A Finite Volume Scheme for a Wave Equation with Several Time Independent Delays

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LSSC2021 (The 13th Int. Conference on Large-Scale Scientific Computations)
June 7–10, Sozopol-Bulgaria
On-Line Presentation

Aim of the presentation

The aim of this talk is to establish a finite volume scheme along with a convergence analysis for a Second Order Hyperbolic Equation with a Several Time Independent Delays.

Plan of this presentation

Problem to be solved

Equation

Second Order Hyperbolic Equation with several time Independent delays, see Nicaise et al. (SIAM J. Control Optim. 45/5, 2006) and Parhi-Kirane (1994), $(\boldsymbol{x}, t) \in \Omega \times (0, T)$:

$$\begin{aligned} u_{tt}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) + \alpha_0 u_t(\boldsymbol{x}, t) + \alpha_1 u_t(\boldsymbol{x}, t - \tau_1) + \alpha_2 u_t(\boldsymbol{x}, t - \tau_2) \\ + \quad \beta_0 u(\boldsymbol{x}, t) + \beta_1 u(\boldsymbol{x}, t - \tau_3) + \beta_2 u(\boldsymbol{x}, t - \tau_4) = f(\boldsymbol{x}, t), \end{aligned} \tag{1}$$

where Ω is an open polygonal bounded subset in $\mathbb{R}^d$, f is a given function defined on $\Omega \times (0, T)$, and $T > 0$, $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2 \geq 0$, $\tau_1, \tau_2, \tau_3, \tau_4 > 0$ are given.

The positive values τ_i called the delays.

Problem to be solved (Suite)

Initial conditions

$$u(\boldsymbol{x},0) = u^0(\boldsymbol{x}) \quad \text{and} \quad u_t(\boldsymbol{x},t) = u^1(\boldsymbol{x},t), \qquad \boldsymbol{x} \in \Omega, \quad -\tau \leq t \leq 0, \qquad (2)$$

where u^0 and u^1 are two given functions defined respectively on Ω and $\Omega \times (-\tau, 0)$ with

$$\tau = \max\{\tau_1, \tau_2, \tau_3, \tau_4\}.$$

Homogeneous Dirichlet boundary

$$u(\boldsymbol{x},t) = 0, \;\; (\boldsymbol{x},t) \in \partial\Omega \times (0,T). \qquad (3)$$

What about Delay differential equations?

Some physics

Delay differential equations occur in many applications such as ecology and biology. They have long played important roles in the literature of theoretical population dynamics, and they have been continuing to serve as useful models.

References

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- Benkhaldoun, F., Bradji, A.: Note on the convergence of a finite volume scheme for a second order hyperbolic equation with a time delay in any space dimension. Finite volumes for complex applications IX–Methods, Theoretical Aspects, Examples, 285–293, Springer Proc. Math. Stat., 323, Springer, Cham, 2020.
- Eymard, R., Gallouët, T., Herbin, R.: Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. IMA J. Numer. Anal. 30/4, 1009–1043 (2010).

References (Suite)

- Nicaise, S., Pignotti, C.: Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. SIAM J. Control Optim. 45/5, 1561–1585 (2006).
- Parhi, N., Kirane, M.: Oscillatory behaviour of solutions of coupled hyperbolic differential equations. Analysis 14/1, 43–56 (1994).

Introduction: Finite Volume from Admissible meshes to Nonconforming meshes

Finite volume methods are numerical methods approximating different types of Partial Differential Equations (PDEs). They are based on three principle ideas:

- Subdivision of the spatial domain into subsets called Control Volumes.
- Integration of the equation to be solved over the Control Volumes.
- Approximation of the derivatives appearing after integration.

[1] We mean here the "pure" finite volume methods and not finite volume-element methods

Introduction (suite): Finite Volume from Admissible meshes to Nonconforming meshes

Finite Volume methods passed by two steps:

First step

Finite Volume methods using Admissible meshes.

Second step

SUSHI method.

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Introduction (suite): Finite Volume methods on admissible meshes

Definition

Let $\mathcal{T}$ be an Admissible Mesh in the sense of Eymard et al. (Handbook, 2000).

$K \in \mathcal{T}$ are the control volumes and σ are the edges of the control volumes K.

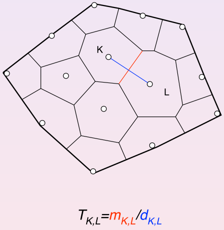

Figure: transmissivity between K and L: $\mathcal{T}_\sigma = \mathcal{T}_{K|L} = \frac{m_{K,L}}{d_{K,L}}$

Introduction (suite): Finite Volume methods on admissible meshes

Main properties of Admissible mesh:

1. Convexity of the Control Volumes.
2. The orthogonality property: the $(\boldsymbol{x}_K \boldsymbol{x}_L)$ is orthogonal to the common edge σ between the control volumes K and L.

