# On the convergence order of gradient schemes for time dependent partial differential equations

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Talk in LAGA (Laboratoire d'Analyse, Géométrie, et Applications)
November 24th-2017, Paris 13-France

# Aim of the presentation

The aim of this presentation is to present some results related to the convergence order of Gradient Schemes for time dependent partial differential equations.

# Plan of this presentation

1. History: FV (Finite Volume) on Admissible meshes, SUSHI, GS
2. References on the subject
3. Some highlights on FV (Finite Volume) on Admissible meshes
4. Some highlights on SUSHI
5. Definition of the approximate gradient discretization
6. Some examples of the approximate gradient discretization
7. Gradient schemes for some known models:
    1. Linear Heat equation (as a model for Parabolic equations)
    2. Linear Wave equation (as a model for Second Order Hyperbolic equations)
    3. Semi-Linear Heat equation
    4. Semi-Linear Wave equation
    5. TDJHS (Time Dependent Joule Heating system )
8. Works under Preparation

# History on the Discrete Gradient Discretization (GDM)

**First Step**

Finite Volume Methods on Admissible meshes: Handbook of EGH (Eymard, Gallouët, Herbin) (2000).

**Second Step**

SUSHI (Scheme using Stabilization and hybrid interfaces) on General Meshes: EGH-IMAJNA (2010).

**Third Step**

GDM: Droniou, Eymard, Gallouët, Guichard, Herbin-HAL (2016).

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## References on the subject

1. Bradji: An analysis of a second order time accurate scheme for a finite volume method for parabolic equations on general nonconforming multidimensional spatial meshes. AMC, 2013.
2. Bradji: Some new first order and second order time accurate gradient schemes for semilinear parabolic equations. Under revision in CMWA, 2016.
3. Bradji: Convergence analysis of some first order and second order time accurate gradient schemes for semilinear second order hyperbolic equations. NMPDE, 2017
4. Bradji: Convergence analysis of some high–order time accurate schemes for a finite volume method for second order hyperbolic equations on general nonconforming multidimensional spatial meshes. NMPDE, 2013

# References (Suite)


5. Droniou, Eymard, Gallouët, Guichard, Herbin: Gradient discretisation method: A framework for the discretisation and numerical analysis of linear and nonlinear elliptic and parabolic problems. 2016. $< hal - 01382358v3 >$.

6. Droniou and Eymard: A mixed finite volume scheme for anisotropic diffusion problems on any grid. Numer Math, 2006.

7. Droniou, Eymard, Gallouët, and Herbin: A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods. M3AS, 2010.

8. Eymard, Gallouët, and R. Herbin: Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. SUSHI: A scheme using stabilization and hybrid interfaces. IMAJNA, 2010.

# Principles of Finite Volume methods

Finite volume methods[1] are numerical methods approximating different types of Partial Differential Equations (PDEs). They are based on three principle ideas:

- Subdivision of the spatial domain into subsets called Control Volumes.
- Integration of the equation to be solved over the Control Volumes.
- Approximation of the derivatives appearing after integration.

[1] We mean here the "pure" finite volume methods and not finite volume-element methods

# Finite Volume methods on admissible meshes

## Definition

Let $\mathcal{T}$ be an Admissible Mesh in the sense of Eymard et al. (Handbook, 2000).

$K \in \mathcal{T}$ are the control volumes and $\sigma$ are the edges of the control volumes $K$.

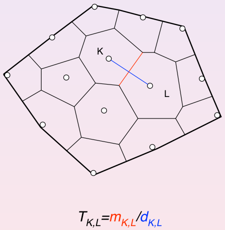


Figure : transmissivity between $K$ and $L$: $\mathcal{T}_\sigma = \mathcal{T}_{K|L} = \frac{m_{K,L}}{d_{K,L}}$

# Finite Volume methods on admissible meshes

**Main properties of Admissible mesh:**

1. Convexity of the Control Volumes.
2. The orthogonality property: the $(\boldsymbol{x}_K\boldsymbol{x}_L)$ is orthogonal to the common edge $\sigma$ between the control volumes $K$ and $L$.

