A Convergence Result of a Linear SUSHI Scheme Using Characteristics Method for a Semi-Linear Parabolic Equation

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Aim of the presentation

The aim of this talk is to establish FVSs (Finite Volume Schemes) using Characteristic Methods along with a convergence analysis for a **Semi-Linear Parabolic Equation**. We set two FVSs: one is linear and the other one is nonlinear. For the sake of simplicity, we focus only on the **Linear FVS**.

The FVS we use here, as discretization in space, is **SUSHI** (Scheme Using Stabilization and Hybrid Interfaces)

Characteristics method is the replacement of the advective part of the equation by total differentiation along characteristics.

Plan of the talk

1. Problem to be solved.
2. Some Literature (References) on the subject.
3. Finite Volume methods on Nonconforming meshes (SUSHI scheme).
4. Definition of the Characteristics methods.
5. Formulation of TWO Finite Volume schemes using Characteristics methods: one is Linear and the other one is NON-Linear.
6. Convergence analysis for the LINEAR scheme.
7. A simple Theoretical Comparison with FV scheme derived directly from a Weak Formulation.
8. Conclusion and Perspectives.

Problem to be solved

Equation

SLUADP (Semi-Linear Unsteady Advection-Diffusion Problem), for $(\boldsymbol{x}, t) \in \Omega \times (0, T)$:

$$u_t(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) + \operatorname{div}(\mathbf{v}u)(\boldsymbol{x}, t) + b(\boldsymbol{x})u(\boldsymbol{x}, t) = f(u)(\boldsymbol{x}, t), \tag{1}$$

where Ω is a polyhedral open bounded connected subset of $\mathbb{R}^d$, $\mathbf{v} = \mathbf{v}(\boldsymbol{x}, t) \in \mathbb{R}^d$ is a vector field, and $b = b(\boldsymbol{x})$.

Initial and Homogeneous Dirichlet boundary conditions

$$u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega \ \text{ and } \ u(\boldsymbol{x}, t) = 0, \ (\boldsymbol{x}, t) \in \partial\Omega \times (0, T). \tag{2}$$

Assumptions on the data of the considered problem

Assumption

We assume that the functions $\boldsymbol{v}$*, b, and f are satisfying:*

$$\boldsymbol{v} \in \mathcal{C}^1(\overline{\Omega} \times [0, T]), \quad \boldsymbol{v}(\boldsymbol{x}, t) = 0, \quad \forall (\boldsymbol{x}, t) \in \partial\Omega \times [0, T], \tag{3}$$

$$b \in \mathcal{C}^1(\overline{\Omega}), \quad \text{and} \quad b(\boldsymbol{x}) + \mathrm{div}\boldsymbol{v}(\boldsymbol{x}, t) \geq 0, \quad \forall (\boldsymbol{x}, t) \in \Omega \times [0, T], \tag{4}$$

and $f \in \mathcal{C}^1(\mathbb{R})$ *such that, for a given* $\kappa > 0$

$$|f'(s)| \leq \kappa, \quad \forall s \in \mathbb{R}. \tag{5}$$

References

1. Benkhaldoun, F., Bradji, A.: Convergence Analysis of a Finite Volume Gradient Scheme for a Linear Parabolic Equation Using Characteristic Methods. LSSC'2019 (Large-Scale Scientific Computations, Lirkov and Margenov (Eds.))
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Principles of Finite Volume methods

Finite volume methods are numerical methods approximating different types of Partial Differential Equations (PDEs). They are based on three principle ideas:

- Subdivision of the spatial domain into subsets called Control Volumes.
- Integration of the equation to be solved over the Control Volumes.
- Approximation of the derivatives appearing after integration.

[1] We mean here the "pure" finite volume methods and not finite volume-element methods

Finite Volume methods using nonconforming grids, SUSHI scheme

Definition (New mesh of Eymard et *al.*, IMAJNA 2010)

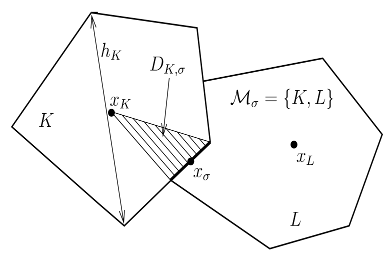

Figure : Notations for two neighbouring control volumes in $d = 2$

Finite Volume methods using nonconforming grids, SUSHI scheme

Main properties of this new mesh:

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (orthogonality property is not required): the orthogonality property which is required in the classical meshes is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convex, whereas the convexity property is not required in this new mesh.