Introduction (suite): Finite Volume methods on admissible meshes

Model to be solved:

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega \quad \text{and} \quad u(\boldsymbol{x}) = 0,\ \boldsymbol{x} \in \partial\Omega. \tag{4}$$

Principles of Finite Volume scheme:

1. Integration on each control volume K : $-\int_K \Delta u(\boldsymbol{x}) d\boldsymbol{x} = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$,
2. Integration by Parts gives : $-\int_{\partial K} \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$
3. Summing on the lines of K: $-\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$

Introduction (suite): Finite Volume methods on admissible meshes

Approximate Finite Volume Solution $u_{\mathcal{T}} = (u_K)_K$

$$-\sum_{\sigma\in\mathcal{E}_K}\frac{\mathrm{m}(\sigma)}{d_{K|L}}(u_L-u_K)=\int_K f(\boldsymbol{x})d\boldsymbol{x}. \tag{5}$$

Matrix Form

$$\mathcal{A}^{\mathcal{T}}u_{\mathcal{T}}=f_{\mathcal{T}}.$$

Introduction (suite): Finite Volume methods on admissible meshes

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Introduction (suite): Finite Volume methods on admissible meshes

Theorem

Let $\mathcal{X}(\mathcal{T})$: functions which are constant on each control volume K. Let $e_\mathcal{T} \in \mathcal{X}(\mathcal{T})$ be defined by $e_K = u(\boldsymbol{x}_K) - u_K$ for any $K \in \mathcal{T}$. Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence results hold:

1. *H_0^1-error estimate*

$$\|e_\mathcal{T}\|_{1,\mathcal{T}} \leq Ch\|u\|_{2,\overline{\Omega}}, \tag{6}$$

where $\|\cdot\|_{1,\mathcal{T}}$ is the H_1^0-norm $\|e_\mathcal{T}\|_{1,\mathcal{T}}^2 = \sum_{\sigma=K|L\in\mathcal{E}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(u_L - u_K)^2.$

2. *L^2-error estimate:*

$$\|e_\mathcal{T}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{7}$$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Definition (New mesh of Eymard et *al.*, IMAJNA 2010)

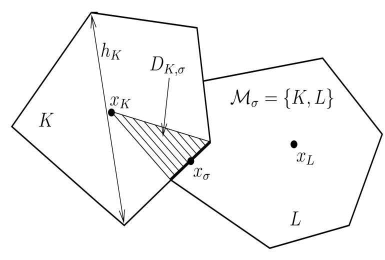

Figure: Notations for two neighbouring control volumes in $d = 2$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Main properties of this new mesh:

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (orthogonality property is not required): the orthogonality property is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convex, whereas the convexity property is not required in this new mesh.

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Principles of discretization for Poisson's equation:

1. Discrete unknowns: the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \{\left((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}\right), \ v_K, v_\sigma \in \mathbb{R}, \ v_\sigma = 0, \forall \sigma \in \mathcal{E}_{\text{ext}}\}$$

2. Discretization of the gradient: the discretization of ∇ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see Eymard et *al.* (IMAJNA, 2010):

    1. The discrete gradient $\nabla_{\mathcal{D}}$ is stable
    2. The discrete gradient $\nabla_{\mathcal{D}}$ is consistent.

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI

Weak formulation for Poisson's equation: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega). \tag{8}$$

SUSHI (Scheme Using stabilized Hybrid Interfaces) for Poisson's equation: Find $u_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\int_\Omega \nabla_{\mathcal{D}} u_{\mathcal{D}}(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{9}$$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI

Theorem

Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence result hold:

1. H_0^1*-error estimate*

$$\|\nabla u - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \le Ch\|u\|_{2,\overline{\Omega}}. \tag{10}$$

2. L^2*-error estimate:*

$$\|u - \Pi_{\mathcal{M}} u_{\mathcal{D}}\|_{L^2(\Omega)} \le Ch\|u\|_{2,\overline{\Omega}}. \tag{11}$$

Principles of the discretization

Definition of a discretization in time and its parameters

- The time discretization is performed with a constrained time step-size k such that $\frac{\tau}{k} \in \mathbb{N}$. We set then $k = \frac{\tau}{M}$, where $M \in \mathbb{N} \setminus \{0\}$. We denote by ∂^1 the discrete first time derivative given by $\partial^1 v^{j+1} = \frac{v^{j+1} - v^j}{k}$.
- The discrete second time derivative $\partial^2 v^{j+1} = \partial^1(\partial^1 v^{j+1})$.
- Denote by N the integer part of $\frac{T}{k}$, i.e. $N = \left[\frac{T}{k}\right]$.
- We shall denote $t_n = nk$, for $n \in [\![-M, N]\!]$.

