Some Convergence Results in Mixed Finite Elements Methods

Abdallah Bradji[1,2]
[1] Department of Mathematics, University of Annaba–Algeria
[2] Visiting Professor to LAGA-USPN-Paris
Based on joint works with Fayssal Benkhaldoun

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Aim of the presentation

We first give an overview on the approaches of MFEMs (Mixed Finite Element Methods): Primal and Dual MFEMs. We review some known convergence results of MFEMs for Elliptic and Parabolic. We then present some new convergence results of MFEMs for Parabolic and Hyperbolique equations. We also give some new obtained results on the super-convergence phenomenon of MFEMs applied to one dimensional Parabolic equations. We finally sketch some interesting perspectives.

Plan of this presentation

1. Overview on the approaches of MFEMs
   1. Primal MFEMs
   2. Dual MFEMs
2. Some known convergence results for Dual MFEMs for Elliptic Equations
3. Some known convergence results Dual MFEMs for Parabolic Equations.
4. New (recent) convergence results for Dual MFEMs for Parabolic Equations.
5. New convergence results Dual MFEMs for Second Order Hyperbolic Equations.
6. Some Super-convergence results.
7. Conclusion and Perspectives

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Primal MFEMs

Model Equation: Poisson Equation

Heat equation:

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is an open domain of $\mathbb{R}^d$, f is given function.

Homogeneous Dirichlet boundary

$$u(\boldsymbol{x}) = 0, \ \boldsymbol{x} \in \partial\Omega. \tag{2}$$

General principles of MFEM

First step: Writing the problem as:

$$p = -\nabla u. \tag{3}$$

and

$$\mathrm{div} p = f. \tag{4}$$

Second step: Weak formulation for [3]–[4]

We have at least two possible weak formulations:

1. Primal Weak Formulation.
2. Dual (or also Primal Dual) Weak Formulation

Primal Weak Formulation

Weak Formulation of Primal MFEMs

Find $(p, u) \in L^2(\Omega)^d \times H_0^1(\Omega)$ such that

$$(p, \tau)_{L^2(\Omega)} + (\nabla u, \tau)_{L^2(\Omega)} = 0, \quad \forall \tau \in L^2(\Omega)^d \tag{5}$$

and

$$-(p, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega). \tag{6}$$

Dual Weak Formulation

Weak Formulation of Dual MFEMs

Find $(p,u) \in H_{\text{div}}(\Omega) \times L^2(\Omega)$ such that

$$(p,\psi)_{L^2(\Omega)} - (u, \text{div}\psi)_{L^2(\Omega)} = 0, \quad \forall \psi \in H_{\text{div}}(\Omega) \tag{7}$$

and

$$(\text{div}p, \varphi)_{L^2(\Omega)} = (f,\varphi)_{L^2(\Omega)}, \quad \forall \varphi \in L^2(\Omega), \tag{8}$$

$H_{\text{div}}(\Omega)$ is the space defined by

$$H_{\text{div}}(\Omega) = \left\{ \xi \in \left(L^2(\Omega)\right)^d : \quad \text{div}\xi \in L^2(\Omega) \right\}.$$

Finite Element spaces

Finite Element spaces

Let $V_h^{\text{div}} \subset H_{\text{div}}(\Omega)$ and $W_h \subset L^2(\Omega)$ be two finite dimensional spaces such that:

- Compatibility condition (also known as the inf − sup-condition). There exists $\beta^\star > 0$ independent of h such that, for all $q \in W_h$

$$\sup_{w \in V_h^{\text{div}} \setminus \{0\}} \frac{1}{\|w\|_{H_{\text{div}}(\Omega)}} \int_\Omega q(\boldsymbol{x}) \text{div} w(\boldsymbol{x}) d\boldsymbol{x} \geq \beta^\star \|q\|_{L^2(\Omega)}. \tag{9}$$

- The subspace G_h of V_h^{div} given by $G_h = \{w \in V_h^{\text{div}} : \int_\Omega q(\boldsymbol{x}) \text{div} w(\boldsymbol{x}) d\boldsymbol{x} = 0, \ \forall q \in W_h\}$ satisfies (This condition can be weakened.)

$$w \in G_h \quad \text{implies} \quad \text{that} \quad \text{div}\, w = 0. \tag{10}$$

