A Second Order Time Accurate Finite Volume Scheme for the Time-Fractional Diffusion Wave Equation on General Nonconforming Meshes

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Aim and Abstract of this presentation

The aim of this work is to establish a second order time accurate finite volume scheme for a Time Fractional Diffusion-Wave equation. The discretization in space is performed using SUSHI (Scheme Using Stabilization and Hybrid Interfaces) developed recently in Eymard et *al.* (IMA J. Numer. Anal., 30/4, 2010).

The approach followed is to write the Time Fractional Diffusion-Wave equation as a system of two low order differential equation and to apply the scheme of Alikhanov (J. Comput. Phys., 280, 2015.

Plan of this presentation

1. Problem to be solved
2. References
3. Discretization in time
4. Discretization in space:
    1. An overview on the standard Finite Volume methods (Admissible Meshes) (Eymard et *al.*, Handbook 2000)
    2. SUSHI method (Eymard et *al.*, IMAJNA 2010).
5. Formulation of a second order time accurate Finite Volume scheme for a time fractional diffusion equation
6. Statement of the Convergence Rate of the numerical scheme
7. Conclusion and Perspectives

Equation to be solved

Equation

We consider the following Time Fractional Diffusion-Wave equation:

$$\partial_t^\alpha u(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t),\ \ (\boldsymbol{x},t) \in \Omega \times (0,T), \tag{1}$$

where Ω an open polygonal bounded subset in $\mathbb{R}^d$ and $1 < \alpha < 2$. The operator ∂_t^α is the Caputo derivative:

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} u''(s) ds. \tag{2}$$

Initial conditions

Initial conditions are given by $u(\boldsymbol{x},0) = u^0(\boldsymbol{x})$ and $u_t(\boldsymbol{x},0) = u^1(\boldsymbol{x})$, for all $\boldsymbol{x} \in \Omega$

Homogeneous Dirichlet boundary

$u(\boldsymbol{x},t) = 0,\ \ (\boldsymbol{x},t) \in \partial\Omega \times (0,T).$

What about Fractional differential equations?

Some physics

Fractional differential equations have been successfully used in the modeling of many different processes and systems. They are used, for instance, to describe anomalous transport in disordered semiconductors, penetration of light beam through a turbulent medium, transport of resonance radiation in plasma, blinking fluorescence of quantum dots, penetration and acceleration of cosmic ray in the Galaxy, and large-scale statistical Cosmography.

The TFDWE (Time Fractional Diffusion-Wave Equation) can be used to model the propagation of mechanical waves in viscoelastic media, see Jin et *al.* (Siam J. Sci. Comp., 38/1, 2016).

We refer to the monograph Uchaikin (Fractional Derivatives for Physicists and Engineers, Springer-Verlag Heidelberg, 2013) where we find many details.

References

References

1. Alikhanov, A.-A.: A new difference scheme for the fractional diffusion equation. J. Comput. Phys. 280, 424–438 (2015).
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Principles of the time discretization

Steps of Discretization in time

- First step: Definition of mesh points. We define $k = T/(M+1)$ and mesh points $t_n = nk$. We denote by ∂^1 and ∂^2 the discrete first time derivative and discrete second time derivative given respectively by

$$\partial^1 v^{j+1} = \frac{v^{j+1} - v^j}{k} \quad \text{and} \quad \partial^2 v^{j+1} = \partial^1(\partial^1 v^{j+1}) = \frac{v^{j+1} - 2v^j + v^{j-1}}{k^2}.$$

- Second step: Equation to be solved on the mesh points. Writing (1) with $t = t_{n+\sigma} = (n+\sigma)k = t_n + \sigma k$ with $0 < \sigma < 1$ will be chosen later (recall that $1 < \alpha < 2$) under the system:

$$\partial_t^{\alpha-1}\overline{u}(t_{n+\sigma}) - \Delta u(t_{n+\sigma}) = f(t_{n+\sigma}) \quad \text{and} \quad \overline{u}(t_{n+\sigma}) = u_t(t_{n+\sigma}). \tag{3}$$

Principles of the time discretization (Suite)

