Convergence Analysis of a Finite Volume Gradient Scheme for a Linear Parabolic Equation Using Characteristic Methods

Fayssal Benkhaldoun [1] and Abdallah Bradji[2]

[1] LAGA (Laboratoire d'Analyse, Géométrie, et Applications)-Paris 13–France

[2] Department of Mathematics, University of Annaba–Algeria

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Aim of the presentation

The aim of this talk is to establish a Finite Volume Scheme using Characteristic Methods along with a convergence analysis for Parabolic Equation.

Characteristics method is the replacement of the advective part of the equation by total differentiation along characteristics.

Plan of the talk

1. Problem to be solved.
2. Some Literature (References) on the subject.
3. Introduction: Finite Volume methods from Admissible to Nonconforming meshes (SUSHI scheme).
4. Definition of the Characteristics methods.
5. Formulation of a Finite Volume scheme using Characteristics methods.
6. Convergence analysis for the numerical scheme.
7. A simple Theoretical Comparison with FV scheme derived directly from a Weak Formulation.
8. Conclusion and Perspectives.

Problem to be solved

Equation

UADP (Unsteady Advection-Diffusion Problem), for $(\boldsymbol{x}, t) \in \Omega \times (0, T)$:

$$u_t(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) + \operatorname{div}(\mathbf{v}u)(\boldsymbol{x}, t) + b(\boldsymbol{x})u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t), \quad (1)$$

where Ω is a polyhedral open bounded connected subset of $\mathbb{R}^d$, $\mathbf{v} = \mathbf{v}(\boldsymbol{x}, t) \in \mathbb{R}^d$ is a vector field, and $b = b(\boldsymbol{x})$.

Initial and Homogeneous Dirichlet boundary conditions

$$u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega \ \text{ and } \ u(\boldsymbol{x}, t) = 0, \ (\boldsymbol{x}, t) \in \partial\Omega \times (0, T). \quad (2)$$

Assumptions on the data of the considered problem

Assumption

We assume that the functions $\boldsymbol{v}$ *and* b *are satisfying:*

$$\boldsymbol{v} \in \mathcal{C}^1(\overline{\Omega} \times [0,T]), \quad \boldsymbol{v}(\boldsymbol{x},t) = 0, \quad \forall (\boldsymbol{x},t) \in \partial\Omega \times [0,T], \tag{3}$$

and

$$b \in \mathcal{C}^1(\overline{\Omega}), \quad \text{and} \quad b(\boldsymbol{x}) + \operatorname{div}\boldsymbol{v}(\boldsymbol{x},t) \geq 0, \quad \forall (\boldsymbol{x},t) \in \Omega \times [0,T]. \tag{4}$$

References

1. Bradji, A., Fuhrmann, J.: Some abstract error estimates of a finite volume scheme for a nonstationary heat equation on general nonconforming multidimensional spatial meshes. Appl. Math. 58/1, 1–38 (2013).

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3. Eymard, R., Gallouët, T., Herbin, R.: Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. IMA J. Numer. Anal. 30/4, 1009–1043 (2010).

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Introduction: Finite Volume from Admissible meshes to Nonconforming meshes

Finite volume methods are numerical methods approximating different types of Partial Differential Equations (PDEs). They are based on three principle ideas:

- Subdivision of the spatial domain into subsets called Control Volumes.
- Integration of the equation to be solved over the Control Volumes.
- Approximation of the derivatives appearing after integration.

[1] We mean here the "pure" finite volume methods and not finite volume-element methods

Introduction (suite): Finite Volume from Admissible meshes to Nonconforming meshes

Finite Volume methods passed by two steps:

First step

Finite Volume methods using Admissible meshes.

Second step

SUSHI method.

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Introduction (suite): Finite Volume methods on admissible meshes

Definition

Let $\mathcal{T}$ be an Admissible Mesh in the sense of Eymard et al. (Handbook, 2000).

$K \in \mathcal{T}$ are the control volumes and σ are the edges of the control volumes K.