Finite Volume methods using nonconforming grids, SUSHI scheme

Principles of discretization for Poisson's equation:

1. Discrete unknowns: the space of solution as well as the space of test functions are in (are located on the centers and on the edges of the control volumes)

$$\mathcal{X}_{\mathcal{D},0} = \{ \left((v_K)_{K\in\mathcal{M}} \,,\, (v_\sigma)_{\sigma\in\mathcal{E}} \right),\ \ v_K, v_\sigma \in \mathbb{R},\ \ v_\sigma = 0,\ \forall\sigma \in \mathcal{E}_{\text{ext}} \}$$

2. Discretization of the gradient: the discretization of ∇ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see Eymard et *al.* (IMAJNA, 2010):

   1. The discrete gradient $\nabla_{\mathcal{D}}$ is stable
   2. The discrete gradient $\nabla_{\mathcal{D}}$ is consistent.

Finite Volume methods using nonconforming grids, SUSHI

Weak formulation for Poisson's equation: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega). \tag{6}$$

SUSHI (Scheme Using stabilized Hybrid Interfaces) for Poisson's equation: Find $u_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\int_\Omega \nabla_{\mathcal{D}} u_{\mathcal{D}}(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{7}$$

Finite Volume methods using nonconforming grids, SUSHI

Theorem

Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence result hold:

1. H_0^1*-error estimate*

$$\|\nabla u - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{8}$$

2. L^2*-error estimate:*

$$\|u - \Pi_{\mathcal{M}} u_{\mathcal{D}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{9}$$

Definition of the Characteristics

Time discretization

We consider a constant time step $k = \dfrac{T}{N+1}$, where $N \in \mathbb{N}^\star$. The mesh points are $t_n = nk$, for $n \in [\![0, N+1]\!]$ and we denote by ∂^1 the discrete first time derivative:

$$\partial^1 v^{j+1} = \frac{v^{j+1} - v^j}{k}. \tag{10}$$

Definition of the Characteristics

For any $s \in [0, T]$ and $\boldsymbol{x} \in \Omega$, we define the characteristic lines associated to $\mathbf{v} = \mathbf{v}(\boldsymbol{x}, t)$ as the vector functions $\Phi = \Phi(t; \boldsymbol{x}, s) : [0, T] \longrightarrow \Omega$ satisfying the following differential equation:

$$\begin{cases} \dfrac{d\Phi}{dt}(t; \boldsymbol{x}, s) = \mathbf{v}(\Phi(t; \boldsymbol{x}, s), t), \quad t \in (0, T) \\[2ex] \Phi(s; \boldsymbol{x}, s) = \boldsymbol{x}. \end{cases} \tag{11}$$

Properties of the Characteristics

- The existence and uniqueness of the characteristic lines for each choice of s and and $\boldsymbol{x}$ hold under suitable assumptions on $\mathbf{v}$, for instance $\mathbf{v}$ continuous in $\overline{\Omega} \times [0, T]$ and Lipschitz continuous in $\overline{\Omega}$, uniformly with respect to $t \in [0, T]$.
- The uniqueness stated in the previous item implies that

$$\Phi(t; \Phi(s; \boldsymbol{x}, \tau), s) = \Phi(t; \boldsymbol{x}, \tau). \tag{12}$$

- Taking $t = \tau$ in (12) yields

$$\Phi(\tau; \Phi(s; \boldsymbol{x}, \tau), s) = \Phi(\tau; \boldsymbol{x}, \tau) = \boldsymbol{x}. \tag{13}$$

- For any t and s, the inverse of the function $\boldsymbol{x} \mapsto \Phi(t; \boldsymbol{x}, s)$ is $\boldsymbol{x} \mapsto \Phi(s; \boldsymbol{x}, t)$

Principles of scheme in time using Characteristics

1. Let us define
$$\overline{u}(\boldsymbol{x},t) = u(\Phi(t;\boldsymbol{x},0),t). \tag{14}$$

2. We have
$$\frac{\partial \overline{u}}{\partial t} - \overline{\Delta u} + \left(\overline{b} + \overline{\mathrm{div}\mathbf{v}}\right)\overline{u} = f(\overline{u}). \tag{15}$$
Taking $t = t_{n+1}$ as argument in equation (15) leads to
$$\frac{\partial \overline{u}}{\partial t}(t_{n+1}) - \overline{\Delta u}(t_{n+1}) + \left(\overline{b} + \overline{\mathrm{div}\mathbf{v}}(t_{n+1})\right)\overline{u}(t_{n+1}) = f(\overline{u}(t_{n+1})). \tag{16}$$