Steps of Discretization in time (Suite)

- Third step: Principle idea of a second order approximation $\partial_t^{\alpha-1}\overline{u}(t_{n+\sigma})$ (Idea of Alikhanov and Gao et al.). We write $\partial_t^{\alpha-1}\overline{u}(t_{n+1})$ as (Notation $\beta = \alpha - 1$)

$$\frac{1}{\Gamma(1-\beta)}\left(\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(t_{n+\sigma}-s)^{-\beta}\overline{u}_s(s)ds + \int_{t_n}^{t_{n+\sigma}}(t_{n+\sigma}-s)^{-\beta}\overline{u}_s(s)\right) ds. \tag{4}$$

For each $j \in [\![1, N+1]\!]$, let $\Pi_{2,j}\overline{u}$ be the quadratic interpolation defined on (t_{j-1}, t_j) on the points t_{j-1}, t_j, t_{j+1} of $\overline{u}$. An explicit expansion for $\Pi_{2,j}\overline{u}'$ yields:

$$\partial^1\overline{u}(t_{j+1}) + \partial^2\overline{u}(t_{j+1})\left(s - t_{j+\frac{1}{2}}\right) = \partial^1\overline{u}(t_j) + \partial^2\overline{u}(t_{j+1})\left(s - t_{j-\frac{1}{2}}\right). \tag{5}$$

Principles of the time discretization (Suite)

Steps of Discretization in time (Suite)

- Fifth step: Computation of a second order approximation for $\partial_t^\beta \overline{u}(t_{n+\sigma})$. When approximating the terms of the sum (resp. the last term) using quadratic interpolations (resp. a linear interpolation) in (4) of $\partial_t^\beta \overline{u}(t_{n+\sigma})$, we have to compute the following integrals:

  1. First set of integrals:

$$\int_{t_{j-1}}^{t_j} \left(s - t_{j-\frac{1}{2}}\right) (t_{n+\sigma} - s)^{-\beta} ds = \frac{k^{2-\beta}}{1-\beta} b_{n-j}^{\sigma}, \tag{6}$$

where

$$\begin{aligned} b_l^{\sigma} &= \frac{1}{2-\alpha} \left((l+\sigma+1)^{2-\beta} - (l+\sigma)^{2-\beta} \right) \\ &- \frac{1}{2} \left((l+\sigma+1)^{1-\beta} + (l+\sigma)^{1-\beta} \right). \end{aligned} \tag{7}$$

Principles of the time discretization (Suite)

Steps of Discretization in time (Suite)

- Fifth step: Computation of a second order approximation for $\partial_t^\beta \overline{u}(t_{n+\sigma})$ (Suite).
  2. Second set of integrals:

$$\int_{t_{j-1}}^{t_j} (t_{n+\sigma} - s)^{-\beta} ds = \frac{k^{1-\beta}}{1-\beta} d_{n+\sigma-j,\beta}, \tag{8}$$

  with, for all $s > 0$, $d_{s,\alpha}$ is given by

$$d_{s,\alpha} = (s+1)^{1-\beta} - s^{1-\beta}. \tag{9}$$

  3. Third set of integrals:

$$\int_{t_n}^{t_{n+\sigma}} (t_{n+\sigma} - s)^{-\beta} ds = \frac{k^{1-\beta}}{1-\beta} \sigma^{1-\beta}. \tag{10}$$

Formulation of a second order approximation for $\partial_t^\beta \overline{u}(t_{n+\sigma})$

Formulation of a second order approximation for $\partial_t^\beta \overline{u}(t_{n+\sigma})$

We then obtained approximation for the fractional derivative $\partial_t^\beta \overline{u}(t_{n+\sigma})$ using (4)–(10). This approximation is of order two if

$$\sigma = 1 - \frac{\beta}{2} = \frac{3}{2} - \frac{\alpha}{2} < 1. \tag{11}$$

This approximation is given by

$$\frac{1}{\Gamma(1-\beta)} \left(\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (t_{n+\sigma} - s)^{-\beta} \left(\Pi_{2,j}\overline{u}(s)\right)' ds + \frac{k^{1-\beta}}{1-\beta} \sigma^{1-\beta} \partial^1 \overline{u}(t_{n+1}) \right)$$
$$= \sum_{j=0}^{n} k \lambda_j^{n+1} \partial^1 \overline{u}(t_{j+1}). \tag{12}$$

Let us denote $\Lambda_{n+\sigma}\overline{u} = \sum_{j=0}^{n} k \lambda_j^{n+1} \partial^1 \overline{u}(t_{j+1})$.