## Finite Volume methods on admissible meshes

Model to be solved:

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega \quad \text{and} \quad u(\boldsymbol{x}) = 0,\ \boldsymbol{x} \in \partial\Omega. \tag{1}$$

**Principles of Finite Volume scheme**:

1. Integration on each control volume $K$ :$-\displaystyle\int_K \Delta u(\boldsymbol{x}) d\boldsymbol{x} = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$,
2. Integration by Parts gives :$-\displaystyle\int_{\partial K} \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$
3. Summing on the lines of $K$: $-\displaystyle\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$

# Finite Volume methods on admissible meshes

## Approximate Finite Volume Solution $u_{\mathcal{T}} = (u_K)_K$

$$-\sum_{\sigma\in\mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{d_{K|L}}(u_L - u_K) = \int_K f(\boldsymbol{x})d\boldsymbol{x}. \qquad (2)$$

## Matrix Form

$$\mathcal{A}^{\mathcal{T}} u_{\mathcal{T}} = f_{\mathcal{T}}.$$

# Finite Volume methods on admissible meshes

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## Matrix Form

$$\mathcal{A}^{\mathcal{T}} u_{\mathcal{T}} = f_{\mathcal{T}}.$$

# Finite Volume methods on admissible meshes

**Theorem**

*Let $\mathcal{X}(\mathcal{T})$: functions which are constant on each control volume $K$. Let $e_{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$ be defined by $e_K = u(\boldsymbol{x}_K) - u_K$ for any $K \in \mathcal{T}$. Assume that the exact solution $u$ satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence results hold:*

1. *$H_0^1$-error estimate*

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \le Ch\|u\|_{2,\overline{\Omega}}, \tag{3}$$

*where $\|\cdot\|_{1,\mathcal{T}}$ is the $H_1^0$-norm* $\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 = \displaystyle\sum_{\sigma=K|L\in\mathcal{E}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(u_L - u_K)^2.$

2. *$L^2$-error estimate:*

$$\|e_{\mathcal{T}}\|_{L^2(\Omega)} \le Ch\|u\|_{2,\overline{\Omega}}. \tag{4}$$

# Finite Volume methods using nonconforming grids, SUSHI scheme

## Definition (New mesh of Eymard et *al.*, IMAJNA 2010)

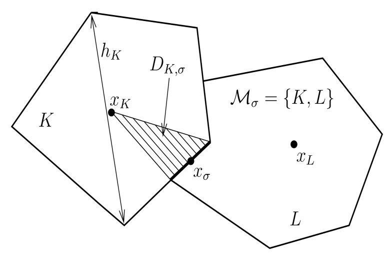


Figure : Notations for two neighbouring control volumes in $d = 2$

# Finite Volume methods using nonconforming grids, SUSHI scheme

**Main properties of this new mesh:**

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (orthogonality property is not required): the orthogonality property is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convex, whereas the convexity property is not required in this new mesh.

# Finite Volume methods using nonconforming grids, SUSHI scheme

## Principles of discretization for Poisson's equation:

1. Discrete unknowns: the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \{\left((v_K)_{K\in\mathcal{M}}\,,\,(v_\sigma)_{\sigma\in\mathcal{E}}\right),\;\; v_K, v_\sigma \in \mathbb{R},\;\; v_\sigma = 0,\; \forall\sigma \in \mathcal{E}_{\text{ext}}\}$$

2. Discretization of the gradient: the discretization of $\nabla$ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see Eymard et *al.* (IMAJNA, 2010):
   1. The discrete gradient $\nabla_{\mathcal{D}}$ is stable
   2. The discrete gradient $\nabla_{\mathcal{D}}$ is consistent.

# Finite Volume methods using nonconforming grids, SUSHI

**Weak formulation for Poisson's equation:** Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega). \tag{5}$$

**SUSHI (Scheme Using stabilized Hybrid Interfaces) for Poisson's equation:** Find $u_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\int_\Omega \nabla_{\mathcal{D}} u_{\mathcal{D}}(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{6}$$

# Finite Volume methods using nonconforming grids, SUSHI

## Theorem

*Assume that the exact solution $u$ satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence result hold:*

1. $H_0^1$*-error estimate*

$$\|\nabla u - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{7}$$

2. $L^2$*-error estimate:*

$$\|u - \Pi_{\mathcal{M}} u_{\mathcal{D}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{8}$$