Principles of scheme in time using Characteristics (Suite)

3. Let us set

$$
\begin{aligned}
&\frac{\partial \overline{u}}{\partial t}(\Phi(0;\boldsymbol{x},t_{n+1}),t_{n+1})\\
&= \frac{u(\Phi(t_{n+1};\Phi(0;\boldsymbol{x},t_{n+1}),0),t_{n+1}) - u(\Phi(t_n;\Phi(0;\boldsymbol{x},t_{n+1}),0),t_n)}{k}\\
&+ \mathbb{T}_1^{n+1}(\boldsymbol{x})\\
&= \frac{u(\boldsymbol{x},t_{n+1}) - u(\Phi(t_n;\boldsymbol{x},t_{n+1}),t_n)}{k} + \mathbb{T}_1^{n+1}(\boldsymbol{x}). \qquad (17)
\end{aligned}
$$

Using a Taylor expansion to get

$$
|\mathbb{T}_1^{n+1}| \le Ck\|u\|_{\mathcal{C}^1([0,T];\,\mathcal{C}(\overline{\Omega}))}. \qquad (18)
$$

Principles of scheme in time using Characteristics (Suite)

4. Taking $\boldsymbol{x} = \Phi(0; \boldsymbol{x}, t_{n+1})$ in (16) and gathering the result with (17) yield

First approximation of the equation using Characteristics

$$\frac{u(t_{n+1}) - u(t_n)(\Phi(t_n; \boldsymbol{x}, t_{n+1}))}{k} - \Delta u(t_{n+1})$$
$$+ \quad (b + \operatorname{div}\mathbf{v}(t_{n+1}))\, u(t_{n+1}) = f(u(t_{n+1})) - \mathbb{T}_1^{n+1}(\boldsymbol{x}). \qquad (19)$$

This implies that

Second approximation of the equation using Characteristics

$$\frac{u(t_{n+1}) - u(t_n)(\Phi(t_n; \boldsymbol{x}, t_{n+1}))}{k} - \Delta u(t_{n+1})$$
$$+ \quad (b + \operatorname{div}\mathbf{v}(t_{n+1}))\, u(t_{n+1}) = f(u(t_n)) + \overline{\mathbb{T}}_1^{n+1}(\boldsymbol{x}), \qquad (20)$$

where $\overline{\mathbb{T}}_1^{n+1}$ is of order k.

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where $\overline{\mathbb{T}}_1^{n+1}$ is of order k.

Principles of scheme in time using Characteristics (Suite)

5. Approximation of the Characteristics $\Phi(t_n; \boldsymbol{x}, t_{n+1})$. Let us set

$$\Phi(t_n; \boldsymbol{x}, t_{n+1}) = \omega^{n+1}(\boldsymbol{x}) + \mathbb{T}_2^{n+1}(\boldsymbol{x}) \tag{21}$$

and

$$u(t_n)(\Phi(t_n; \boldsymbol{x}, t_{n+1})) = u(t_n)(\omega^{n+1}(\boldsymbol{x})) + \mathbb{T}_3^{n+1}(\boldsymbol{x}), \tag{22}$$

where

$$\omega^{n+1}(\boldsymbol{x}) = \boldsymbol{x} - k\mathbf{v}(\boldsymbol{x}, t_{n+1}). \tag{23}$$

We can check that ω^{n+1} is a second order accurate approximation for $\Phi(t_n; \cdot, t_{n+1})$. This implies that $\mathbb{T}_3^{n+1}$ is of order two, i.e.

$$|\mathbb{T}_3^{n+1}| \le Ck^2 \|u\|_{\mathcal{C}^1([0,T];\, \mathcal{C}(\overline{\Omega}))}.$$

Principles of scheme in time using Characteristics (Suite)

6. Approximation of the parabolic equation using Characteristics. Under Assumption 1 and the assumption that k is sufficiently small, we prove that

$$\omega^{n+1}(\boldsymbol{x}) \in \Omega.$$

From (19) and (21), we deduce that

First approximation using Characteristics

$$\Delta u(t_{n+1}) + f(u(t_{n+1})) = \frac{u(t_{n+1}) - u(t_n)(\omega^{n+1})}{k} + (b + \mathrm{div}\mathbf{v}(t_{n+1}))\, u(t_{n+1})$$
$$+ \quad \mathbb{T}_1^{n+1}(\boldsymbol{x}) - \frac{\mathbb{T}_3^{n+1}(\boldsymbol{x})}{k}. \qquad (24)$$

