Some Convergence Results for Mixed Finite Element Methods in the Divergence Norm

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Séminaire de Modélisation et Calcul Scientifique
Laboratoire LAGA-University of Paris Nord
June 10, Paris-France

On-Line Presentation

Aim of the presentation

We first give an overview on the approaches of MFEMs (Mixed Finite Element Methods): Primal and Dual MFEMs. We review some known convergence results of MFEMs for Elliptic and Parabolic. We then present some new convergence results of MFEMs for Parabolic and Second Order Hyperbolic Equations. We finally sketch some interesting perspectives.

Plan of this presentation

1. Overview on the approaches of MFEMs
   1. Primal MFEMs
   2. Dual MFEMs
2. Some known convergence results for Dual MFEMs for Elliptic Equations
3. Some known convergence results Dual MFEMs for Parabolic Equations.
4. New (recent) convergence results Dual MFEMs for Parabolic Equations.
5. New convergence results Dual MFEMs for Second Order Hyperbolic Equations.
6. Conclusion and Perspectives

References

- Benkhaldoun and Bradji, Two new error estimates of a fully discrete primal-dual mixed finite element scheme for parabolic equations in any space dimension. Results Math, 2021.
- Benkhaldoun and A. Bradji, A New Error Estimate for a Primal-Dual Crank-Nicolson Mixed Finite Element using Lowest Degree Raviart-Thomas Spaces for Parabolic Equations. Large-scale scientific computing, 489–497, LNCS, 13127, Springer, Cham, 2022.
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References (Suite)

- P. G. Ciarlet, The Finite Element Method for Elliptic Problems. Classics in Applied Mathematics, 40. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (2002).
- C. Johnson and V. Thomee, Error estimates for some mixed finite element methods for parabolic type problems. RAIRO Anal. Numér. 15/1, 41–78 (1981).
- A. K. Pani, An H^1-Galerkin mixed finite element method for parabolic partial differential equations. SIAM J. Numer. Anal. 35/2 (1998) 712–727.
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Primal MFEMs

Model Equation: Poisson Equation

Heat equation:

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is an open domain of $\mathbb{R}^d$, f is given function.

Homogeneous Dirichlet boundary

$$u(\boldsymbol{x}) = 0, \ \boldsymbol{x} \in \partial\Omega. \tag{2}$$

General principles of MFEM

First step: Writing the problem as:

$$p = -\nabla u. \tag{3}$$

and

$$\mathrm{div} p = f. \tag{4}$$

Second step: Weak formulation for [3]–[4]

We have at least two possible weak formulations:

1. Primal Weak Formulation.
2. Dual (or also Primal Dual) Weak Formulation

Primal Weak Formulation

Weak Formulation of Primal MFEMs

Find $(p,u) \in L^2(\Omega)^d \times H_0^1(\Omega)$ such that

$$(p,\tau)_{L^2(\Omega)} + (\nabla u,\tau)_{L^2(\Omega)} = 0, \quad \forall \tau \in L^2(\Omega)^d \tag{5}$$

and

$$(p,\nabla v)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega). \tag{6}$$

Dual Weak Formulation

Weak Formulation of Dual MFEMs

Find $(p,u) \in H_{\text{div}}(\Omega) \times L^2(\Omega)$ such that

$$(p,\psi)_{L^2(\Omega)} - (u, \text{div}\psi)_{L^2(\Omega)} = 0, \quad \forall \psi \in H_{\text{div}}(\Omega) \tag{7}$$

and

$$(\text{div}p, \varphi)_{L^2(\Omega)} = (f,\varphi)_{L^2(\Omega)}, \quad \forall \varphi \in L^2(\Omega), \tag{8}$$

$H_{\text{div}}(\Omega)$ is the space defined by

$$H_{\text{div}}(\Omega) = \left\{ \xi \in \left(L^2(\Omega)\right)^d : \quad \text{div}\xi \in L^2(\Omega) \right\}.$$

Finite Element spaces

Finite Element spaces

We consider two finite dimensional spaces $V_h^{\text{div}} \subset H_{\text{div}}(\Omega)$ and $W_h \subset L^2(\Omega)$ such that the following two hypotheses hold:

- Compatibility condition (also known as the inf − sup-condition). There exists $\beta^\star > 0$ independent of h such that, for all $q \in W_h$

$$\sup_{w \in V_h^{\text{div}} \setminus \{0\}} \frac{1}{\|w\|_{H_{\text{div}}(\Omega)}} \int_\Omega q(\boldsymbol{x}) \text{div} w(\boldsymbol{x}) d\boldsymbol{x} \geq \beta^\star \|q\|_{L^2(\Omega)}. \tag{9}$$