# Definition of the approximate gradient discretization

**Definition (Definition of a generic approximate gradient discretization, Droniou et al. (M2AS, 2013))**

Let $\Omega$ be an open domain of $\mathbb{R}^d$, where $d \in \mathbb{N} \setminus \{0\}$. An approximate gradient discretization $\mathcal{D}$ is defined by $\mathcal{D} = (\mathcal{X}_{\mathcal{D},0}, h_{\mathcal{D}}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$, where

1. The set of discrete unknowns $\mathcal{X}_{\mathcal{D},0}$ is a finite dimensional vector space on $\mathbb{R}$.
2. The space step $h_{\mathcal{D}} \in (0, +\infty)$ is a positive real number.
3. The linear mapping $\Pi_{\mathcal{D}} : \mathcal{X}_{\mathcal{D},0} \to L^2(\Omega)$ is the reconstruction of the approximate function.
4. The mapping $\nabla_{\mathcal{D}} : \mathcal{X}_{\mathcal{D},0} \to L^2(\Omega)^d$ is the reconstruction of the gradient of the function; it must be chosen such that $\|\nabla_{\mathcal{D}} \cdot \|_{L^2(\Omega)^d}$ is a norm on $\mathcal{X}_{\mathcal{D},0}$.

# Additional hypotheses on the approximate gradient discretization

## Definition (Additional hypotheses on the approximate gradient discretization)

- The **coercivity** of the discretization is measured through the the constant $C_{\mathcal{D}}$ given by:

$$C_{\mathcal{D}} = \max_{v \in \mathcal{X}_{\mathcal{D},0} \setminus \{0\}} \frac{\|\Pi_{\mathcal{D}} v\|_{L^2(\Omega)}}{\|\nabla_{\mathcal{D}} v\|_{L^2(\Omega)^d}}. \tag{9}$$

- The **strong consistency**: $S_{\mathcal{D}} : H_0^1(\Omega) \to [0, +\infty)$ defined by, for all $\varphi \in H_0^1(\Omega)$

$$S_{\mathcal{D}}(\varphi) = \min_{v \in \mathcal{X}_{\mathcal{D},0}} \left( \|\Pi_{\mathcal{D}} v - \varphi\|^2_{L^2(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla \varphi\|^2_{L^2(\Omega)^d} \right)^{\frac{1}{2}}. \tag{10}$$

- The **dual consistency**: For all $\varphi \in H_{\mathrm{div}}(\Omega)$, $W_{\mathcal{D}}(\varphi)$ is given by

$$\max_{u \in \mathcal{X}_{\mathcal{D},0} \setminus \{0\}} \frac{1}{\|\nabla_{\mathcal{D}} u\|_{L^2(\Omega)^d}} \left| \int_{\Omega} \left( \nabla_{\mathcal{D}} u(\boldsymbol{x}) \cdot \varphi(\boldsymbol{x}) + \Pi_{\mathcal{D}} u(\boldsymbol{x}) \mathrm{div} \varphi(\boldsymbol{x}) \right) d\boldsymbol{x} \right|.$$

# First example on the approximate gradient discretization: Conforming finite element method

Let $\{\mathcal{T}_h; h > 0\}$ be a family of shape regular and quasi-uniform triangulations of the domain $\Omega$. Let $\mathcal{V}^h$ be the standard finite element space of continuous, piecewise polynomial functions of degree less or equal $l \in \mathbb{N} \setminus \{0\}$ and we denote by $\mathcal{V}_0^h = \mathcal{V}^h \cap H_0^1(\Omega)$. Assume that $\mathcal{V}_0^h$ is spanned by the usual basis functions $\varphi_1, \ldots, \varphi_M$. The space $\mathcal{X}_{\mathcal{D},0}$ can be $\mathbb{R}^M$ and for any $(u_1, \ldots, u_M) \in \mathcal{X}_{\mathcal{D},0}$, we define $\Pi_{\mathcal{D}} u = \sum_{i=1}^M u_i \varphi \in \mathcal{V}_0^h \subset H_0^1(\Omega)$ and $\nabla_{\mathcal{D}} u = \sum_{i=1}^M u_i \nabla \varphi = \nabla \Pi_{\mathcal{D}} u$. Using the Poincaré inequality, we have for all $u \in \mathcal{X}_{\mathcal{D},0}$, $\|\Pi_{\mathcal{D}} u\|_{\mathbb{L}^2(\Omega)} \leq C(\Omega) \|\nabla_{\mathcal{D}} u\|_{\mathbb{L}^2(\Omega)}$.

