{-\alpha}u(t)\right)$ with $\partial_t^{-\alpha}u(t)$ is the fractional integral operator:

$$\partial_t^{-\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds. \quad (2)$$

- Initial condition is given by

$$u(\boldsymbol{x},0)=0, \qquad \boldsymbol{x}\in\Omega. \quad (3)$$

- Homogeneous Dirichlet boundary conditions are given by

$$u(\boldsymbol{x},t)=0, \qquad (\boldsymbol{x},t)\in\partial\Omega\times(0,T). \quad (4)$$

## Some Physics and others about the problem

- The Fokker–Planck equations describe:
  **1.** The time evolution of the probability density function of the position.
  **2.**The velocity of a particle.
- When $\alpha$ tends to one and the diffusion coefficient is constant, we get the standard Fokker–Planck equation.

## Simple version for the Fokker-Planck equation

We assume that the driving force $\mathbf{F}$ is independent of time, $\mathbf{F}=\mathbf{F}(\boldsymbol{x})$.
The equation can be written then as:

$$\partial_t u - {}_{\mathrm{RL}}\partial_t^{1-\alpha}\nabla\cdot(\kappa_\alpha\nabla u - \mathbf{F}u) = g.$$

By acting the operator $\partial_t^{\alpha-1}$ on the both sides of the last equation yields

$$\partial_t^{\alpha}u - \nabla\cdot(\kappa_\alpha\nabla u - \mathbf{F}u) = f, \quad (5)$$

where $f=\partial_t^{\alpha-1}g$ and $\partial_t^\alpha u$ is the Caputo derivative of order $\alpha$ given by

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u_t(s)ds. \quad (6)$$

## Definition of an approximate gradient discretization, cf. [2]

**Approximate gradient discretization $\mathcal{D}$ is defined by**

$$\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$$

**1.** The set of discrete unknowns $X_{\mathcal{D},0}$ is a finite dimensional vector space on $\mathbb{R}$.

**2.** The linear mapping $\Pi_{\mathcal{D}} : X_{\mathcal{D},0}\to L^2(\Omega)$ is the reconstruction of the approximate function.

**3.** The gradient reconstruction $\nabla_{\mathcal{D}} : X_{\mathcal{D},0}\to L^2(\Omega)^d$ is a linear mapping which reconstructs, from an element of $X_{\mathcal{D},0}$, a "gradient" (vector-valued function) over $\Omega$. The gradient reconstruction must be chosen such that

$$\|\nabla_{\mathcal{D}}\cdot\|_{L^2(\Omega)^d} \quad \text{is a norm on } X_{\mathcal{D},0}.$$

## Parameters of an approximate gradient discretization $\mathcal{D}$

**1. Coercivity. Poincaré inequality:**

$$\|\Pi_{\mathcal{D}}v\|_{L^2(\Omega)} \le C_{\mathcal{D}}\|\nabla_{\mathcal{D}}v\|_{L^2(\Omega)^d}, \quad \forall v\in X_{\mathcal{D},0}. \quad (7)$$

**2. Strong consistency.**

$$S_{\mathcal{D}}(\varphi) = \min_{v\in X_{\mathcal{D},0}}\left(\|\Pi_{\mathcal{D}}v-\varphi\|^2_{L^2(\Omega)} + \|\nabla_{\mathcal{D}}v-\nabla\varphi\|^2_{L^2(\Omega)^d}\right)^{\frac12}.$$

**3. Dual consistency**

$$W_{\mathcal{D}}(\varphi) = \max_{u\in X_{\mathcal{D},0}\setminus\{0\}}\frac{1}{\|\nabla_{\mathcal{D}}u\|_{L^2(\Omega)^d}}\left|\int_\Omega\left(\nabla_{\mathcal{D}}u(\boldsymbol{x})\cdot\varphi(\boldsymbol{x}) + \Pi_{\mathcal{D}}u(\boldsymbol{x})\mathrm{div}\varphi(\boldsymbol{x})\right)d\boldsymbol{x}\right|.$$