Properties of the time approximation

Properties of the time approximation, cf. Alikhanov (2015)

Recall that

$$\Lambda_{n+\sigma}\overline{u} = \sum_{j=0}^{n} k\lambda_j^{n+1}\partial^1\overline{u}(t_{j+1}) \approx \partial_t^\beta \overline{u}(t_{n+\sigma}). \tag{13}$$

1. Properties of λ_j^{n+1}.

$$\sum_{j=0}^{n} k\lambda_j^{n+1} \leq \frac{T^{1-\beta}}{\Gamma(2-\beta)}, \quad \sum_{n=0}^{N} k\lambda_1^{n+1} \leq \frac{T^{1-\beta}}{\Gamma(2-\beta)}$$

and

$$\lambda_n^{n+1} > \lambda_{n-1}^{n+1} > \ldots > \lambda_0^{n+1} > \lambda_0 = \frac{1}{2T^\beta\Gamma(1-\beta)}.$$

Properties of the time approximation (Suite)

Properties of the time approximation (Suite)

2. Stability result. For all $\left(\gamma^j\right)_{j=0}^{N+1} \in \mathbb{R}^{N+2}$, for any $n \in [\![0, N+1]\!]$:

$$\left(\sigma\gamma^{n+1} + (1-\sigma)\gamma^n\right) \sum_{j=0}^{n} \lambda_j^{n+1}(\gamma^{j+1} - \gamma^j) \geq \frac{1}{2} \sum_{j=0}^{n} \lambda_j^{n+1} \left((\gamma^{j+1})^2 - (\gamma^j)^2\right).$$

3. Consistency result. For any $\varphi \in \mathcal{C}^3([0,T])$:

$$\left|\partial^\beta \varphi(t_{n+\sigma}) - \Lambda_{n+\sigma}\varphi\right| \leq Ck^{3-\beta} \left|\varphi^{(3)}\right|_{\mathcal{C}([0,T])}. \tag{14}$$

Formulation of a suitable approximation for $\partial_t^\beta \overline{u}(t_{n+\sigma}) - \Delta u(t_{n+\sigma}) = f(t_{n+\sigma})$

Formulation of a suitable approximation for $\partial_t^\beta \overline{u}(t_{n+\sigma}) - \Delta u(t_{n+\sigma}) = f(t_{n+\sigma})$.

- First fact. As justified before

$$\partial^\beta \overline{u}(t_{n+\sigma}) = \sum_{j=0}^{n} k\lambda_j^{n+1}\partial^1\overline{u}(t_{j+1}) + \mathcal{O}(k^{3-\beta}). \tag{15}$$

- Second fact. Using a Taylor expansion yields

$$\Delta\overline{u}(t_{n+\sigma}) = \sigma\Delta\overline{u}(t_{n+1}) + (1-\sigma)\Delta\overline{u}(t_n) + \mathcal{O}(k^2). \tag{16}$$

From (15) and (16), we deduce that

$$\sum_{j=0}^{n} k\lambda_j^{n+1}\partial^1\overline{u}(t_{j+1}) - \sigma\Delta u(t_{n+1}) - (1-\sigma)\Delta u(t_n) = f(t_{n+\sigma}) + \mathcal{O}(k^2). \tag{17}$$

Formulation of a suitable approximation for $\partial_t^{\beta}\overline{u}(t_{n+\sigma}) - \Delta u(t_{n+\sigma}) = f(t_{n+\sigma})$ (Suite)

Formulation of a suitable approximation for $\partial_t^{\beta}\overline{u}(t_{n+\sigma}) - \Delta u(t_{n+\sigma}) = f(t_{n+\sigma})$