An example of Finite Element spaces: $\mathbb{RT}_l$ (Raviart-Thomas) finite Element spaces

Example of space discretization: $\mathbb{RT}_l$ MFE

We define the Raviart-Thomas Mixed FE spaces, for $l \in \mathbb{N}$:

$$V_h^{\text{div}} = \{v \in H_{\text{div}}(\Omega) : \quad v|_K \in \mathbb{D}_l, \quad \forall K \in \mathcal{T}_h\}, \tag{11}$$

where $\mathcal{T}_h$ is a family of triangulations of $\overline{\Omega}$ with d-simplex and

$$W_h = \{p \in L^2(\Omega) : \quad p|_K \in \mathbb{P}_l, \quad \forall K \in \mathcal{T}_h\}, \tag{12}$$

where $\mathbb{P}_l$ is the space of d-variate polynomials on K having degree less than or equal to l and

$$\mathbb{D}_l = (\mathbb{P}_l)^d \oplus \boldsymbol{x}\mathbb{P}_l.$$

Nice property of $\mathbb{RT}_l$

Nice property of $\mathbb{RT}_l$

One of the main properties of the spaces $\mathbb{RT}_l$ is that

$$\operatorname{div} V_h^{\text{div}} \subset W_h.$$

DMFE (Dual Mixed Finite Element) scheme for the Poisson equation

Formulation of the DMFE scheme for the Poisson equation

Find $(p_h, u_h) \in V_h^{\text{div}} \times W_h$ such that

$$(p_h, \psi)_{L^2(\Omega)} - (u_h, \text{div}\psi)_{L^2(\Omega)} = 0, \quad \forall \psi \in V_h^{\text{div}} \tag{13}$$

and

$$(\text{div}p_h, \varphi)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)}, \quad \forall \varphi \in W_h. \tag{14}$$

Well-Posedness and convergence result for the DMFES for the Poisson equation

Theorem (cf. Quarteroni and Valli, 2008)

- *The well-posedness result:*

$$\|p_h\|_{H_{\text{div}}(\Omega)} + \|u_h\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{15}$$

- *Error estimate:*

$$\|\nabla u + p_h\|_{H_{\text{div}}(\Omega)} + \|u - u_h\|_{L^2(\Omega)} \leq C\mathbb{E}_h(-\nabla u, u), \tag{16}$$

where $\mathbb{E}_h$ *is the error given by*

$$\mathbb{E}_h(P, U) = \inf_{\psi \in V_h^{\text{div}}} \|P - \psi\|_{H_{\text{div}}(\Omega)} + \inf_{\varphi \in W_h} \|U - \varphi\|_{L^2(\Omega)}. \tag{17}$$

Problem to be solved

Heat equation

$$u_t(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega \times (0,T). \tag{18}$$

Initial and Dirichlet boundary conditions

$$u(\boldsymbol{x},0) = u^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \quad \text{and} \quad u(\boldsymbol{x},t) = 0, \; (\boldsymbol{x},t) \in \partial\Omega \times (0,T). \tag{19}$$

Space and time discretizations

Space discretization

The FE spaces $V_h^{\text{div}} \subset H_{\text{div}}(\Omega)$ and $W_h \subset L^2(\Omega)$ satisfy the hypotheses (9) and (10).

Time discretization

Constant time step $k = \dfrac{T}{N+1}$, where $N \in \mathbb{N}^\star$. The mesh points are $t_n = nk$.

- **Discrete temporal derivative**

$$\partial^1 v^{n+1} = \frac{v^{n+1} - v^n}{k}.$$

- **Arithmetic mean value** (it serves when we use Crank-Nicolson method):

$$v^{n+\frac{1}{2}} = \frac{v^{n+1} + v^n}{2}.$$

Formulation of a MFE scheme for the Heat Equation

The unknowns of this scheme are the set of the couples

$$\left\{(p_h^n, u_h^n) \in V_h^{\mathrm{div}} \times W_h; n \in [\![0, N+1]\!]\right\}.$$

These unknowns are expected to approximate the set of the unknowns

$$\{(-\nabla u(t_n), u(t_n)); \quad n \in [\![0, N+1]\!]\}.$$

Formulation of a DMFE scheme for the Heat Equation (Suite)