Therefore, the assumption (9) of Definition 5 holds with constant $C_{\mathcal{D}}$ only depending on $\Omega$. In addition to this, we have $W_{\mathcal{D}}(\varphi) = 0$, for all $\varphi \in H_{\mathrm{div}}(\Omega)$, and $S_{\mathcal{D}}(\varphi)$ is bounded above by (up to a multiplicative constant independent of the mesh) $h^l |\varphi|_{l+1,\Omega}$, for all $\varphi \in H^{l+1}(\Omega)$.

# Other example on the approximate gradient discretization: SUSHI method

## Second example

SUSHI method, cf. Eymard et al. (IMAJNA, 2010).

## Third example

Mimetic Finite Difference methods, cf. Brezzi et al. (Math. Models Methods Appl. Sci., 2005).

## Fourth example

Mixed Finite Volume method, cf. Droniou et al. (Numer. Math., 2006).

## Nice remark

It is shown in Droniou et al. (Math. Models Methods Appl. Sci., 2010) that the Second example–Fourth example mentioned can be identified to each other.

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# How to use GS: an example of application

## Weak formulation for Poisson's equation

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega). \tag{11}$$

## GS for Poisson's equation

Find $u_\mathcal{D} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\int_\Omega \nabla_\mathcal{D} u_\mathcal{D}(\boldsymbol{x}) \cdot \nabla_\mathcal{D} v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in \in \mathcal{X}_{\mathcal{D},0}. \tag{12}$$

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# Control of the error, cf. Eymard, Guichard, and R. Herbin (M2AN, 2012)

## Theorem

*Assume that $u \in H^2(\Omega)$. The following convergence results hold:*

1. *$H_0^1$-error estimate*

$$\|\nabla u - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \leq W_{\mathcal{D}}(\nabla u) + 2S_{\mathcal{D}}(u). \tag{13}$$

2. *$L^2$-error estimate:*

$$\|u - \Pi_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)} \leq C_{\mathcal{D}} W_{\mathcal{D}}(\nabla u) + (C_{\mathcal{D}} + 1) S_{\mathcal{D}}(u). \tag{14}$$

# Gradient schemes for linear parabolic equations

## Model to be solved:

- Equation:

$$u_t(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t), \ (\boldsymbol{x},t) \in \Omega \times (0,T), \tag{15}$$

  where, $\Omega \subset \mathbb{R}^d$, with $d \in \mathbb{N}^\star$.

- Initial condition:

$$u(\boldsymbol{x},0) = u^0(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega. \tag{16}$$

- Dirichlet boundary conditions:

$$u(\boldsymbol{x},t) = 0, \ (\boldsymbol{x},t) \in \partial\Omega \times (0,T), \tag{17}$$

  where, $\partial\Omega = \overline{\Omega} \setminus \Omega$ the boundary of $\Omega$.

# About Heat equation?

1. (some physics): Heat equation $u_t - \Delta\, u$ is typically used in different applications, such as *fluid mechanics*, *heat and mass transfer*,...
2. (existence and uniqueness): existence and uniqueness of a weak solution of heat equation, with (16) (*initial condition*) and (17) (*Dirichlet boundary condition*) can be formulated using Bochner spaces; see for instance Evans book of partial differential equation

# Discretization of the domain $\Omega$ and time interval $(0, T)$

1. Spatial domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is discretized using the GS.
2. The time interval $(0, T)$ constant step $k = T/(N+1)$, $N \in \mathbb{N}$.

# Principles of scheme

**Principles of scheme:**

1. discretization of heat equation: the discretization of $u_t - \Delta u = f$ stems from weak formulation (like in finite element method)

$$\int_\Omega u_t(\boldsymbol{x},t)v(\boldsymbol{x})dx - \int_\Omega \nabla u(\boldsymbol{x},t)\cdot\nabla v(\boldsymbol{x})dx = \int_\Omega f(\boldsymbol{x},t)v(x)dx, \quad \forall v \in H_0^1(\Omega).$$

2. (discretization of initial condition $u(\boldsymbol{x},0) = u^0(\boldsymbol{x})$): using a suitable discrete projection
3. (discretization of boundary condition $u(x,t) = 0$, $x \in \partial\Omega$ and $t \in (0,T)$): will be in the definition of discrete space
4. The time interval $(0,T)$ constant step $k = T/(N+1)$, $N \in \mathbb{N}$.