Principles of scheme in time using Characteristics (Suite)

6. From (20) and (21), we deduce that

Second approximation using Characteristics

$$\Delta u(t_{n+1}) + f(u(t_n)) = \frac{u(t_{n+1}) - u(t_n)(\omega^{n+1})}{k} + \left(b + \mathrm{div}\mathbf{v}(t_{n+1})\right) u(t_{n+1}) - \overline{\mathbb{T}}_1^{n+1}(\boldsymbol{x}) - \frac{\mathbb{T}_3^{n+1}(\boldsymbol{x})}{k}. \tag{25}$$

Principles of the discretization

Discretization in time

As stated before, uniform mesh.

Discretization in space

We use SUSHI scheme

Formulation of a First FVS (is **Non-Linear**)

Formulation of a NON-Linear FVS

- Discretization of initial condition:

$$\left(\nabla_{\mathcal{D}} u^0_{\mathcal{D}}, \nabla_{\mathcal{D}}\, v\right)_{(\mathbb{L}^2(\Omega))^d} = -\left(\Delta u^0(t_n), \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)}, \quad \forall\, v \in \mathcal{X}_{\mathcal{D},0}, \quad (26)$$

- Discretization of the Parabolic equation: For any $n \in [\![0, N]\!]$, find $u^{n+1}_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that, for all $v \in \mathcal{X}_{\mathcal{D},0}$

$$\begin{aligned} &\frac{1}{k}\left(\Pi_{\mathcal{M}} u^{n+1}_{\mathcal{D}} - \Pi_{\mathcal{M}} u^n_{\mathcal{D}}(\omega^{n+1}), \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)} + \left(\nabla_{\mathcal{D}} u^{n+1}_{\mathcal{D}}, \nabla_{\mathcal{D}}\, v\right)_{(\mathbb{L}^2(\Omega))^d} \\ &+ \quad \left(\left(b + \mathrm{div}\mathbf{v}(t_{n+1})\right) \Pi_{\mathcal{M}} u^{n+1}_{\mathcal{D}}, \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)} \\ &= \quad \left(f(u(t_{n+1})),\ \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)}. \end{aligned} \quad (27)$$

Formulation of a Second FVS scheme (is **Linear**)

Formulation of a Linear FVS

- Discretization of initial condition:

$$\left(\nabla_{\mathcal{D}} u^0_{\mathcal{D}}, \nabla_{\mathcal{D}}\, v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\left(\Delta u^0(t_n), \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)}, \quad \forall\, v \in \mathcal{X}_{\mathcal{D},0}, \quad (28)$$

- Discretization of the Parabolic equation: For any $n \in [\![0, N]\!]$, find $u^{n+1}_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that, for all $v \in \mathcal{X}_{\mathcal{D},0}$

$$\begin{aligned} \frac{1}{k}\left(\Pi_{\mathcal{M}} u^{n+1}_{\mathcal{D}} - \Pi_{\mathcal{M}} u^n_{\mathcal{D}}(\omega^{n+1}), \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)} + \left(\nabla_{\mathcal{D}} u^{n+1}_{\mathcal{D}}, \nabla_{\mathcal{D}}\, v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} \\ + \quad \left(\left(b + \mathrm{div}\mathbf{v}(t_{n+1})\right) \Pi_{\mathcal{M}} u^{n+1}_{\mathcal{D}}, \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)} \\ = \quad \left(f(u(t_n)), \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)}. \qquad (29) \end{aligned}$$

Statement of the convergence results for the Linear Scheme

Theorem (Error estimates)

We assume that u is sufficiently smooth and k is sufficiently small, there exists a unique solution and the following $L^\infty(L^2)$–error estimate holds:

$$\max_{n=0}^{N+1} \|\Pi_{\mathcal{M}} u_{\mathcal{D}}^n - u(t_n)\|_{L^2(\Omega)} \leq C \left(k + h_{\mathcal{D}} + \frac{h_{\mathcal{D}}}{k} \right) \|u\|_{\mathcal{C}^1([0,T];\, \mathcal{C}^2(\overline{\Omega}))}. \tag{30}$$