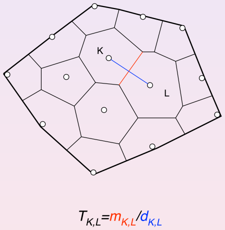

Figure : transmissivity between K and L: $\mathcal{T}_\sigma = \mathcal{T}_{K|L} = \frac{m_{K,L}}{d_{K,L}}$

Introduction (suite): Finite Volume methods on admissible meshes

Main properties of Admissible mesh:

1. Convexity of the Control Volumes.
2. The orthogonality property: the $(\boldsymbol{x}_K \boldsymbol{x}_L)$ is orthogonal to the common edge σ between the control volumes K and L.

Introduction (suite): Finite Volume methods on admissible meshes

Model to be solved:

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega \quad \text{and} \quad u(\boldsymbol{x}) = 0,\ \boldsymbol{x} \in \partial\Omega. \tag{5}$$

Principles of Finite Volume scheme:

1. Integration on each control volume K : $-\int_K \Delta u(\boldsymbol{x}) d\boldsymbol{x} = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$,
2. Integration by Parts gives : $-\int_{\partial K} \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$
3. Summing on the lines of K: $-\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$

Introduction (suite): Finite Volume methods on admissible meshes

Approximate Finite Volume Solution $u_{\mathcal{T}} = (u_K)_K$

$$-\sum_{\sigma \in \mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{d_{K|L}}(u_L - u_K) = \int_K f(\boldsymbol{x})d\boldsymbol{x}. \quad (6)$$

Matrix Form

$$\mathcal{A}^{\mathcal{T}} u_{\mathcal{T}} = f_{\mathcal{T}}.$$

Introduction (suite): Finite Volume methods on admissible meshes

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Introduction (suite): Finite Volume methods on admissible meshes

Theorem

Let $\mathcal{X}(\mathcal{T})$: functions which are constant on each control volume K. Let $e_{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$ be defined by $e_K = u(\boldsymbol{x}_K) - u_K$ for any $K \in \mathcal{T}$. Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence results hold:

1. *H_0^1-error estimate*

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \leq Ch\|u\|_{2,\overline{\Omega}}, \tag{7}$$

where $\|\cdot\|_{1,\mathcal{T}}$ is the H_1^0-norm $\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 = \sum_{\sigma=K|L\in\mathcal{E}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(u_L - u_K)^2.$

2. *L^2-error estimate:*

$$\|e_{\mathcal{T}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{8}$$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Definition (New mesh of Eymard et *al.*, IMAJNA 2010)

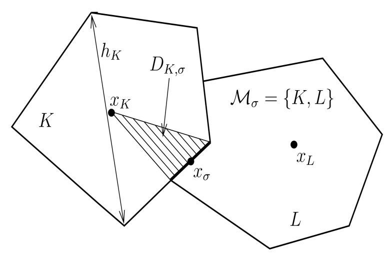

Figure : Notations for two neighbouring control volumes in $d = 2$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Main properties of this new mesh:

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (orthogonality property is not required): the orthogonality property is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convex, whereas the convexity property is not required in this new mesh.

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Principles of discretization for Poisson's equation:

1. Discrete unknowns: the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \left\{ \left((v_K)_{K\in\mathcal{M}} \,,\, (v_\sigma)_{\sigma\in\mathcal{E}} \right) ,\;\; v_K, v_\sigma \in \mathbb{R},\;\; v_\sigma = 0,\, \forall \sigma \in \mathcal{E}_{\text{ext}} \right\}$$

2. Discretization of the gradient: the discretization of ∇ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see Eymard et *al.* (IMAJNA, 2010):

   1. The discrete gradient $\nabla_{\mathcal{D}}$ is stable
   2. The discrete gradient $\nabla_{\mathcal{D}}$ is consistent.