# Formulation of GS for Heat equation

## Weak Formulation

$$\int_\Omega u_t(\boldsymbol{x},t)v(\boldsymbol{x})dx - \int_\Omega \nabla u(\boldsymbol{x},t)\cdot\nabla v(\boldsymbol{x})dx = \int_\Omega f(\boldsymbol{x},t)v(\boldsymbol{x})d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega).$$

## GS for Heat equation

For any $n \in [\![0,N]\!]$, find $u_\mathcal{D}^{n+1} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\partial^1 \Pi_\mathcal{D} u_\mathcal{D}^{n+1}, \Pi_\mathcal{D}\, v\right)_{\mathbb{L}^2(\Omega)} + \left(\nabla_\mathcal{D} u_\mathcal{D}^{n+1}, \nabla_\mathcal{D} v\right)_{(\mathbb{L}^2(\Omega))^d} = \left(f(t_{n+1}), v\right)_{\mathbb{L}^2(\Omega)}, \quad (18)$$

where $\partial^1$ is a discrete time derivative.

# Formulation of GS for Heat equation

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where $\partial^1$ is a discrete time derivative.

# Formulation of GS for Heat equation (suite)

## GS for Heat equation: discrete equation

For any $n \in [\![0, N]\!]$, find $u_{\mathcal{D}}^{n+1} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\partial^1 \Pi_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{D}}\, v\right)_{\mathbb{L}^2(\Omega)} + \left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}} v\right)_{(\mathbb{L}^2(\Omega))^d} = \left(f(t_{n+1}), v\right)_{\mathbb{L}^2(\Omega)}, \quad (19)$$

where $\partial^1$ is a discrete time derivative.

## GS for Heat equation: discrete initial condition $u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x})$ is an orthogonal projection

Find $u_{\mathcal{D}}^0 \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^0, \nabla_{\mathcal{D}}\, v\right)_{(\mathbb{L}^2(\Omega))^d} = -\left(\Delta u^0, \Pi_{\mathcal{D}}\, v\right)_{\mathbb{L}^2(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \quad (20)$$

# Formulation of GS for Heat equation (suite)

## GS for Heat equation: discrete equation

For any $n \in [\![0, N]\!]$, find $u_{\mathcal{D}}^{n+1} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\partial^1 \Pi_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{D}}\, v\right)_{\mathbb{L}^2(\Omega)} + \left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = \left(f(t_{n+1}), v\right)_{\mathbb{L}^2(\Omega)}, \quad (19)$$

where $\partial^1$ is a discrete time derivative.

## GS for Heat equation: discrete initial condition $u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x})$ is an orthogonal projection

Find $u_{\mathcal{D}}^0 \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^0, \nabla_{\mathcal{D}}\, v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\left(\Delta u^0, \Pi_{\mathcal{D}}\, v\right)_{\mathbb{L}^2(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \quad (20)$$

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# Error estimates for the GS for Heat equation, cf. Bradji and Fuhrmann (AM-Praha, 2013)

## Theorem (Error estimates)

- *Control of the error in the gradient approximation: For all* $n \in [\![0, N+1]\!]$

$$\|\nabla_{\mathcal{D}} u_{\mathcal{D}}^n - \nabla u(t_n)\|_{\mathbb{L}^2(\Omega)^d} \leq C \exp\left(C C_{\mathcal{D}}^2\right)\left(k\|u\|_{\mathcal{C}^2(\mathbb{L}^2)} + \mathbb{E}_{\mathcal{D}}^k(u)\right),$$

- $W^{1,\infty}(0,T;L^2(\Omega))$*–estimate: For all* $n \in [\![1, N+1]\!]$

$$\|u_t(t_n) - \partial^1 \Pi_{\mathcal{D}} u_{\mathcal{D}}^n\|_{\mathbb{L}^2(\Omega)} \leq C\left(1 + C_{\mathcal{D}} \exp\left(C_{\mathcal{D}}^2\right)\right)\left(k\|u\|_{\mathcal{C}^2(\mathbb{L}^2)} + \mathbb{E}_{\mathcal{D}}^k(u)\right),$$

