On the convergence of a finite volume scheme for a parabolic equation

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Aim of the presentation

The aim of the present contribution is to deal with some error estimates of an implicit finite volume scheme for the heat equation.

Plan of this presentation

Problem to be solved

Non Stationary Heat equation:

$$u_t(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t), \;\; (\boldsymbol{x},t) \in \Omega \times (0,T), \tag{1}$$

where, Ω is an open bounded polyhedral subset in $\mathbb{R}^d$, with $d \in \mathbb{N}^\star$, $T > 0$, and f is a given function. An initial condition is given by:

$$u(\boldsymbol{x},0) = u^0(\boldsymbol{x}), \;\; \boldsymbol{x} \in \Omega. \tag{2}$$

A Dirichlet boundary condition is defined by

$$u(\boldsymbol{x},t) = 0, \;\; (\boldsymbol{x},t) \in \partial\Omega \times (0,T), \tag{3}$$

where, we denote by $\partial\Omega = \overline{\Omega} \setminus \Omega$ the boundary of Ω.

Some highlights on the problem to be solved

1. (Some physics): Heat equation $u_t - \Delta u$ is typically used in different applications, such as *fluid mechanics*, *heat and mass transfer*,...

2. (Existence and Uniqueness): existence and uniqueness of a weak solution of heat equation, with (2) (*initial condition*) and (3) (*Dirichlet boundary condition*) can be formulated using Bochner spaces; see for instance Evans book of partial differential equation

References on the subject

- Bradji, A.: Some simples error estimates for finite volume approximation of parabolic equations. Comptes Rendus de l'Académie de Sciences, Paris, 346/9-10, 571–574 (2008).
- (Handbook) Eymard, R., Gallouët T., Herbin, R.: Finite volume methods. Handbook of Numerical Analysis. P. G. Ciarlet and J. L. Lions (eds.), VII , 723–1020 (2000).

Finite Volume methods on Admissible meshes

Definition

Let $\mathcal{T}$ be an Admissible Mesh in the sense of Eymard et al. (Handbook, 2000).

$K \in \mathcal{T}$ are the control volumes and σ are the edges of the control volumes K.

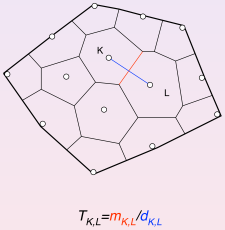

Figure : transmissivity between K and L: $\mathcal{T}_\sigma = \mathcal{T}_{K|L} = \frac{m_{K,L}}{d_{K,L}}$

Finite Volume methods on admissible meshes

Main properties of Admissible mesh:

1. Convexity of the Control Volumes.
2. The orthogonality property: the $(\boldsymbol{x}_K \boldsymbol{x}_L)$ is orthogonal to the common edge σ between the control volumes K and L.

Principles of Finite Volume methods

Finite volume methods [1] are numerical methods approximating different types of Partial Differential Equations (PDEs). They are based on three principle ideas:

- Subdivision of the spatial domain into subsets called Control Volumes.
- Integration of the equation to be solved over the Control Volumes.
- Approximation of the derivatives appearing after integration.

[1] We mean here the "pure" finite volume methods and not finite volume-element methods

Definition (Description of the method)

There are two variables in $u_t(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t),\ \boldsymbol{x} \in \Omega,\ t \in (0,T), \Omega \subset \mathbb{R}^d$

1. Domain space Ω is discretized using Admissible Mesh.
2. Uniform mesh on $(0,T)$ with constant step $k = T/(N+1)$. Mesh points $t_n = nk$ with $n = 0, \ldots, N$