Advantages of this time discretization

The point $t = 0$ is a mesh point which is suitable since we have equation (1) defined for $t \in (0, T)$ and initial condition (2) defined for $t \in (-\tau, 0)$.

Principles of the discretization (suite)

Discretization in space

We use SUSHI scheme

Formulation of scheme

The set of unknowns

The unknowns of this scheme are $\{u_{\mathcal{D}}^n; n \in [\![-M, N]\!]\}$.

The set of unknowns

These unknowns are expected to approximate the unknowns $\{u(t_n); n \in [\![-M, N]\!]\}$.

Approximation of initial conditions (2)

Find $u_{\mathcal{D}}^n$ for $n \in [\![-M, 0]\!]$ such that for all $v \in \mathcal{X}_{\mathcal{D},0}$, for all $n \in [\![-M+1, 0]\!]$

$$\langle u_{\mathcal{D}}^0, v\rangle_F = -\left(\Delta u^0, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} \tag{12}$$

and

$$\langle \partial^1 u_{\mathcal{D}}^n, v\rangle_F = -\left(\Delta u^1(t_n), \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}. \tag{13}$$

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Formulation of scheme (Suite)

Approximation of (1) and (3)

For any $n \in [\![0, N-1]\!]$, find $u_{\mathcal{D}}^{n+1} \in \mathcal{X}_{\mathcal{D},0}$ such that, for all $v \in \mathcal{X}_{\mathcal{D},0}$

$$\begin{aligned}
&\left(\partial^2 \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} + \langle u_{\mathcal{D}}^{n+1}, v\rangle_F + \alpha_0 \left(\partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} \\
&+ \quad \alpha_1 \left(\partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1-M_1}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} + \alpha_2 \left(\partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1-M_2}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} \\
&+ \quad \beta_0 \left(\Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} + \beta_1 \left(\Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1-M_3}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} \\
&+ \quad \beta_2 \left(\Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1-M_4}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} = \left(f(t_{n+1}), \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}, \qquad (14)
\end{aligned}$$

where, for $i \in \{1,2,3,4\}$, $M_i = \left[\dfrac{\tau_i}{k}\right]$.

Useful Assumption

Assumption (Assumption on the time step k)

$$k < \min\{\tau_1, \tau_2, \tau_3, \tau_4\}. \tag{15}$$

This implies that $M_i \geq 1$, for all $i \in \{1, 2, 3, 4\}$.

Statement of the convergence results

Theorem (Error estimates)

We assume that u is sufficiently smooth.

- $\mathbb{L}^\infty(H_0^1)$–*estimate. For all* $n \in [\![-M, N]\!]$

$$\|\nabla_{\mathcal{D}} u_{\mathcal{D}}^n - \nabla u(t_n)\|_{\mathbb{L}^2(\Omega)} \le C(k + h_{\mathcal{D}}). \tag{16}$$

- $W^{1,2}(\mathbb{L}^2)$–*estimate.*

$$\left(\sum_{n=-M+1}^{N} k \left\| u_t(t_n) - \Pi_{\mathcal{M}} \partial^1 u_{\mathcal{D}}^n \right\|_{\mathbb{L}^2(\Omega)}^2 \right)^{\frac{1}{2}} \le C(k + h_{\mathcal{D}}). \tag{17}$$

Idea on the proof

The proof is manly based on two facts:

- Comparison with an optimal scheme : for any $n \in [\![0, N+1]\!]$, find $\bar{u}^n_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\nabla_{\mathcal{D}} \bar{u}^n_{\mathcal{D}}, \nabla_{\mathcal{D}}\, v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\sum_{K\in\mathcal{M}} v_K \int_K \Delta u(x, t_n)dx, \;\; \forall\, v \in \mathcal{X}_{\mathcal{D},0}. \qquad (18)$$

- A convenient a priori estimate.

Conclusion

First result obtained

We developed a new finite volume scheme for a second hyperbolic equation with several time independent delays. These delays are involved in both the the exact solution and its time derivative.

Second result obtained

The order is proved to be one in time and one in space.

The third result obtained

The analysis is performed in several discrete norms.

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Perspectives

First perspective

The use of Crank Nicolson method in order to improve the order in time.

Perspectives

Second perspective

Extension to the the case when the right hand side involves the exact solution and its gradient

Third perspective

Delays are not numbers but functions depending on t (time dependent delays).

Perspectives

Second perspective

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