*where we denote by, for any function* $u \in \mathcal{C}([0,T];\ H^2(\Omega))$

$$\mathbb{E}_{\mathcal{D}}^k(u) = \max_{j\in\{0,1\}} \max_{n\in[\![j,N+1]\!]} \mathbb{E}_{\mathcal{D}}(\partial^j u(t_n))$$

$$\mathbb{E}_{\mathcal{D}}(\bar{u}) = \max\left(W_{\mathcal{D}}(\nabla \bar{u}) + 2S_{\mathcal{D}}(\bar{u}),\ C_{\mathcal{D}} W_{\mathcal{D}}(\nabla \bar{u}) + (C_{\mathcal{D}} + 1) S_{\mathcal{D}}(\bar{u})\right).$$

# Error estimates for the GS for Heat equation, cf. Bradji and Fuhrmann (AM-Praha, 2013)

## Theorem (Error estimates (Suite))

- $L^\infty(0,T;L^2(\Omega))$*–estimate: For all* $n \in [\![0, N+1]\!]$

$$\|\Pi_{\mathcal{D}} u_{\mathcal{D}}^n - u(t_n)\|_{\mathbb{L}^2(\Omega)} \leq C\left(1 + C_{\mathcal{D}} \exp\left(CC_{\mathcal{D}}^2\right)\right)\left(k\|u\|_{\mathcal{C}^2(\mathbb{L}^2)} + \mathbb{E}_{\mathcal{D}}^k(u)\right),$$

# Remarks on error estimates for the GS for Heat equation

## Approximation of $u$ and its first derivatives

Error estimates obtained do not allow to approximate the exact solution for the Heat equation but also its first derivatives both temporal and spatial.

## Possibility to improve the order in time

The convergence order in time of the suggested scheme is one. But it is possible to construct another scheme using the Crank-Nicolson method whose the order in time is two.

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# Gradient schemes for linear Second Order Hyerbolic equations

Model to be solved:

- Equation:

$$u_{tt}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t), \ \ (\boldsymbol{x},t) \in \Omega \times (0,T), \tag{21}$$

  where, $\Omega \subset \mathbb{R}^d$, with $d \in \mathbb{N}^\star$.

- Initial conditions:

$$u(\boldsymbol{x},0) = u^0(\boldsymbol{x}) \quad \text{and} \quad u_t(\boldsymbol{x},0) = u^1(\boldsymbol{x}), \ \ \boldsymbol{x} \in \Omega. \tag{22}$$

- Dirichlet boundary conditions:

$$u(\boldsymbol{x},t) = 0, \ \ (\boldsymbol{x},t) \in \partial\Omega \times (0,T). \tag{23}$$

# About Wave equation

1. (some physics): The wave equation occur in physics such as sound waves, light waves and water waves. It arises in fields like acoustics, electromagnetics, and fluid dynamics, ...
2. (as model): The wave equation is an important model of second-order hyperbolic equations
3. (existence and uniqueness): existence and uniqueness of a weak solution of wave equation (21), with (22) (*initial conditions*) and (23) (*Dirichlet boundary condition*) can be formulated using Bochner spaces; see for instance Evans book of partial differential equation

# Discretization of the domain $\Omega$ and time interval $(0, T)$

1. Spatial domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is discretized using the GS.
2. The time interval $(0, T)$ constant step $k = T/(N+1)$, $N \in \mathbb{N}$.

# Principles of scheme

**Principles of scheme:**

1. discretization of heat equation: the discretization of $u_t - \Delta u = f$ stems from weak formulation (like in finite element method)

$$\int_\Omega u_{tt}(\boldsymbol{x},t)v(\boldsymbol{x})dx - \int_\Omega \nabla u(\boldsymbol{x},t)\cdot\nabla v(\boldsymbol{x})d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x},t)v(\boldsymbol{x})d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega).$$

2. (discretization of initial conditions $u(\boldsymbol{x},0) = u^0(\boldsymbol{x})$ and $u_t(\boldsymbol{x},0) = u^1(\boldsymbol{x})$): using a suitable discrete projection
3. (discretization of boundary condition $u(x,t) = 0$, $x \in \partial\Omega$ and $t \in (0,T)$): will be in the definition of discrete space
4. The time interval $(0,T)$ constant step $k = T/(N+1)$, $N \in \mathbb{N}$.