- For any $n \in [\![0, N]\!]$ and for all $\varphi \in W_h$:

$$\left(\partial^1 u_h^{n+1}, \varphi\right)_{L^2(\Omega)} + \left(\nabla \cdot p_h^{n+1}, \varphi\right)_{L^2(\Omega)} = (f(t_{n+1}), \varphi)_{L^2(\Omega)}\,, \tag{20}$$

- For any $n \in [\![0, N+1]\!]$:

$$(p_h^n, \psi)_{L^2(\Omega)^d} = (u_h^n, \nabla \cdot \psi)_{L^2(\Omega)}\,, \qquad \forall \psi \in V_h^{\mathrm{div}}, \tag{21}$$

where

$$\left(\nabla \cdot p_h^0, \varphi\right)_{L^2(\Omega)} = \left(-\Delta u^0, \varphi\right)_{L^2(\Omega)}\,, \qquad \forall \varphi \in W_h. \tag{22}$$

Known convergence result for the DMFES for the Heat Equation, cf. Johnson and Thomee 1981

Theorem ($L^\infty(L^2(\Omega)^d) \times L^\infty(L^2(\Omega))$–error estimate, Johnson and Thomé 1981)

For all $n \in [\![0, N+1]\!]$

$$\|\nabla u(t_n) + p_h^n\|_{L^2(\Omega)^d} + \|u(t_n) - u_h^n\|_{L^2(\Omega)} \leq C\left(\max_{j\in\{0,1\}} \max_{n=j}^{N+1} \mathbb{E}_h(-\nabla\partial^j u(t_n), \partial^j u(t_n)) + k\right), \tag{23}$$

where $\mathbb{E}_h$ *is the error given by (17).*

Principal and nice remark

Principal and nice remark

The error estimate (23) of Theorem 2 does not include the divergence of the velocity $p(t_n)$ whereas this divergence is present in the Elliptic case.

Our aim...

Our aim is to prove error estimates which include the divergence of the velocity $p(t_n)$ for:

- Heat Equation (as model of Parabolic Equations): Done
- Superconvergence of MFEMs for Parabolic equations: Some Done and other in Progress
- Wave Equation (as model of Second Order Hyperbolic Equations). In Progress
- The Evolutionary Stokes Equations. In Progress

New convergence result, cf. Benkhaldoun and Bradji 2021

Theorem

$L^\infty(H_{\mathrm{div}}(\Omega)) \times L^\infty(L^2(\Omega))$–error estimate, cf. Benkhaldoun and Bradji 2021

$$\max_{n=0}^{N+1} \|\nabla u(t_n) + p_h^n\|_{H_{\mathrm{div}}(\Omega)} + \max_{n=1}^{N+1} \|u_t(t_n) - \partial^1 u_h^n\|_{L^2(\Omega)}$$

$$\leq \quad C\left(\max_{j\in\{0,1,2\}} \max_{n=j}^{N+1} \mathbb{E}_h(-\nabla \partial^j u(t_n), \partial^j u(t_n)) + k\right). \qquad (24)$$

Idea on the proof of Theorem 3

Lemma (New a priori estimate)

Assume that $\left((\eta_{\mathcal{D}}^n)_{n=0}^{N+1}, (\overline{\eta}_{\mathcal{D}}^n)_{n=0}^{N+1}\right) \in \left(V_h^{\mathrm{div}}\right)^{N+2} \times W_h^{N+2}$ *such that* $\overline{\eta}_h^0 = 0$ *and for any* $n \in [\![0, N]\!]$, *for all* $\varphi \in W_h$

$$\left(\partial^1 \overline{\eta}_h^{n+1}, \varphi\right)_{L^2(\Omega)} + \left(\mathrm{div}\eta_h^{n+1}, \varphi\right)_{L^2(\Omega)} = \left(\mathcal{S}^{n+1}, \varphi\right)_{L^2(\Omega)}, \tag{25}$$

where $\mathcal{S}^{n+1} \in L^2(\Omega)$ *is given and for any* $n \in [\![0, N+1]\!]$

$$(\eta_h^n, \psi)_{L^2(\Omega)^d} = (\overline{\eta}_h^n, \mathrm{div}\psi)_{L^2(\Omega)}, \quad \forall \psi \in V_h^{\mathrm{div}}.$$