- The subspace G_h of V_h^{div} given by $G_h = \{w \in V_h^{\text{div}} : \int_\Omega q(\boldsymbol{x}) \text{div} w(\boldsymbol{x}) d\boldsymbol{x} = 0,\ \forall q \in W_h\}$ satisfies (This condition can be weakened.)

$$w \in G_h \quad \text{implies} \quad \text{that} \quad \text{div}\, w = 0. \tag{10}$$

An example of Finite Element spaces: $\mathbb{RT}_l$ (Raviart-Thomas) finite Element spaces

Example of space discretization: $\mathbb{RT}_l$ MFE

We define the Raviart-Thomas Mixed FE spaces, for $l \in \mathbb{N}$:

$$V_h^{\text{div}} = \{v \in H_{\text{div}}(\Omega) : \quad v|_K \in \mathbb{D}_l, \quad \forall K \in \mathcal{T}_h\}, \tag{11}$$

where $\mathcal{T}_h$ is a family of triangulations of $\overline{\Omega}$ with d-simplex and

$$W_h = \{p \in L^2(\Omega) : \quad p|_K \in \mathbb{P}_l, \quad \forall K \in \mathcal{T}_h\}, \tag{12}$$

where $\mathbb{P}_l$ is the space of d-variate polynomials on K having degree less than or equal to l and

$$\mathbb{D}_l = (\mathbb{P}_l)^d \oplus \boldsymbol{x}\mathbb{P}_l.$$

Nice property of $\mathbb{RT}_l$

Nice property of $\mathbb{RT}_l$

One of the main properties of the spaces $\mathbb{RT}_l$ is that

$$\operatorname{div} V_h^{\text{div}} \subset W_h.$$

DMFE (Dual Mixed Finite Element) scheme for the Poisson equation

The unknowns of the DMFE scheme for the Poisson equation

The unknowns of this scheme are the set of the couples

$$\left\{(p_h, u_h) \in V_h^{\text{div}} \times W_h\right\}.$$

Formulation of the DMFE scheme for the Poisson equation

Find $(p_h, u_h) \in V_h^{\text{div}} \times W_h$ such that

$$(p_h, \psi)_{L^2(\Omega)} - (u_h, \text{div}\psi)_{L^2(\Omega)} = 0, \quad \forall \psi \in V_h^{\text{div}} \tag{13}$$

and

$$(\text{div} p_h, \varphi)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)}, \quad \forall \varphi \in W_h. \tag{14}$$

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$$(p_h, \psi)_{L^2(\Omega)} - (u_h, \mathrm{div}\psi)_{L^2(\Omega)} = 0, \quad \forall \psi \in V_h^{\mathrm{div}} \tag{15}$$

and

$$(\mathrm{div} p_h, \varphi)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)}, \quad \forall \varphi \in W_h. \tag{16}$$

Well-Posedness and convergence result for the DMFES for the Poisson equation

Theorem (cf. Quarteroni and Valli, 2008)

- *The well-posedness result:*

$$\|p_h\|_{H_{\text{div}}(\Omega)} + \|u_h\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{17}$$

- *Error estimate:*

$$\|\nabla u + p_h\|_{H_{\text{div}}(\Omega)} + \|u - u_h\|_{L^2(\Omega)} \leq C\mathbb{E}_h(-\nabla u, u), \tag{18}$$

where $\mathbb{E}_h$ *is the error given by*

$$\mathbb{E}_h(P, U) = \inf_{\psi \in V_h^{\text{div}}} \|P - \psi\|_{H_{\text{div}}(\Omega)} + \inf_{\varphi \in W_h} \|U - \varphi\|_{L^2(\Omega)}. \tag{19}$$

Problem to be solved

Heat equation

$$u_t(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega \times (0,T). \tag{20}$$

Initial conditions

$$u(\boldsymbol{x},0) = u^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{21}$$

Homogeneous Dirichlet boundary

$$u(\boldsymbol{x},t) = 0, \;\; (\boldsymbol{x},t) \in \partial\Omega \times (0,T). \tag{22}$$

Space and time discretizations

Space discretization

The FE spaces $V_h^{\text{div}} \subset H_{\text{div}}(\Omega)$ and $W_h \subset L^2(\Omega)$ satisfy the hypotheses (9) and (10).

Time discretization

We consider constant time step $k = \dfrac{T}{N+1}$, where $N \in \mathbb{N}^\star$. The mesh points are denoted by $t_n = nk$, for $n \in [\![0, N+1]\!]$.