# Formulation of GS for Wave equation

## Weak Formulation

$$\int_\Omega u_{tt}(\boldsymbol{x},t)v(\boldsymbol{x})dx - \int_\Omega \nabla u(\boldsymbol{x},t)\cdot\nabla v(\boldsymbol{x})d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x},t)v(\boldsymbol{x})d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega).$$

## GS for Wave equation

For any $n \in [\![1,N]\!]$, find $u_{\mathcal{D}}^{n+1} \in \mathcal{X}_{\mathcal{D},0}$ such that

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where $\partial^2$ is a discrete second time derivative.

# Formulation of GS for Wave equation

## Weak Formulation

$$\int_\Omega u_{tt}(\boldsymbol{x},t)v(\boldsymbol{x})dx - \int_\Omega \nabla u(\boldsymbol{x},t)\cdot\nabla v(\boldsymbol{x})d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x},t)v(\boldsymbol{x})d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega).$$

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**Discrete initial conditions $u(\boldsymbol{x},0) = u^0(\boldsymbol{x})$ and $u_t(\boldsymbol{x},0) = u^1(\boldsymbol{x})$ are orthogonal projections**

Find $u^0_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ and $u^1_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\Big(\nabla_{\mathcal{D}} u^0_{\mathcal{D}}, \nabla_{\mathcal{D}}\, v\Big)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\Big(\Delta u^0, \Pi_{\mathcal{D}}\, v\Big)_{\mathbb{L}^2(\Omega)}, \quad \forall\, v \in \mathcal{X}_{\mathcal{D},0}, \tag{25}$$

$$\Big(\nabla_{\mathcal{D}} \partial^1 u^1_{\mathcal{D}}, \nabla_{\mathcal{D}}\, v\Big)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\Big(\Delta u^1, \Pi_{\mathcal{D}}\, v\Big)_{\mathbb{L}^2(\Omega)}, \quad \forall\, v \in \mathcal{X}_{\mathcal{D},0}, \tag{26}$$

# Error estimates for the GS for Wave equation, Bradji (NMPDE, 2013)

## Theorem (Error estimates)

- *Control of the error in the gradient approximation: For all* $n \in [\![0, N+1]\!]$

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- $W^{1,\infty}(0,T;L^2(\Omega))$*–estimate: For all* $n \in [\![1, N+1]\!]$

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# Gradient schemes for Semi-linear parabolic equations

Model to be solved:

- Equation:

$$u_t(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t,u(\boldsymbol{x},t)),\ (\boldsymbol{x},t) \in \Omega \times (0,T). \tag{27}$$

- Initial condition:

$$u(\boldsymbol{x},0) = u^0(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega. \tag{28}$$

- Dirichlet boundary conditions:

$$u(\boldsymbol{x},t) = 0,\ (\boldsymbol{x},t) \in \partial\Omega \times (0,T), \tag{29}$$

# Formulation of GS for Semi-linear Heat equation

## Weak Formulation

$$\int_\Omega u_t(\boldsymbol{x},t)v(\boldsymbol{x})dx - \int_\Omega \nabla u(\boldsymbol{x},t)\cdot\nabla v(\boldsymbol{x})d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x},t,u(\boldsymbol{x},t))v(\boldsymbol{x})d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega).$$

## GS for Semi-linear Heat equation

For any $n \in [\![0,N]\!]$, find $u_{\mathcal{D}}^{n+1} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\begin{aligned}&\left(\partial^1 \Pi_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{D}}\, v\right)_{\mathbb{L}^2(\Omega)} + \left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}} v\right)_{(\mathbb{L}^2(\Omega))^d}\\ = \quad &\left(f(t_{n+1}, \Pi_{\mathcal{D}} u_{\mathcal{D}}^{n+1}(t_{n+1})), v\right)_{\mathbb{L}^2(\Omega)},\end{aligned} \tag{30}$$

where $f(t_{n+1}, u^{n+1}(t_{n+1}))$ represents the function $\boldsymbol{x} \mapsto f(\boldsymbol{x}, t_{n+1}, \Pi_{\mathcal{D}} u_{\mathcal{D}}^{n+1}(\boldsymbol{x}))$.