Recall that

$$\sum_{j=0}^{n} k\lambda_j^{n+1}\partial^1\overline{u}(t_{j+1}) - \sigma\Delta u(t_{n+1}) - (1-\sigma)\Delta u(t_n) = f(t_{n+\sigma}) + \mathcal{O}(k^2). \quad (18)$$

The derivative $\overline{u}(t_{n+\sigma}) = u_t(t_{n+\sigma})$ is approximated by

$$\frac{(2\sigma+1)u(t_{n+1}) - 4\sigma u(t_n) + (2\sigma-1)u(t_{n-1})}{2k} = \overline{u}(t_{n+\sigma}) + \mathcal{O}(k^2). \quad (19)$$

Introduction: Finite Volume from Admissible meshes to Nonconforming meshes

Finite volume methods are numerical methods approximating different types of Partial Differential Equations (PDEs). They are based on three principle ideas:

- Subdivision of the spatial domain into subsets called Control Volumes.
- Integration of the equation to be solved over the Control Volumes.
- Approximation of the derivatives appearing after integration.

[1] We mean here the "pure" finite volume methods and not finite volume-element methods

Introduction (suite): Finite Volume from Admissible meshes to Nonconforming meshes

Finite Volume methods passed by two steps:

First step

Finite Volume methods using Admissible meshes.

Second step

SUSHI method.

Introduction (suite): Finite Volume from Admissible meshes to Nonconforming meshes

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Introduction (suite): Finite Volume from Admissible meshes to Nonconforming meshes

Finite Volume methods passed by two steps:

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Introduction (suite): Finite Volume methods on admissible meshes

Definition

Let $\mathcal{T}$ be an Admissible Mesh in the sense of Eymard et al. (Handbook, 2000).

$K \in \mathcal{T}$ are the control volumes and σ are the edges of the control volumes K.

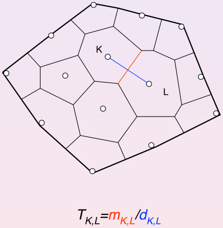

Figure : transmissivity between K and L: $\mathcal{T}_\sigma = \mathcal{T}_{K|L} = \frac{m_{K,L}}{d_{K,L}}$

Introduction (suite): Finite Volume methods on admissible meshes

Main properties of Admissible mesh:

1. Convexity of the Control Volumes.
2. The orthogonality property: the $(\boldsymbol{x}_K\boldsymbol{x}_L)$ is orthogonal to the common edge σ between the control volumes K and L.

Introduction (suite): Finite Volume methods on admissible meshes

Model to be solved:

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega \quad \text{and} \quad u(\boldsymbol{x}) = 0,\ \boldsymbol{x} \in \partial\Omega. \tag{20}$$

Principles of Finite Volume scheme:

1. Integration on each control volume K :$-\int_K \Delta u(\boldsymbol{x}) d\boldsymbol{x} = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$,
2. Integration by Parts gives :$-\int_{\partial K} \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$
3. Summing on the lines of K: $-\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$

Introduction (suite): Finite Volume methods on admissible meshes

Approximate Finite Volume Solution $u_{\mathcal{T}} = (u_K)_K$

$$-\sum_{\sigma\in\mathcal{E}_K}\frac{\mathrm{m}(\sigma)}{d_{K|L}}(u_L-u_K)=\int_K f(\boldsymbol{x})d\boldsymbol{x}. \tag{21}$$

Introduction (suite): Finite Volume methods on admissible meshes

Approximate Finite Volume Solution $u_{\mathcal{T}} = (u_K)_K$

$$-\sum_{\sigma\in\mathcal{E}_K}\frac{\mathrm{m}(\sigma)}{d_{K|L}}(u_L-u_K)=\int_K f(\boldsymbol{x})d\boldsymbol{x}. \tag{21}$$

Introduction (suite): Finite Volume methods on admissible meshes

Theorem

Let $\mathcal{X}(\mathcal{T})$: functions which are constant on each control volume K. Let $e_{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$ be defined by $e_K = u(\boldsymbol{x}_K) - u_K$ for any $K \in \mathcal{T}$. Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence results hold:

1. *H_0^1-error estimate*

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \leq Ch\|u\|_{2,\overline{\Omega}}, \tag{22}$$

where $\|\cdot\|_{1,\mathcal{T}}$ is the H_1^0-norm $\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 = \sum_{\sigma=K|L\in\mathcal{E}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(u_L - u_K)^2.$

2. *L^2-error estimate:*

$$\|e_{\mathcal{T}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{23}$$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Definition (New mesh of Eymard et *al.*, IMAJNA 2010)

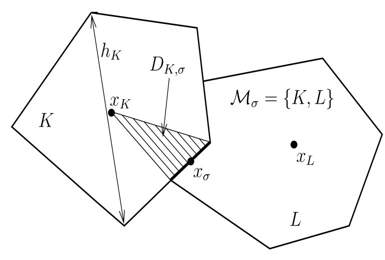

Figure : Notations for two neighbouring control volumes in $d = 2$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Main properties of this new mesh:

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (orthogonality property is not required): the orthogonality property is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convex, whereas the convexity property is not required in this new mesh.

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Principles of discretization for Poisson's equation:

1. Discrete unknowns: the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \{\left((v_K)_{K\in\mathcal{M}}\,,\,(v_\sigma)_{\sigma\in\mathcal{E}}\right),\;\; v_K, v_\sigma \in \mathbb{R},\;\; v_\sigma = 0,\, \forall\sigma \in \mathcal{E}_{\rm ext}\}$$

2. Discretization of the gradient: the discretization of ∇ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see Eymard et *al.* (IMAJNA, 2010):

   1. The discrete gradient $\nabla_{\mathcal{D}}$ is stable
   2. The discrete gradient $\nabla_{\mathcal{D}}$ is consistent.

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI

Weak formulation for Poisson's equation: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega). \tag{24}$$

SUSHI (Scheme Using stabilized Hybrid Interfaces) for Poisson's equation: Find $u_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\int_\Omega \nabla_{\mathcal{D}} u_{\mathcal{D}}(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{25}$$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI

Theorem

Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence result hold:

1. *H_0^1-error estimate*

$$\|\nabla u - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \le Ch\|u\|_{2,\overline{\Omega}}. \tag{26}$$

2. *L^2-error estimate:*

$$\|u - \Pi_{\mathcal{M}} u_{\mathcal{D}}\|_{L^2(\Omega)} \le Ch\|u\|_{2,\overline{\Omega}}. \tag{27}$$

Discretization in space

Discretization in space

We use SUSHI scheme

Main properties of this new mesh:

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (orthogonality property is not required): the orthogonality property is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convex, whereas the convexity property is not required in this new mesh.

Discretization in space for Time Fractional Diffusion-Wave equation, using SUSHI

1. Discrete unknowns: the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \left\{ \left((v_K)_{K \in \mathcal{M}}, \, (v_\sigma)_{\sigma \in \mathcal{E}} \right), \;\; v_K, v_\sigma \in \mathbb{R}, \;\; v_\sigma = 0, \, \forall \sigma \in \mathcal{E}_{\mathrm{ext}} \right\}$$

2. Discretization of the gradient: the discretization of ∇ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see Eymard et *al.* (IMAJNA, 2010):
   1. The discrete gradient $\nabla_{\mathcal{D}}$ is stable
   2. The discrete gradient $\nabla_{\mathcal{D}}$ is consistent.

Formulation of scheme

The finite volume scheme can then be defined as:

1. Discretization of initial conditions: Find $u^0_{\mathcal{D}}, \bar{u}^0_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that, for all $v \in \mathcal{X}_{\mathcal{D},0}$

$$\left(\nabla_{\mathcal{D}} u^0_{\mathcal{D}}, \nabla_{\mathcal{D}}\, v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\left(\Delta u^0, \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)} \tag{28}$$

and

$$\left(\nabla_{\mathcal{D}} \bar{u}^0_{\mathcal{D}}, \nabla_{\mathcal{D}}\, v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\left(\Delta u^1, \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)}. \tag{29}$$