- **Discrete temporal derivative**

$$\partial^1 v^{n+1} = \frac{v^{n+1} - v^n}{k}.$$

- **Arithmetic mean value** (it serves when we use Crank-Nicolson method):

$$v^{n+\frac{1}{2}} = \frac{v^{n+1} + v^n}{2}.$$

Formulation of a MFE scheme for the Heat Equation

The unknowns of this scheme are the set of the couples

$$\left\{(p_h^n, u_h^n) \in V_h^{\text{div}} \times W_h; n \in [\![0, N+1]\!]\right\}.$$

These unknowns are expected to approximate the set of the unknowns

$$\{(-\nabla u(t_n), u(t_n)); \quad n \in [\![0, N+1]\!]\}.$$

Formulation of a DMFE scheme for the Heat Equation (Suite)

- For any $n \in [\![0, N]\!]$ and for all $\varphi \in W_h$:

$$\left(\partial^1 u_h^{n+1}, \varphi\right)_{L^2(\Omega)} + \left(\nabla \cdot p_h^{n+1}, \varphi\right)_{L^2(\Omega)} = (f(t_{n+1}), \varphi)_{L^2(\Omega)}\,, \tag{23}$$

- For any $n \in [\![0, N+1]\!]$:

$$(p_h^n, \psi)_{L^2(\Omega)^d} = (u_h^n, \nabla \cdot \psi)_{L^2(\Omega)}\,, \qquad \forall \psi \in V_h^{\mathrm{div}}, \tag{24}$$

where

$$\left(\nabla \cdot p_h^0, \varphi\right)_{L^2(\Omega)} = \left(-\Delta u^0, \varphi\right)_{L^2(\Omega)}\,, \qquad \forall \varphi \in W_h. \tag{25}$$

Known convergence result for the DMFES for the Heat Equation, cf. Johnson and Thomee 1981

Theorem ($L^\infty(L^2(\Omega)^d) \times L^\infty(L^2(\Omega))$–error estimate, Johnson and Thomé 1981)

For all $n \in [\![0, N+1]\!]$

$$\|\nabla u(t_n) + p_h^n\|_{L^2(\Omega)^d} + \|u(t_n) - u_h^n\|_{L^2(\Omega)}$$
$$\leq \quad C\left(\max_{j\in\{0,1\}} \max_{n=j}^{N+1} \mathbb{E}_h(-\nabla \partial^j u(t_n), \partial^j u(t_n)) + k\right), \tag{26}$$

where $\mathbb{E}_h$ *is the error given by (19).*

Principal and nice remark

Principal and nice remark

The error estimate (26) of Theorem 2 does not include the divergence of the velocity $p(t_n)$ whereas this divergence is present in the Elliptic case.

Our aim...

Our aim is to prove error estimates which include the divergence of the velocity $p(t_n)$ for:

- Heat Equation (as model of Parabolic Equations): Done
- Wave Equation (as model of Second Order Hyperbolic Equations). In Progress
- The Evolutionary Stokes Equations. In Progress

New convergence result, cf. Benkhaldoun and Bradji 2020

Theorem

$L^\infty(H_{\mathrm{div}}(\Omega)) \times L^\infty(L^2(\Omega))$–error estimate, cf. Benkhaldoun and Bradji 2020

$$\begin{aligned} &\max_{n=0}^{N+1} \|\nabla u(t_n) + p_h^n\|_{H_{\mathrm{div}}(\Omega)} + \max_{n=1}^{N+1} \|u_t(t_n) - \partial^1 u_h^n\|_{L^2(\Omega)} \\ \leq \quad & C\left(\max_{j\in\{0,1,2\}} \max_{n=j}^{N+1} \mathbb{E}_h(-\nabla \partial^j u(t_n), \partial^j u(t_n)) + k \right). \end{aligned} \tag{27}$$

Idea on the proof of Theorem 3

Lemma (New a priori estimate)

Assume that $\left((\eta^n_{\mathcal{D}})_{n=0}^{N+1}, (\overline{\eta}^n_{\mathcal{D}})_{n=0}^{N+1}\right) \in \left(V_h^{\mathrm{div}}\right)^{N+2} \times W_h^{N+2}$ *such that* $\overline{\eta}^0_h = 0$

- *For any* $n \in [\![0, N]\!]$, *for all* $\varphi \in W_h$:

$$\left(\partial^1 \overline{\eta}^{n+1}_h, \varphi\right)_{L^2(\Omega)} + \left(\mathrm{div}\eta^{n+1}_h, \varphi\right)_{L^2(\Omega)} = \left(\mathcal{S}^{n+1}, \varphi\right)_{L^2(\Omega)}, \tag{28}$$

- *For any* $n \in [\![0, N+1]\!]$:

$$(\eta^n_h, \psi)_{L^2(\Omega)^d} = (\overline{\eta}^n_h, \mathrm{div}\psi)_{L^2(\Omega)}, \qquad \forall \psi \in V_h^{\mathrm{div}}. \tag{29}$$

Then, the following $L^2(H_{\mathrm{div}})$*–a priori estimate holds:*

$$\max_{n=0}^{N+1} \|\mathrm{div}\eta^n_h\|_{L^2(\Omega)} \le C \max_{n=0}^{N} \|\mathcal{S}^{n+1}\|_{L^2(\Omega)}. \tag{30}$$