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**GS for Semi-linear Heat equation: discrete initial condition**

Discrete initial condition can be defined as done for the linear case.

# Error estimates for GS for Semi-linear Heat equation

## Error estimates for GS for Semi-linear Heat equation

Assume that the source function $(\boldsymbol{x}, t, s) \mapsto f(\boldsymbol{x}, t, s)$ is Lipschitz continuous with respect to $s$ with a constant $\kappa$ independent of $(\boldsymbol{x}, t) \in \Omega \times (0, T)$, i.e.

$$|f(\boldsymbol{x}, t, s) - f(\boldsymbol{x}, t, r)| \leq \kappa |s - r|, \quad \forall (s, r) \in \mathbb{R} \times \mathbb{R}, \quad \forall (\boldsymbol{x}, t) \in \Omega \times (0, T). \quad (31)$$

Then, we obtain convergence results similar to those obtained for the linear case.

# Gradient schemes for Semi-linear second order hyperbolic equations

## Model to be solved:

- Equation:

$$u_{tt}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t,u(\boldsymbol{x},t)),\ (\boldsymbol{x},t) \in \Omega \times (0,T). \tag{32}$$

- Initial condition:

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$$|f(\boldsymbol{x}, t, s) - f(\boldsymbol{x}, t, r)| \leq \kappa |s - r|, \quad \forall (s, r) \in \mathbb{R} \times \mathbb{R}, \quad \forall (\boldsymbol{x}, t) \in \Omega \times (0, T). \quad (36)$$

Then, we obtain convergence results similar to those obtained for the linear case.

# Problem to be solved: Time Dependent Joule Heating system (TDJAS)

Model to be solved: We seek a couple of real valued functions $(u, \varphi)$ defined on $\Omega \times [0, T]$ and satisfying:

1. Temperature equation:

$$u_t(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = \kappa(u(\boldsymbol{x}, t)) \left|\nabla\varphi\right|^2 (\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega \times (0, T), \tag{37}$$

2. Electric potential equation:

$$-\nabla \cdot \left(\kappa(u(\boldsymbol{x}, t))\nabla\varphi\right)(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \Omega \times (0, T), \tag{38}$$

3. An initial condition is given by:

$$u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}), \ x \in \Omega, \tag{39}$$

4. Dirichlet boundary conditions:

$$u(\boldsymbol{x}, t) = 0 \ \text{ and } \ \varphi(\boldsymbol{x}, t) = g(\boldsymbol{x}, t), \ (\boldsymbol{x}, t) \in \partial\Omega \times (0, T). \tag{40}$$

# Some physics on TDJHS

## Some physics on TDJHS

The nonlinear system models electric heating of a conducting body, where $u$ is the temperature, $\varphi$ is the electric potential.

# An assumption

## Assumption on the function $\kappa$

We assume that the function $\kappa$ is satisfying $\kappa \in \mathcal{C}^2(\mathbb{R})$ and that for some two positive constants $K_1$ and $K_2$, we have for all $s \in \mathbb{R}$

$$K_1 < \kappa(s) + |\kappa'(s)| + |\kappa''(s)| \leq K_2. \tag{41}$$

# Some literature


1. Bradji, A., Herbin, R.: Discretization of coupled heat and electrical diffusion problems by finite–element and finite-volume methods. IMA J. Numer. Anal. 28/3, 469–495 (2008).
2. Elliott, Ch.M., Larsson, S.: A finite element model for the time-dependent Joule heating problem. Math. Comput. 64/112, 1433–1453 (1995).
3. Glitzky, A., Gärtner, K.: Energy estimates for continuous and discretized electro-reaction-diffusion systems. Nonlinear Anal. 70, 788–805 (2009).
4. Li, B., Sun, W.: Error analysis of linearized semi implicit Galerkin finite element methods for non linear parabolic equations. International Journal of Numerical Analysis and Modeling. 10/3 622–633 (2013).

# GS for TDJHS

**Till now**

A formulation for the GS can constructed as for semilinear heat equation. However, the convergence order is not yet well proved

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