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI

Weak formulation for Poisson's equation: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega). \tag{9}$$

SUSHI (Scheme Using stabilized Hybrid Interfaces) for Poisson's equation: Find $u_\mathcal{D} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\int_\Omega \nabla_\mathcal{D} u_\mathcal{D}(\boldsymbol{x}) \cdot \nabla_\mathcal{D} v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{10}$$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI

Theorem

Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence result hold:

1. H_0^1*-error estimate*

$$\|\nabla u - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{11}$$

2. L^2*-error estimate:*

$$\|u - \Pi_{\mathcal{M}} u_{\mathcal{D}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{12}$$

Definition of the Characteristics

Time discretization

We consider a constant time step $k = \dfrac{T}{N+1}$, where $N \in \mathbb{N}^\star$. The mesh points are $t_n = nk$, for $n \in [\![0, N+1]\!]$ and we denote by ∂^1 the discrete first time derivative:

$$\partial^1 v^{j+1} = \frac{v^{j+1} - v^j}{k}. \tag{13}$$

Definition of the Characteristics

For any $s \in [0, T]$ and $\boldsymbol{x} \in \Omega$, we define the characteristic lines associated to $\mathbf{v} = \mathbf{v}(\boldsymbol{x}, t)$ as the vector functions $\Phi = \Phi(t; \boldsymbol{x}, s) : [0, T] \longrightarrow \Omega$ satisfying the following differential equation:

$$\begin{cases} \dfrac{d\Phi}{dt}(t; \boldsymbol{x}, s) = \mathbf{v}(\Phi(t; \boldsymbol{x}, s), t), \quad t \in (0, T) \\ \Phi(s; \boldsymbol{x}, s) = \boldsymbol{x}. \end{cases} \tag{14}$$

Properties of the Characteristics

- The existence and uniqueness of the characteristic lines for each choice of s and and $\boldsymbol{x}$ hold under suitable assumptions on $\mathbf{v}$, for instance $\mathbf{v}$ continuous in $\overline{\Omega} \times [0, T]$ and Lipschitz continuous in $\overline{\Omega}$, uniformly with respect to $t \in [0, T]$.
- The uniqueness stated in the previous item implies that

$$\Phi(t; \Phi(s; \boldsymbol{x}, \tau), s) = \Phi(t; \boldsymbol{x}, \tau). \tag{15}$$

- Taking $t = \tau$ in (15) yields

$$\Phi(\tau; \Phi(s; \boldsymbol{x}, \tau), s) = \Phi(\tau; \boldsymbol{x}, \tau) = \boldsymbol{x}. \tag{16}$$

- For any t and s, the inverse of the function $\boldsymbol{x} \mapsto \Phi(t; \boldsymbol{x}, s)$ is $\boldsymbol{x} \mapsto \Phi(s; \boldsymbol{x}, t)$

Principles of scheme in time using Characteristics

1. Let us define

$$\overline{u}(\boldsymbol{x},t) = u(\Phi(t;\boldsymbol{x},0),t). \tag{17}$$

2. We have

$$\frac{\partial \overline{u}}{\partial t} - \overline{\Delta u} + \left(\overline{b} + \overline{\mathrm{div}\mathbf{v}}\right)\overline{u} = \overline{f}. \tag{18}$$

Taking $t = t_{n+1}$ as argument in equation (18) leads to

$$\frac{\partial \overline{u}}{\partial t}(t_{n+1}) - \overline{\Delta u}(t_{n+1}) + \left(\overline{b} + \overline{\mathrm{div}\mathbf{v}}(t_{n+1})\right)\overline{u}(t_{n+1}) = \overline{f}(t_{n+1}). \tag{19}$$

Principles of scheme in time using Characteristics (Suite)

3. Let us set

$$\begin{aligned}
&\frac{\partial \overline{u}}{\partial t}(\Phi(0;\boldsymbol{x},t_{n+1}),t_{n+1})\\
&\quad= \frac{u(\Phi(t_{n+1};\Phi(0;\boldsymbol{x},t_{n+1}),0),t_{n+1})-u(\Phi(t_n;\Phi(0;\boldsymbol{x},t_{n+1}),0),t_n)}{k}+\mathbb{T}_1^{n+1}(\boldsymbol{x})\\
&\quad= \frac{u(\boldsymbol{x},t_{n+1})-u(\Phi(t_n;\boldsymbol{x},t_{n+1}),t_n)}{k}+\mathbb{T}_1^{n+1}(\boldsymbol{x}). \qquad (20)
\end{aligned}$$