Formulation of a MFE scheme using Crank-Nicolson method (Suite)

- For any $n \in [\![0, N]\!]$ and for all $\varphi \in W_h$:

$$\left(\partial^1 u_h^{n+1}, \varphi\right)_{L^2(\Omega)} + \left(\nabla \cdot p_h^{n+\frac{1}{2}}, \varphi\right)_{L^2(\Omega)} = \left(\frac{f(t_{n+1}) + f(t_n)}{2}, \varphi\right)_{L^2(\Omega)}, \tag{31}$$

- For any $n \in [\![0, N+1]\!]$:

$$(p_h^n, \psi)_{L^2(\Omega)^d} = (u_h^n, \nabla \cdot \psi)_{L^2(\Omega)}, \qquad \forall \psi \in V_h^{\text{div}}, \tag{32}$$

where

$$\left(\nabla \cdot p_h^0, \varphi\right)_{L^2(\Omega)} = \left(-\Delta u^0, \varphi\right)_{L^2(\Omega)}, \qquad \forall \varphi \in W_h. \tag{33}$$

New convergence result, cf. Benkhaldoun and Bradji 2022

Theorem (New error estimate for scheme (31)–(31))

The following $L^2(H_{\text{div}})$–error estimate holds:

$$\max_{n=0}^{N+1} \|\text{div} p_h^n + \Delta u(t_n)\|^2_{L^2(\Omega)} \le C(k^2 + h). \tag{34}$$

DMFEMs for the Wave Equation

Wave equation

$$u_{tt}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega \times (0,T). \tag{35}$$

Initial conditions

$$u(0) = u^0 \quad \text{and} \quad u_t(0) = u^1. \tag{36}$$

Homogeneous Dirichlet boundary

$$u(\boldsymbol{x},t) = 0, \ (\boldsymbol{x},t) \in \partial\Omega \times (0,T). \tag{37}$$

Formulation of a DMFE scheme for the Wave Equation

- For any $n \in [\![0, N]\!]$ and for all $\varphi \in W_h$:

$$\left(\partial^2 u_h^{n+1}, \varphi\right)_{L^2(\Omega)} + \left(\nabla \cdot p_h^{n+1}, \varphi\right)_{L^2(\Omega)} = \left(f(t_{n+1}), \varphi\right)_{L^2(\Omega)}, \tag{38}$$

- For any $n \in [\![0, N+1]\!]$:

$$(p_h^n, \psi)_{L^2(\Omega)^d} = (u_h^n, \nabla \cdot \psi)_{L^2(\Omega)}, \qquad \forall \psi \in V_h^{\mathrm{div}}, \tag{39}$$

where, for $\varphi \in W_h$

$$\left(\nabla \cdot p_h^0, \varphi\right)_{L^2(\Omega)} = \left(-\Delta u^0, \varphi\right)_{L^2(\Omega)} \tag{40}$$

$$\left(\nabla \cdot p_h^1, \varphi\right)_{L^2(\Omega)} = \left(-k\Delta u^1 + u^0, \varphi\right)_{L^2(\Omega)}. \tag{41}$$

Formulation of another DMFE scheme for the Wave Equation: using Newmark's method

Definition of the main scheme

For any $n \in [\![0, N]\!]$ and for all $\varphi \in W_h$:

$$\left(\partial^2 u_h^{n+1}, \varphi\right)_{L^2(\Omega)} + \frac{1}{2}\left(\nabla \cdot (p_h^{n+1} + p_h^{n-1}), \varphi\right)_{L^2(\Omega)} = \frac{1}{2}\left(f(t_{n+1}) + f(t_{n-1}), \varphi\right)_{L^2(\Omega)}. \quad (42)$$

Discrete initial conditions

The discrete initial conditions should be chosen carefully to get second order time accurate.

Formulation of another DMFE scheme for the Wave Equation: using Newmark's method

Definition of the main scheme

For any $n \in [\![0, N]\!]$ and for all $\varphi \in W_h$:

$$\begin{aligned} \left(\partial^2 u_h^{n+1}, \varphi\right)_{L^2(\Omega)} &+ \frac{1}{2}\left(\nabla \cdot (p_h^{n+1} + p_h^{n-1}), \varphi\right)_{L^2(\Omega)} \\ &= \frac{1}{2}\left(f(t_{n+1}) + f(t_{n-1}), \varphi\right)_{L^2(\Omega)}. \end{aligned} \tag{42}$$

Discrete initial conditions

The discrete initial conditions should be chosen carefully to get second order time accurate.

Works, related to the subject, under preparation

First work

DMFE for the Wave Equation

Second work

DMFE for the Evolutionary Stokes Equations.

Third work

DMFE for the Time Fractional Diffusion Equations.

Fourth work

Extension to Non-Uniform temporal mesh..

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