Some convergence results of a multi-dimensional finite volume scheme for a time-fractional diffusion-wave equation

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Aim of the presentation

The aim of this talk is to provide a convergence rate for a finite volume scheme approximating a time fractional diffusion-wave equation.

Plan of this presentation

1. Equation to be solved
2. Discretization in time
3. Discretization in space, SUSHI method (Eymard et *al.*, IMAJNA 2010).
4. Finite Volume scheme for a time fractional diffusion-wave equation
5. Convergence rate of the numerical scheme
6. Working on and Perspectives

Equation to be solved

Equation

We consider the following time fractional diffusion equation:

$$\partial_t^\alpha u(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t), \ \ (\boldsymbol{x},t) \in \Omega \times (0,T), \tag{1}$$

where Ω an open polygonal bounded subset in $\mathbb{R}^d$. The operator ∂_t^α is the Caputo derivative:

$$\partial_t^\alpha u(\boldsymbol{x},t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \frac{\partial^2 u(\boldsymbol{x},s)}{\partial s^2} ds, \quad 1 < \alpha < 2, \tag{2}$$

Initial condition

Initial conditions are given by $u(\boldsymbol{x},0) = u^0(\boldsymbol{x})$ and $u_t(\boldsymbol{x},0) = u^1(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega$

Homogeneous Dirichlet boundary

$u(\boldsymbol{x},t) = 0, \ \ (\boldsymbol{x},t) \in \partial\Omega \times (0,T).$

What about time fractional diffusion equation?

Some physics

Fractional differential equations have been successfully used in the modeling of many different processes and systems. They are used, for instance, to describe anomalous transport in disordered semiconductors, penetration of light beam through a turbulent medium, transport of resonance radiation in plasma, blinking fluorescence of quantum dots, penetration and acceleration of cosmic ray in the Galaxy, and large-scale statistical Cosmography. We refer to the monograph Uchaikin (Fractional Derivatives for Physicists and Engineers, Springer-Verlag Heidelberg, 2013) where we find many details.

Principles of the discretization

Discretization in time

We define $k = T/(M+1)$ and mesh points $t_n = nk$. The scheme is based on $(t = t_{n+1})$:

$$\partial_t^\alpha u(t_{n+1}) - \Delta u(t_{n+1}) = f(t_{n+1}). \tag{3}$$

- First step: reduce the order of time derivative, for $\phi = u_t$

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \phi_s(s) ds. \tag{4}$$

- Second step: approximation of the Caputo w.r.t. $\phi = u_t$

$$\partial_t^\alpha u(t_{n+1}) = \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{n} \left(\int_{t_j}^{t_{j+1}} (t-s)^{1-\alpha} \right) \partial^1 \phi(t_{j+1}) + \mathbb{T}_1^{n+1}.$$

where ∂^1 is the discrete time derivative $\partial^1 \phi(t_{j+1}) = (\phi(t_{j+1}) - \phi(t_j))/k$.

Principles of the discretization (Suite): what about the remainder term $\mathbb{T}_1^{n+1}$?

Estimate on the remainder term $\mathbb{T}_1^{n+1}$.

$$|\mathbb{T}_1^{n+1}| \le \frac{\alpha^2 - 2\alpha + 3}{6\Gamma(4-\alpha)} k^{3-\alpha} \|u\|_{\mathcal{C}^3(0,T;\,\mathcal{C}(\overline{\Omega}))}. \tag{5}$$

Principles of the discretization (Suite):

Suite of Steps.

- Third step: Convenient form for the approximation ($\phi = u_t$).

$$\frac{1}{\Gamma(3-\alpha)}\sum_{j=0}^{n} d_{j,\alpha}\overline{\partial}^{\alpha}\varphi(t_{n-j+1}) \approx \partial_t^{\alpha} u(t_{n+1}), \tag{6}$$

where

$$\overline{\partial}^{\alpha} v^{j+1} = \frac{v^{j+1}-v^{j}}{k^{\alpha-1}} \quad \text{and} \quad d_{j,\alpha} = (j+1)^{2-\alpha} - j^{2-\alpha}. \tag{7}$$

Principles of the discretization (Suite):

Suite of Steps.

- Fourth step: Approximation w.r.t. u. Inserting the obtained approximation of $\partial_t^\alpha u(t_{n+1})$ in the equation to be solved and re-ordering the sums yield (recall that $u_t(0) = u^1$)

$$\frac{1}{k^{\alpha-1}\Gamma(3-\alpha)}\left(u_t(t_{n+1}) + \sum_{j=1}^{n}(d_{n-j+1,\alpha} - d_{n-j,\alpha})u_t(t_j)\right) - \Delta u(t_{n+1})$$
$$\approx \quad f(t_{n+1}) + \frac{1}{k^{\alpha-1}\Gamma(3-\alpha)} d_{n,\alpha} u^1, \qquad (8)$$

where this approximation is of order $k^{3-\alpha}$.

Principles of the discretization (Suite):

Suite of the Fourth Step.

Writing (8) in the level n and taking the mean value of the result with (8) gives

$$\frac{1}{k^{\alpha-1}\Gamma(3-\alpha)}\left(u_t^{n+\frac{1}{2}}+\sum_{j=1}^{n}(d_{n-j+1,\alpha}-d_{n-j,\alpha})u_t^{j-\frac{1}{2}}\right)-\Delta u^{n+\frac{1}{2}}$$
$$\approx f^{n+\frac{1}{2}}+\frac{1}{k^{\alpha-1}\Gamma(3-\alpha)}d_{n,\alpha}u^1, \tag{9}$$

where where we have denoted by $v^{n+\frac{1}{2}}$ the mean value

$$v^{n+\frac{1}{2}}=\frac{v^{n+1}+v^n}{2}.$$

Principles of the discretization (Suite):

Suite of the Fourth Step.