Using a Taylor expansion to get

$$|\mathbb{T}_1^{n+1}| \le Ck\|u\|_{\mathcal{C}^1([0,T];\,\mathcal{C}(\overline{\Omega}))}. \qquad (21)$$

Taking $\boldsymbol{x}=\Phi(0;\boldsymbol{x},t_{n+1})$ in (19) and gathering the result with (20) yield

$$\begin{aligned}
&\frac{u(t_{n+1})-u(t_n)(\Phi(t_n;\boldsymbol{x},t_{n+1}))}{k}-\Delta u(t_{n+1})\\
&+\quad (b+\operatorname{div}\mathbf{v}(t_{n+1}))\,u(t_{n+1})=f(t_{n+1})-\mathbb{T}_1^{n+1}(\boldsymbol{x}). \qquad (22)
\end{aligned}$$

Principles of scheme in time using Characteristics (Suite)

4. **Approximation of the Characteristics $\Phi(t_n; \boldsymbol{x}, t_{n+1})$**. Let us set

$$\Phi(t_n; \boldsymbol{x}, t_{n+1}) = \omega^{n+1}(\boldsymbol{x}) + \mathbb{T}_2^{n+1}(\boldsymbol{x}) \tag{23}$$

and

$$u(t_n)(\Phi(t_n; \boldsymbol{x}, t_{n+1})) = u(t_n)(\omega^{n+1}(\boldsymbol{x})) + \mathbb{T}_3^{n+1}(\boldsymbol{x}), \tag{24}$$

where

$$\omega^{n+1}(\boldsymbol{x}) = \boldsymbol{x} - k\mathbf{v}(\boldsymbol{x}, t_{n+1}). \tag{25}$$

We can check that ω^{n+1} is a second order accurate approximation for $\Phi(t_n; \cdot, t_{n+1})$. This implies that $\mathbb{T}_3^{n+1}$ is of order two, i.e.

$$|\mathbb{T}_3^{n+1}| \le Ck^2 \|u\|_{\mathcal{C}^1([0,T];\, \mathcal{C}(\overline{\Omega}))}. \tag{26}$$

Principles of scheme in time using Characteristics (Suite)

5. **Approximation of the parabolic equation using Characteristics**. Under Assumption 1 and the assumption that k is sufficiently small, we prove that

$$\omega^{n+1}(\boldsymbol{x}) \in \Omega.$$

From (22) and (23), we deduce that

$$\Delta u(t_{n+1}) + f(t_{n+1}) = \frac{u(t_{n+1}) - u(t_n)(\omega^{n+1})}{k} + (b + \operatorname{div}\mathbf{v}(t_{n+1}))\, u(t_{n+1}) + \mathbb{T}_1^{n+1}(\boldsymbol{x}) - \frac{\mathbb{T}_3^{n+1}(\boldsymbol{x})}{k}. \qquad (27)$$

Principles of the discretization

Discretization in time

As stated before, uniform mesh.

Discretization in space

We use SUSHI scheme

Formulation of scheme

The finite volume scheme can then be defined as:

- Discretization of initial condition:

$$\left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^0, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\left(\Delta u^0(t_n), \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}, \quad (28)$$

- Discretization of the Parabolic equation: For any $n \in [\![0, N]\!]$, find $u_{\mathcal{D}}^{n+1} \in \mathcal{X}_{\mathcal{D},0}$ such that, for all $v \in \mathcal{X}_{\mathcal{D},0}$

$$\begin{aligned} &\frac{1}{k}\left(\Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1} - \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n}(\omega^{n+1}), \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)} + \left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} \\ &+ \left(\left(b + \mathrm{div}\mathbf{v}(t_{n+1})\right) \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)} \\ &= \left(f(t_{n+1}), \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)}. \end{aligned} \quad (29)$$

Statement of the convergence results

Theorem (Error estimates)

We assume that u is sufficiently smooth and k is sufficiently small:

$$\max_{n=0}^{N+1} \|\Pi_{\mathcal{M}} u_{\mathcal{D}}^n - u(t_n)\|_{L^2(\Omega)} \leq C\left(k + h_{\mathcal{D}} + \frac{h_{\mathcal{D}}}{k}\right)\|u\|_{\mathcal{C}^1([0,T];\,\mathcal{C}^2(\overline{\Omega}))}. \tag{30}$$