If we assume in addition that for some given positive δ, we have $h_{\mathcal{D}} \leq C k^{1+\delta}$, then the error estimate (30) becomes as

$$\max_{n=0}^{N+1} \|\Pi_{\mathcal{M}} u_{\mathcal{D}}^n - u(t_n)\|_{L^2(\Omega)} \leq C \left(k + k^\delta + h_{\mathcal{D}} \right) \|u\|_{\mathcal{C}^1([0,T];\, \mathcal{C}^2(\overline{\Omega}))}. \tag{31}$$

Idea on the proof

The proof is manly based on two facts:

- Comparison with an optimal scheme : for any $n \in [\![0, N+1]\!]$, find $\bar{u}^n_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$(\nabla_{\mathcal{D}} \bar{u}^n_{\mathcal{D}}, \nabla_{\mathcal{D}} v)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\sum_{K \in \mathcal{M}} v_K \int_K \Delta u(x, t_n) dx, \ \forall \, v \in \mathcal{X}_{\mathcal{D},0}. \qquad (32)$$

- A convenient a priori estimate.

A simple Theoretical Comparison with a FV derived directly

Weak Formulation

We multiply (1) by $\varphi \in H_0^1(\Omega)$ and use integration by parts

$$\int_\Omega u_t(t)\varphi + \int_\Omega \nabla u(t)\cdot\nabla\varphi - \sum_{i=1}^{d}\int_\Omega u\mathbf{v}_i(t)\frac{\partial\varphi}{\partial \boldsymbol{x}_i} + \int_\Omega bu(t)\varphi = \int_\Omega f(u(t))\varphi. \quad (33)$$

Scheme

For all $v \in \mathcal{X}_{\mathcal{D},0}$ ($\nabla^i_\mathcal{D} v$ denotes the i-th componenent of $\nabla_\mathcal{D} v$)

$$\left(\Pi_\mathcal{M}\partial^1 u_\mathcal{D}^{n+1}, \Pi_\mathcal{M} v\right)_{L^2(\Omega)} + \left(\nabla_\mathcal{D} u_\mathcal{D}^{n+1}, \nabla_\mathcal{D}\, v\right)_{\mathbb{L}^2(\Omega)} + \left(b\Pi_\mathcal{M} u_\mathcal{D}^{n+1}, \Pi_\mathcal{M} v\right)_{L^2(\Omega)}$$

$$- \sum_{i=1}^{d}\left(\mathbf{v}_i(t_{n+1})\Pi_\mathcal{M} u_\mathcal{D}^{n+1}, \nabla^i_\mathcal{D} v\right)_{L^2(\Omega)} = \left(f(u(t_n)), \Pi_\mathcal{M} v\right)_{L^2(\Omega)}. \quad (34)$$

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$$- \sum_{i=1}^{d}\left(\mathbf{v}_i(t_{n+1})\Pi_{\mathcal M} u_{\mathcal D}^{n+1}, \nabla^i_{\mathcal D} v\right)_{L^2(\Omega)} = \left(f(u(t_n)),\, \Pi_{\mathcal M} v\right)_{L^2(\Omega)}. \quad (34)$$

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$$- \sum_{i=1}^{d}\left(\mathbf{v}_i(t_{n+1})\Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \nabla^i_{\mathcal{D}} v\right)_{L^2(\Omega)} = \left(f(u(t_n)),\, \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)}. \quad (34)$$

Conclusion

We applied SUSHI combined with characteristics method to approximate SLUADP (Semi-Linear Unsteady Advection-Diffusion Problem) (1)–(4). This model is more general than that we considered in some of revious works (quoted in the References) in which we derived schemes directly along with a convergence analysis. It is also a continuation of the previous work of LSSC 2019 in which we studied FVS for the Linear case using the Characteristics method. We considered two schemes. One is linear and the other is Non-Linear Scheme. For the sake of simplicity, we focused on the Linear scheme and we proved an $L^\infty(L^2)$–error estimate which is a conditional convergence. This error estimate is proved thanks to a new well developed *a prior estimate*. This error estimate is similar to the one obtained for the case of Finite Element Methods for the Linear UADP (see the book of Quarteroni and references therein).

Perspectives

First perspective

To try to prove an unconditional convergence instead of the conditional one .

Second perspective

The proof of a convergence in the discrete energy norm of $L^\infty(H^1)$.

Third perspective

We study the NON-Linear scheme.

Fourth perspective

We study the Linearized Cranck-Nicolson method to get high convergence order in time.

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