## Time discretization and discrete temporal derivative

The discretization of $[0,T]$ is performed with a constant time step

$$k=\frac{T}{N+1},$$

where $N\in\mathbb{N}^\star$

$$t_n = nk, \forall n\in[\![0,N+1]\!].$$

The discrete temporal derivative given by

$$\partial^1 v^n = \frac{v^n - v^{n-1}}{k}.$$

## Approximation of the Caputo derivative

$$\partial_t^\alpha u(t_{n+1}) = \sum_{j=0}^n k\lambda_j^{n+1}\partial^1 u(t_{j+1}) + \mathbb{T}_1^{n+1}(u), \quad (8)$$

where

$$\lambda_j^{n+1} = \frac{(n-j+1)^{1-\alpha}-(n-j)^{1-\alpha}}{k^\alpha\Gamma(2-\alpha)} \quad (9)$$

and

$$|\mathbb{T}_1^{n+1}(u)| \le Ck^{2-\alpha}. \quad (10)$$

## Properties of the approximation of the Caputo derivative

$$\frac{k^{-\alpha}}{\Gamma(2-\alpha)} = \lambda_n^{n+1} > \ldots > \lambda_0^{n+1} \ge \lambda_0 = \frac{T^{-\alpha}}{\Gamma(1-\alpha)}.$$

## Formulation of a GS: First attempt

$$\sum_{j=0}^n k\lambda_j^{n+1}\left(\partial^1\Pi_{\mathcal{D}}u_{\mathcal{D}}^{j+1},\Pi_{\mathcal{D}}v\right)_{L^2(\Omega)} + \kappa_\alpha\left(\nabla_{\mathcal{D}}u_{\mathcal{D}}^{n+1},\nabla_{\mathcal{D}}v\right)_{L^2(\Omega)^d}$$
$$-\left(\mathbf{F}\Pi_{\mathcal{D}}u_{\mathcal{D}}^{n+1},\nabla_{\mathcal{D}}v\right)_{L^2(\Omega)^d} = \left(f(t_{n+1}),\Pi_{\mathcal{D}}v\right)_{L^2(\Omega)}, \quad (11)$$

where $u_{\mathcal{D}}^0=0$.

**Disadvantage: No discrete Coercivity.**

## Formulation of a GS: Second attempt-The right choice, see [1]

$$\sum_{j=0}^n k\lambda_j^{n+1}\left(\partial^1\Pi_{\mathcal{D}}u_{\mathcal{D}}^{j+1},\Pi_{\mathcal{D}}v\right)_{L^2(\Omega)} + \kappa_\alpha\left(\nabla_{\mathcal{D}}u_{\mathcal{D}}^{n+1},\nabla_{\mathcal{D}}v\right)_{L^2(\Omega)^d}$$
$$+\frac12\left(\mathbf{F}\cdot\nabla_{\mathcal{D}}u_{\mathcal{D}}^{n+1},\Pi_{\mathcal{D}}v\right)_{L^2(\Omega)} - \frac12\left(\mathbf{F}\Pi_{\mathcal{D}}u_{\mathcal{D}}^{n+1},\nabla_{\mathcal{D}}v\right)_{L^2(\Omega)^d}$$
$$+\frac12\left(\mathrm{div}(\mathbf{F})\Pi_{\mathcal{D}}u_{\mathcal{D}}^{n+1},\Pi_{\mathcal{D}}v\right)_{L^2(\Omega)} = \left(f(t_{n+1}),\Pi_{\mathcal{D}}v\right)_{L^2(\Omega)}, \quad \mathbf{(12)}$$

where $u_{\mathcal{D}}^0=0$.

## $\mathbb{L}^\infty(L^2)$ and $\mathbb{L}^2(H_0^1)$–Error estimates for the GS (12)

$$\max_{n=0}^{N+1}\|\Pi_{\mathcal{D}}u_{\mathcal{D}}^n - u(t_n)\|_{L^2(\Omega)} + \left(\sum_{n=0}^N k\|\nabla_{\mathcal{D}}u_{\mathcal{D}}^n - \nabla u(t_n)\|^2_{L^2(\Omega)^d}\right)^{\frac12}$$
$$\le C(1+C_{\mathcal{D}})\left(\mathbb{E}_{\mathcal{D}}^k(u) + k^{2-\alpha}\right), \quad \mathbf{(13)}$$

where

$$\mathbb{E}_{\mathcal{D}}^k(u) = \max_{j\in\{0,1\}}\max_{n\in[\![j,N+1]\!]}\mathbb{E}_{\mathcal{D}}(\partial^j u(t_n)) \quad (14)$$

and

$$\mathbb{E}_{\mathcal{D}}(\overline{u}) = (1+C_{\mathcal{D}})\left(W_{\mathcal{D}}(\nabla\overline{u}) + W_{\mathcal{D}}(\mathbf{F}\overline{u})\right) + (1+C_{\mathcal{D}}+C_{\mathcal{D}}^2)S_{\mathcal{D}}(\overline{u}). \quad (15)$$

## Main idea on the proof

**A well-developed discrete a priori estimate.**

## In Progress

- The force $\mathbf{F}$ is dependent on time, i.e. $\mathbf{F}=\mathbf{F}(\boldsymbol{x},t)$.
- Second order time accurate.

## References

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