Then, the following $L^2(H_{\mathrm{div}})$*–a priori estimate holds:*

$$\max_{n=0}^{N+1} \|\mathrm{div}\eta_h^n\|_{L^2(\Omega)} \leq C \max_{n=0}^{N} \|\mathcal{S}^{n+1}\|_{L^2(\Omega)}. \tag{26}$$

Formulation of a MFE scheme using Crank-Nicolson method (Suite)

- For any $n \in [\![0, N]\!]$ and for all $\varphi \in W_h$:

$$\left(\partial^1 u_h^{n+1}, \varphi\right)_{L^2(\Omega)} + \left(\nabla \cdot p_h^{n+\frac{1}{2}}, \varphi\right)_{L^2(\Omega)} = \left(\frac{f(t_{n+1}) + f(t_n)}{2}, \varphi\right)_{L^2(\Omega)}, \tag{27}$$

- For any $n \in [\![0, N+1]\!]$:

$$(p_h^n, \psi)_{L^2(\Omega)^d} = (u_h^n, \nabla \cdot \psi)_{L^2(\Omega)}, \qquad \forall \psi \in V_h^{\mathrm{div}}, \tag{28}$$

where

$$\left(\nabla \cdot p_h^0, \varphi\right)_{L^2(\Omega)} = \left(-\Delta u^0, \varphi\right)_{L^2(\Omega)}, \qquad \forall \varphi \in W_h. \tag{29}$$

New convergence result, cf. Benkhaldoun and Bradji 2023

Theorem (New error estimate for scheme (27)–(29))

The following $L^2(H_{\text{div}})$–error estimate holds:

$$\max_{n=0}^{N} \|\text{div} p_h^{n+\frac{1}{2}} + \Delta u(t_{n+\frac{1}{2}})\|^2_{L^2(\Omega)} \leq C(k^2 + h). \tag{30}$$

Definition of Super-convergence, refer to Zlàmal-1977

Assume that $\bar{u}$ is an approximation of u using Finite Differences or Finite Element Methods on a physical domain Ω. Assume in addition that

$$\|\bar{u} - u\| \leq Ch^l, \tag{31}$$

where h is the mesh size of the discretization of ω.
We assume that there exists interpolation operator Π such that, for some $\sigma > 0$

$$\|\bar{u} - \Pi u\| \leq Ch^{l+\sigma}, \tag{32}$$

where Πu is a some interpolation of u.
The estimate (32) means that the convergence is better on the points in which u concides with its interpolation. This is called a Super-convergence phenomenon.

How the Super-convergence can serve us?

Super-convergence serves us...

Super-convergence serves us to improve the convergence order using a Local Post Processing for the approximate solution $\overline{u}$ (as in Durán-1999), or also using an iteration of approximations (using the original matrix that was used in the begining) as in Defect Correction (as in Bradji and Chibi 2007) .

Introduction: Super-convergence in Piece-wise Linear FE in 1D.

Consider the one dimensional stationary equation

$$-u_{xx}(\boldsymbol{x}) + u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathrm{I} = (0,1) \quad \text{and} \quad u(0) = u(1) = 0. \tag{33}$$

The mesh points of I are denoted by $0 = \boldsymbol{x}_0 < \boldsymbol{x}_1 \ldots < \boldsymbol{x}_{M+1} = 1$, with $M \in \mathbb{N} \setminus \{0\}$, and the constant step is given by $h = \boldsymbol{x}_{i+1} - \boldsymbol{x}_i = 1/(M+1)$. We consider the sub-intervals $\mathrm{I}_i = (\boldsymbol{x}_i, \boldsymbol{x}_{i+1})$, for $i \in [\![0, M]\!]$. Let V^h be the Piece-Linear FE, i.e.

$$V^h = \left\{ v \in \mathcal{C}(\bar{\mathrm{I}}) : v \in |_{\mathrm{I}_i} \in \mathcal{P}_1, \forall i \in [\![0, M]\!] \quad \text{and} \quad v(0) = v(1) = 0 \right\}. \tag{34}$$

Define the approximate FE solution by: Find $u^h \in V^h$ such that

$$\int_{\mathrm{I}} \left(u^h_{\boldsymbol{x}}(\boldsymbol{x}) v_{\boldsymbol{x}}(\boldsymbol{x}) + u^h(\boldsymbol{x}) v(\boldsymbol{x}) \right) d\boldsymbol{x} = \int_{\mathrm{I}} f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in V^h. \tag{35}$$

Introduction: Super-convergence in Piece-wise Linear FE in 1D (suite).