2. Discretization of the time factional diffusion equation. For any $n \in [\![0, N]\!]$, find $u^{n+1}_{\mathcal{D}}, \bar{u}^{n+1}_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that, for all $v \in \mathcal{X}_{\mathcal{D},0}$

$$\sum_{j=0}^{n} \lambda_j^{n+1} \left(\Pi_{\mathcal{M}}(\bar{u}^{j+1}_{\mathcal{D}} - \bar{u}^{j}_{\mathcal{D}}), \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} + \left(\nabla_{\mathcal{D}} u^{n+\sigma}_{\mathcal{D}}, \nabla_{\mathcal{D}} v\right)_{\mathbb{L}^2(\Omega)}$$
$$= \left(f(t_{n+\sigma}), \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}, \tag{30}$$

where $v^{n+\sigma}$ is the two-point barycentric element $\sigma v^{n+1} + (1-\sigma)v^n$.

Formulation of scheme (Suite)

3. Discretization of the derivatives $\overline{u}(t_{n+\sigma}) = u_t(t_{n+\sigma})$.

$$\overline{u}_{\mathcal{D}}^{\frac{1}{2}} = \partial^1 u_{\mathcal{D}}^1 \quad (31)$$

and

$$\overline{u}_{\mathcal{D}}^{n+\sigma} = \frac{1}{2k}\left((2\sigma+1)u_{\mathcal{D}}^{n+1} - 4\sigma u_{\mathcal{D}}^{n} + (2\sigma-1)u_{\mathcal{D}}^{n-1}\right), \quad \forall n \in [\![1, N]\!]. \quad (32)$$

Convergence result

Theorem ($L^\infty(H^1)$ and $H^1(L^2)$–error estimates)

The following $L^\infty(H^1)$ and $H^1(L^2)$–error estimates hold:

$$\max_{n=0}^{N+1} \|\nabla_{\mathcal{D}} u_{\mathcal{D}}^n - \nabla u(t_n)\|_{L^2(\Omega)^d} + \left(\sum_{n=0}^{N+1} k \|\Pi_{\mathcal{M}} \overline{u}_{\mathcal{D}}^n - u_t(t_n)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$$
$$\leq \quad C(h_{\mathcal{D}} + k^2)\|u\|_{\mathcal{C}^4(0,T;\,\mathcal{C}^2(\overline{\Omega}))}. \tag{33}$$

Idea on the proof

Idea on the proof

1. Comparison with an auxiliary schemes: For any $n \in [\![0, N+1]\!]$, find $\Xi_{\mathcal{D}}^n, \Upsilon_{\mathcal{D}}^n \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\nabla_{\mathcal{D}}\Xi_{\mathcal{D}}^n, \nabla_{\mathcal{D}}\, v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\left(\Delta u(t_n), \Pi_{\mathcal{D}} v\right)_{L^2(\Omega)} \tag{34}$$

   and

$$\left(\Upsilon_{\mathcal{D}}^n, \nabla_{\mathcal{D}}\, v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\left(\Delta u_t(t_n), \Pi_{\mathcal{D}} v\right)_{L^2(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{35}$$

2. A well developed discrete *a priori estimate*.
3. Other technical details can be found in Proceedings LSSC2019.

Conclusion

Conclusion

Using an equivalent system of low order equations for TFDWE, we established a second order time accurate finite volume scheme a stable and consistent discrete gradient. The time discretization uses the approximation of Caputo derivative of order $0 < \beta < 1$ developed in Alikhanov (J. Comp. Phy, 2015). We proved the convergence under the *strong regularity assumption* $\mathcal{C}^4(\mathcal{C}^2)$. This regularity can be weakened to $\mathcal{C}^3(H^2)$ in the particular cases when $d = 2$ or $d = 3$. The convergence stated in this note includes a convergence in $L^\infty(H^1)$ and $H^1(L^2)$ discrete semi-norms.

Perspectives

First perspective

We detail this note in a full future paper.

Second perspective

Address the case of TFDWE with variable coefficients.

Perspectives

First perspective

We detail this note in a full future paper.

Second perspective

Address the case of TFDWE with variable coefficients.