Since $u_t(t_{j+\frac{1}{2}}) - \partial u^1(t_{j+1})$ is of order two, then (9) becomes as

$$\frac{1}{k^{\alpha-1}\Gamma(3-\alpha)}\left(\partial^1 u(t_{n+1}) + \sum_{j=1}^{n}(d_{n-j+1,\alpha} - d_{n-j,\alpha})\partial^1 u(t_j)\right) - \Delta u^{n+\frac{1}{2}}$$

$$\approx \quad f^{n+\frac{1}{2}} + \frac{1}{k^{\alpha-1}\Gamma(3-\alpha)} d_{n,\alpha} u^1, \tag{10}$$

Multiplying by $\gamma = k^{\alpha-1}\Gamma(3-\alpha)$, (10) leads to

$$\partial^1 u(t_{n+1}) + \sum_{j=1}^{n}(d_{n-j+1,\alpha} - d_{n-j,\alpha})\partial^1 u(t_j) - \gamma\Delta u^{n+\frac{1}{2}}$$

$$\approx \quad \gamma f^{n+\frac{1}{2}} + d_{n,\alpha} u^1. \tag{11}$$

Discretization in space

Discretization in space

We use SUSHI scheme

Main properties of this new mesh:

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (orthogonality property is not required): the orthogonality property is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convex, whereas the convexity property is not required in this new mesh.

Discretization in space

1. Discrete unknowns: the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \{\left(\left(v_K\right)_{K\in\mathcal{M}},\ \left(v_\sigma\right)_{\sigma\in\mathcal{E}}\right),\ \ v_K, v_\sigma \in \mathbb{R},\ \ v_\sigma = 0,\ \forall\sigma \in \mathcal{E}_{\mathrm{ext}}\}$$

2. Discretization of the gradient: the discretization of ∇ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see Eymard et *al.* (IMAJNA, 2010):
    1. The discrete gradient $\nabla_{\mathcal{D}}$ is stable
    2. The discrete gradient $\nabla_{\mathcal{D}}$ is consistent.

Formulation of scheme

The finite volume scheme can then be defined as:

- Discretization of initial condition: Find $u_{\mathcal{D}}^{0} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\nabla_{\mathcal{D}}\, u_{\mathcal{D}}^{0}, \nabla_{\mathcal{D}}\, v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\left(\Delta u^{0}, \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)}, \quad \forall\, v \in \mathcal{X}_{\mathcal{D},0}, \tag{12}$$

- Discretization of the factional wave equation: and for any $n \in [\![0, M]\!]$, find $u_{\mathcal{D}}^{n+1} \in \mathcal{X}_{\mathcal{D},0}$ such that, for all $v \in \mathcal{X}_{\mathcal{D},0}$ for any $n \in [\![0, N]\!]$, find $u_{\mathcal{D}}^{n+1} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)} + \sum_{j=1}^{n} (d_{n-j+1,\alpha} - d_{n-j,\alpha}) \left(\partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^{j}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}$$

$$+ \quad \gamma \left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+\frac{1}{2}}, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = \left(\gamma f^{n+\frac{1}{2}} + d_{n,\alpha} u^{1}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} \tag{13}$$

Convergence result

Theorem (An error estimate for the gradient)

$$\|\nabla_{\mathcal{D}} u_{\mathcal{D}}^n - \nabla u(t_n)\|_{\mathbb{L}^2(\Omega)} \le C\left(C(\alpha)(k^{1-\alpha} h_{\mathcal{D}} + k^{3-\alpha}) + h_{\mathcal{D}}\right). \tag{14}$$

Assume in addition that for some given positive δ, the following relation holds:

$$k^{-(1+\alpha)} h_{\mathcal{D}} \le \delta. \tag{15}$$

Then the error estimate (14) implies that

$$\|\nabla_{\mathcal{D}} u_{\mathcal{D}}^n - \nabla u(t_n)\|_{\mathbb{L}^2(\Omega)} \le C(\alpha, \delta)\left(k^{3-\alpha} + h_{\mathcal{D}}\right). \tag{16}$$

Convergence result

Corollary (An L^2–error estimate)

If the hypothesis (15) is satisfied, the following $\mathbb{L}^\infty(\mathbb{L}^2)$-error estimate holds, for all $n \in [\![0, N+1]\!]$:

$$\|\Pi_{\mathcal{M}} u_{\mathcal{D}}^n - u(t_n)\|_{\mathbb{L}^2(\Omega)} \leq CC(\alpha, \delta) \left(k^{3-\alpha} + h_{\mathcal{D}} \right). \tag{17}$$

Idea on the proof

Idea on the proof

1. Comparison with an auxiliary scheme: for any $n \in [\![0, N+1]\!]$, find $\bar{u}_{\mathcal{D}}^n \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\nabla_{\mathcal{D}} \bar{u}_{\mathcal{D}}^n, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\left(\Delta u(t_n), \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{18}$$

2. A discrete *a priori estimate*
3. Other technical details can be found in Proceedings FVCA8, V1, Pages 391–399.

Conclusion

We considered an implicit finite volume scheme, involving the discrete gradient of SUSHI, to approximate a time-fractional diffusion-wave equation in any space dimension. We derived $L^\infty(H^1)$ and $L^\infty(L^2)$-error estimates.

First work under preparation

The order in time obtained is only $k^{3-\alpha}$ (between k and k^2, since $1 < \alpha < 2$). We have recently obtained full second order k^2.

Second work under preparation

Gradient schemes framework for time fractional diffusion equations and convergence under weak regularity on the exact solution.

Perspective

Gradient schemes framework for space fractional diffusion equations

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