Error estimate.

Assume that $u \in H^2(\mathrm{I})$, we have the following error estimate

$$\|u - u^h\|_{H^1(\mathrm{I})} \leq Ch. \tag{36}$$

Superconvergence result.

$$\|\Pi u - u^h\|_{H^1(\mathrm{I})} \leq Ch^2, \tag{37}$$

where Π is the piece-wise linear interpolation over V^h.

Introduction: Super-convergence in Piece-wise Linear FE in 1D (suite).

Superconvergence result.

The super-convergence estimate can be written as

$$\left(\sum_{i=0}^{M} h\left(\frac{u(\boldsymbol{x}_{i+1})-u(\boldsymbol{x}_i)}{h}-\frac{u_{i+1}-u_i}{h}\right)^2\right)^{\frac{1}{2}} \leq Ch^2, \tag{38}$$

where u_i are the components of u^h in the usual basis of V^h.

Usefulness of this Superconvergence result.

As stated before, the super-convergence estimate can help us to derive a high order (> 2) approximation; we refer for instance to Duran-1990, Bradji and Chibi-2007, and Bradji-Thesis-2005.

Super-convergence for MFEMs applied to 1D-Elliptic equation; cf. Benkhadloun and Bradji-2023.

Comment.

The super-convergence of MFEMs applied to 1D-Elliptic equation is not stated explicitly but it can be deduced from some known a priori estimates in Quarteroni and Valli-2008.

Super-convergence for MFEMs applied to 1D-Elliptic equation (Suite).

Let us consider the following second order elliptic equation in 1D:

1D Elliptic model

$$-\omega_{xx}(\boldsymbol{x}) = F(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathrm{I} = (0,1) \quad \text{and} \quad \omega(0) = \omega(1) = 0. \tag{39}$$

Mixed Formulation for (39)

Find $(p,\omega) \in H_{\mathrm{div}}(\mathrm{I}) \times L^2(\mathrm{I})$ such that, for all $(\varphi,\psi) \in L^2(\mathrm{I}) \times H_{\mathrm{div}}(\mathrm{I})$

$$(p_{\boldsymbol{x}},\varphi)_{L^2(\mathrm{I})} = (F,\varphi)_{L^2(\mathrm{I})} \quad \text{and} \quad (p,\psi)_{L^2(\mathrm{I})} = (\omega,\psi_{\boldsymbol{x}})_{L^2(\mathrm{I})}\,. \tag{40}$$

The space $H_{\mathrm{div}}(\mathrm{I})$ in the case of 1D is given by the Sobolev space $H_{\mathrm{div}}(\mathrm{I}) = H^1(\mathrm{I})$.

MFEMs for 1D-Elliptic equation (Suite).

Definition of $\mathbb{RT}_0$-MFEs. First step: Mesh

$\mathrm{I} = (0, 1)$ is meshed by $0 = \boldsymbol{x}_0 < \boldsymbol{x}_1 \ldots < \boldsymbol{x}_{M+1} = 1$, with $M \in \mathbb{N} \setminus \{0\}$, and the constant step is given by $h = 1/(M+1)$. We set $\mathrm{I}_i = (\boldsymbol{x}_i, \boldsymbol{x}_{i+1})$.

Definition of $\mathbb{RT}_0$-MFEs. Second step: Discrete spaces

$$V_h^{\mathrm{div}} = \{v \in H_{\mathrm{div}}(\mathrm{I}) : \quad v|_{\mathrm{I}_i} \in \mathcal{P}_0 \oplus \boldsymbol{x}\mathcal{P}_0, \quad \forall i \in [\![0, M]\!]\} \tag{41}$$

and

$$W_h = \{u \in L^2(\mathrm{I}) : \quad u|_{\mathrm{I}_i} \in \mathcal{P}_0, \quad \forall i \in [\![0, M]\!]\}, \tag{42}$$

The space V_h^{div} (resp. W_h) is the set of continuous functions (resp. functions of $L^2(\mathrm{I})$) which are linear (resp. constant) over each I_i.