Steps to derive the Finite Volume Scheme

1. Integration of $u_t - \Delta u$ on each $K \times (t_n, t_{n+1})$.
2. Integration by parts: $\int_K (u(\boldsymbol{x}, t_{n+1}) - u(\boldsymbol{x}, t_n)) - \int_{t_n}^{t_{n+1}} \int_{\partial K} \nabla u(\boldsymbol{x}, t) \cdot \mathbf{n}_K$, where $\mathbf{n}_K$ normal to ∂K outward to K
3. Summing on the edges of K:
$$\int_K (u(\boldsymbol{x}, t_{n+1}) - u(\boldsymbol{x}, t_n)) - \int_{t_n}^{t_{n+1}} \sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla u(\boldsymbol{x}, t) \cdot \mathbf{n}_{K,\sigma}.$$
4. We use the orthogonality property of admissible mesh and implicit choice:
$$\mathrm{m}(K)(u(\boldsymbol{x}_K, t_{n+1}) - u(\boldsymbol{x}_K, t_n)) - k \sum_{\sigma \in \mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{d_\sigma} (u(\boldsymbol{x}_L, t_{n+1}) - u(\boldsymbol{x}_K, t_{n+1})).$$

Definition of the Finite Volume Scheme

Definition (Definition of the Scheme)

The unknowns of the scheme are $\{u_K^n; K \in \mathcal{T}, \ n = 0, \ldots, N\}$, which are expected to approximates $u(\boldsymbol{x}_K, t_n)$, and satisfying:

1. Approximation of the equation $u_t - \Delta u = f$ on each $K \times (t_n, t_{n+1})$:

$$\mathrm{m}(K)(u_K^{n+1} - u_K^n) - k \sum_{\sigma \in \mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{d_\sigma} \left(u_L^{n+1} - u_K^n\right) = k \int_K f(\boldsymbol{x}, t_{n+1}) d\boldsymbol{x}. \tag{4}$$

2. Discretization of initial condition:

$$u_K^0 = u^0(\boldsymbol{x}_K), \ \forall K \in \mathcal{T}. \tag{5}$$

3. (discretization of the homogeneous Dirichlet boundary condition): $u_L^{n+1} = 0$ if $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}$.

A first convergence result

Theorem (A first convergence result: $L^\infty(L^2)$–error estimate)

The Scheme has a unique solution. Assume that the solution is sufficiently smooth. Then, the following $L^\infty(L^2)$–error estimate holds (cf. Handbook of Eymard et al. (2000)):

$$\sum_{K\in\mathcal{T}} \mathrm{m}(K)\left(u(x_K,t_n)-u_K^n\right)^2 \le C(h+k)^2, \quad \forall\, n=0,\ldots,N, \tag{6}$$

where h is the mesh size of the space discretization, that is $h=\sup\{\mathrm{diam}(K),\ K\in\mathcal{T}\}$.

A second convergence result

Theorem (A second convergence result: $L^\infty(H^1)$–error estimate)

Assume that the solution is sufficiently smooth. If the discrete initial condition (5) is chosen using an Orthogonal Projection for u^0, then the following $L^\infty(H^1)$–error estimate (cf. Bradji-2008), for all $n \in [\![0, N+1]\!]$

$$\sum_{\substack{\sigma\in\mathcal{E}\\ \sigma=K|L}} \mathrm{m}(\sigma) d_\sigma \left(\frac{u_L^n - u_K^n}{d_\sigma} - \frac{1}{\mathrm{m}(\sigma)} \int_\sigma \nabla u(x, t_n) \cdot \boldsymbol{n}_{K,\sigma} d\gamma(x) \right)^2 \leq C(h+k)^2. \tag{7}$$

Main ideas of the proof of the results

Two paths can be followed:

1. Using the truncation error.
2. Comparing with auxiliary scheme $\{\bar{u}_K^n : K \in \mathcal{T}, n \in [\![0, N+1]\!]\}$ defined by, for all $K \in \mathcal{T}$:

$$-\sum_{\sigma \in \mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{d_\sigma} \left(\bar{u}_L^n - \bar{u}_K^n\right) = -\int_K \Delta u(x, t_n) dx, \ \forall n = 0, \ldots, N+1. \tag{8}$$

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Applying these results to obtain error estimates of finite volume schemes for complex problems involving parabolic equations, for instance Navier Stokes equation.

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