If we assume in addition that for some given positive δ, we have $h_{\mathcal{D}} \leq Ck^{1+\delta}$, then the error estimate (30) becomes as

$$\max_{n=0}^{N+1} \|\Pi_{\mathcal{M}} u_{\mathcal{D}}^n - u(t_n)\|_{L^2(\Omega)} \leq C\left(k + k^{\delta} + h_{\mathcal{D}}\right)\|u\|_{\mathcal{C}^1([0,T];\,\mathcal{C}^2(\overline{\Omega}))}. \tag{31}$$

Idea on the proof

The proof is manly based on two facts:

- Comparison with an optimal scheme : for any $n \in [\![0, N+1]\!]$, find $\bar{u}^n_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$(\nabla_{\mathcal{D}} \bar{u}^n_{\mathcal{D}}, \nabla_{\mathcal{D}} v)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\sum_{K\in\mathcal{M}} v_K \int_K \Delta u(x, t_n) dx, \;\; \forall\, v \in \mathcal{X}_{\mathcal{D},0}. \qquad (32)$$

- A convenient a priori estimate.

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- A convenient a priori estimate.

A simple Theoretical Comparison with a FV derived directly

Weak Formulation

We multiply (1) by $\varphi \in H_0^1(\Omega)$ and use integration by parts

$$\int_\Omega u_t(t)\varphi + \int_\Omega \nabla u(t)\cdot\nabla\varphi - \sum_{i=1}^{d}\int_\Omega u\mathbf{v}_i(t)\frac{\partial\varphi}{\partial \mathbf{x}_i} + \int_\Omega bu(t)\varphi = \int_\Omega f(t)\varphi. \quad (34)$$

Scheme

For all $v \in \mathcal{X}_{\mathcal{D},0}$ ($\nabla^i_{\mathcal{D}} v$ denotes the i-th componenent of $\nabla_{\mathcal{D}} v$)

$$\left(\Pi_{\mathcal{M}}\partial^1 u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)} + \left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}}\, v\right)_{\mathbb{L}^2(\Omega)} + \left(b\Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)}$$

$$- \sum_{i=1}^{d}\left(\mathbf{v}_i(t_{n+1})\Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \nabla^i_{\mathcal{D}} v\right)_{L^2(\Omega)} = \left(f(t_{n+1}),\, \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)}. \quad (35)$$

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$$-\sum_{i=1}^{d}\left(\mathbf{v}_i(t_{n+1})\Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \nabla^i_{\mathcal{D}} v\right)_{L^2(\Omega)} = \left(f(t_{n+1}), \Pi_{\mathcal{M}} v\right)_{L^2(\Omega)}. \quad (35)$$

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Conclusion

We applied SUSHI combined with characteristics method to approximate UADP (1)–(4). This model is more general than that we considered in our previous works in which we derived schemes directly along with a convergence analysis. We proved the $L^\infty(L^2)$–error estimate (30) which is a conditional convergence. This error estimate is proved thanks to the new a prior estimate. Both, scheme (28)–(29) and convergence order (30) are new. The convergence order $k + h_{\mathcal{D}} + \dfrac{h_{\mathcal{D}}}{k}$ of (30) is similar to the one obtained in the context of FEM, that is $k + h + \dfrac{h^2}{k}$, see Quarteronit it is different because of the presence of $\dfrac{h_{\mathcal{D}}}{k}$ instead of $\dfrac{h^2}{k}$. This difference stems from the first order $L^\infty(L^2)$–error estimate in FV which is second order in FEM. This note is an initiation in the application of FV methods combined with the method of characteristics for UADP.

Perspectives

First perspective

To try to prove an unconditional convergence instead of the conditional one .

Second perspective

The proof of a convergence in the discrete energy norm of $L^{\infty}(H^{1})$ and extension of the present work to semi-linear UADP are also interesting perspectives to work on.

Third perspective

We address the non-linear case.

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