MFEMs for 1D-Elliptic equation (Suite)

Definition of $\mathbb{RT}_0$-MFEs. Third step: Formulation of scheme

Find $(p_h, \omega_h) \in V_h^{\text{div}} \times W_h$ such that, for all $(\varphi, \psi) \in W_h \times V_h^{\text{div}}$

$$\left((p_h)_{\boldsymbol{x}}, \varphi\right)_{L^2(\mathrm{I})} = (F, \varphi)_{L^2(\mathrm{I})} \quad \text{and} \quad (p_h, \psi)_{L^2(\mathrm{I})} = (\omega_h, \psi_{\boldsymbol{x}})_{L^2(\mathrm{I})}. \tag{43}$$

Error estimate of scheme; see Quarteroni and Valli-2008

$$\|p_h + u_x\|_{1,\mathrm{I}} + \|\omega_h - \omega\|_{L^2(\mathrm{I})} \leq Ch. \tag{44}$$

MFEMs for 1D-Elliptic equation (Suite)

Superconvergence of $\mathbb{RT}_0$. First step: Interpolation operator; see Yang and Shi-2020

- The usual linear interpolation operator Π_h over V_h^{div}.
- The interpolation operator J_h over W_h of ω:

$$J\omega|_{\mathrm{I}_i} = J_i\omega = \frac{1}{h}\int_{\mathrm{I}_i} \omega(\boldsymbol{x})d\boldsymbol{x}. \tag{45}$$

Superconvergence of $\mathbb{RT}_0$-MFEs. Second step: Superconvergence estimate

$$\|p_h + \Pi_h u_x\|_{1,\mathrm{I}} + \|\omega_h - J_h\omega\|_{L^2(\mathrm{I})} \leq Ch^2. \tag{46}$$

Super-convergence for MFEMs applied to 1D-Heat equation: Problem to be solved.

1D Heat model

$$u_t(\boldsymbol{x},t) - u_{xx}(\boldsymbol{x},t) = f(\boldsymbol{x},t), \qquad (\boldsymbol{x},t) \in \mathrm{I} = (0,1) \times (0,T), \tag{47}$$

This equation is equipped with an initial condition and Dirichlet boundary conditions:

$$u(0) = u^0 \quad \text{and} \quad u(0,t) = u(1,t) = 0, \quad t \in (0,T). \tag{48}$$

Super-convergence for MFEMs applied to 1D-Heat equation: Mixed Formulation

A "formal" mixed formulation:

For each $t \in (0, T)$, find $(p(t), u(t)) \in H_{\text{div}}(\text{I})) \times L^2(\text{I})$ such that, for all $(\varphi, \psi) \in L^2(\text{I}) \times H_{\text{div}}(\text{I})$

$$(u_t(t), \varphi)_{L^2(\text{I})} + (\varphi, \operatorname{div} p(t))_{L^2(\text{I})} = (\varphi, f(t))_{L^2(\text{I})}\,, \tag{49}$$

$$(\psi, p(t))_{L^2(\text{I})} = (\operatorname{div} \psi, u(t))_{L^2(\text{I})}\,, \tag{50}$$

and

$$u(0) = u^0. \tag{51}$$

In 1D, $H_{\text{div}}(\text{I}) = H^1(\text{I})$.

Super-convergence for MFEMs applied to 1D-Heat equation: Meshes

Meshes

- Uniform Mesh in time:
  $0 = t_0 < t_1 = k < t_2 = 2k \ldots t_n = nk < \ldots t_{N+1} = (N+1)k = T$
- Descretization in space: as above using $\mathbb{RT}_0$-MFEs.
- Descrete spaces: as above using $\mathbb{RT}_0$-MFEs

Super-convergence for MFEMs applied to 1D-Heat equation: MFE Scheme

MFE Scheme:

Find $(p_h^n, u_h^n) \in V_h^{\text{div}} \times W_h$ such that:

- For any $n \in [\![0, N]\!]$ and for all $\varphi \in W_h$:

$$\left(\partial^1 u_h^{n+1}, \varphi\right)_{L^2(\mathrm{I})} + \left(\left(p_h^{n+\frac{1}{2}}\right)_x, \varphi\right)_{L^2(\mathrm{I})} = \left(f(t_{n+\frac{1}{2}}), \varphi\right)_{L^2(\mathrm{I})}, \tag{52}$$

- For any $n \in [\![0, N+1]\!]$:

$$(p_h^n, \psi)_{L^2(\mathrm{I})} = (u_h^n, \psi_x)_{L^2(\mathrm{I})}, \qquad \forall \psi \in V_h^{\text{div}}, \tag{53}$$

where $p_h^0 = -\Pi_h(u^0)_x$.

Super-convergence for MFEMs applied to 1D-Heat equation: known error estimate; Benkhaldoun and Bradji (2023)

known error estimate.

$$\max_{n=0}^{N} \|u_x(t_{n+\frac{1}{2}}) + p_h^{n+\frac{1}{2}}\|_{H^1(\mathrm{I})} \le C\left(h + k^2\right). \tag{54}$$

Super-convergence for of $\mathbb{RT}_0$ applied to 1D-Heat equation: Superconvergence result; Benkhaldoun and Bradji-FVCA-2023

New super-convergence result.

$$\left(\sum_{n=0}^{N} k \left\| \Pi_h u_x(t_{n+\frac{1}{2}}) + p_h^{n+\frac{1}{2}} \right\|_{1,I}^2 \right)^{\frac{1}{2}} \leq C(h+k)^2. \tag{55}$$

DMFEMs for the Wave Equation

Wave equation

$$u_{tt}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega \times (0, T). \tag{56}$$

Initial and Dirichlet boundary conditions

$$(u(0), u_t(0)) = (u^0, u^1) \quad \text{and} \quad u(\boldsymbol{x}, t) = 0, \ (\boldsymbol{x}, t) \in \partial\Omega \times (0, T). \tag{57}$$

Formulation of a DMFE scheme for the Wave Equation

- For any $n \in [\![0, N]\!]$ and for all $\varphi \in W_h$:

$$\left(\partial^2 u_h^{n+1}, \varphi\right)_{L^2(\Omega)} + \left(\nabla \cdot p_h^{n+1}, \varphi\right)_{L^2(\Omega)} = (f(t_{n+1}), \varphi)_{L^2(\Omega)}\,, \tag{58}$$

- For any $n \in [\![0, N+1]\!]$:

$$(p_h^n, \psi)_{L^2(\Omega)^d} = (u_h^n, \nabla \cdot \psi)_{L^2(\Omega)}\,, \qquad \forall \psi \in V_h^{\mathrm{div}}, \tag{59}$$

where, for $\varphi \in W_h$

$$\left(\nabla \cdot p_h^0, \varphi\right)_{L^2(\Omega)} = \left(-\Delta u^0, \varphi\right)_{L^2(\Omega)} \tag{60}$$

$$\left(\nabla \cdot p_h^1, \varphi\right)_{L^2(\Omega)} = \left(-k\Delta u^1 + u^0, \varphi\right)_{L^2(\Omega)}\,. \tag{61}$$

Formulation of another DMFE scheme for the Wave Equation: using Newmark's method

Definition of the main scheme

For any $n \in [\![0, N]\!]$ and for all $\varphi \in W_h$:

$$\left(\partial^2 u_h^{n+1}, \varphi\right)_{L^2(\Omega)} + \frac{1}{2}\left(\nabla \cdot (p_h^{n+1} + p_h^{n-1}), \varphi\right)_{L^2(\Omega)} = \frac{1}{2}\left(f(t_{n+1}) + f(t_{n-1}), \varphi\right)_{L^2(\Omega)} . \tag{62}$$

Discrete initial conditions

The discrete initial conditions should be chosen carefully to get second order time accurate.

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Works, related to the subject, under preparation

First work

Extend the Superconvergence, obtained in 1D, to Multi-dimensional Parabolic equations.

Second work

What about the Superconvergence for time derivative u_t of Pressure ?

Third work

Proof the convergence in $L^{\infty}(H_{\text{div}})$ without assumption that the exact solution is smooth and without a need to obtain a convergence rate.

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Works, related to the subject, under preparation (Suite)

Fourth work

DMFE for the Wave Equation

Fifth work

DMFE for the Evolutionary Stokes Equations.

Sixth work

DMFE for the Time Fractional Diffusion Equations.

Works, related to the subject, under preparation (Suite)

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Seventh work

Extension to Non-Uniform temporal mesh..