# A New Error Estimate for a Primal-Dual Crank-Nicolson Mixed Finite Element using Lowest Degree Raviart-Thomas Spaces for Parabolic Equations

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LSSC2021 (The 13th Int. Conference on Large-Scale Scientific Computations)
June 7–10, Sozopol-Bulgaria
On-Line Presentation

# Aim of the presentation

We prove a new error estimate for a Lowest Order Raviart-Thomas Mixed method combined with the Crank-Nicolson method for Parabolic equations. This new convergence result states the convergence rate towards the “velocity” $P(t) = -\nabla u(t)$ in the norm of $L^2(H_{\text{div}})$

# Plan of this presentation

1. Problem to be solved
2. Space and time discretizations
3. Formulation of the scheme (based on Lowest Order Raviart-Thomas Mixed method combined with the Crank Nicolson finite difference method)
4. Known convergence results and a novel error estimate
5. Conclusion and Perspectives

# Problem to be solved

## Equation

Heat equation:

$$u_t(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t), \ \ (\boldsymbol{x},t) \in \Omega \times (0,T), \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is an open domain of $\mathbb{R}^d$, $f$ is given function, and $T > 0$ is given.

## Initial conditions

$$u(\boldsymbol{x},0) = u^0(\boldsymbol{x}), \ \ \boldsymbol{x} \in \Omega. \tag{2}$$

## Homogeneous Dirichlet boundary

$$u(\boldsymbol{x},t) = 0, \ \ (\boldsymbol{x},t) \in \partial\Omega \times (0,T). \tag{3}$$

# About Heat equation?

## Some physics

Heat equation $u_t - \Delta u$ is typically used in different applications, such as *fluid mechanics*, *heat and mass transfer*,...

## Existence and uniqueness

Existence and uniqueness of a weak solution of heat equation, with (2) ( *initial condition*) and (3) ( *Dirichlet boundary condition*) can be formulated using Bochner spaces; see for instance Evans book of partial differential equation

# Space and time discretizations

## Space discretization

We define the lowest order Raviart-Thomas Mixed FE spaces:

$$V_h^{\mathrm{div}} = \{v \in H_{\mathrm{div}}(\Omega) : \quad v|_K \in \mathbb{D}_0, \quad \forall K \in \mathcal{T}_h\}, \tag{4}$$

where $\mathcal{T}_h$ is a family of triangulations of $\overline{\Omega}$ with $d$-simplex and

$$W_h = \{p \in L^2(\Omega) : \quad p|_K \in \mathbb{P}_0, \quad \forall K \in \mathcal{T}_h\}, \tag{5}$$

where $\mathbb{P}_0$ is the space of constant functions and $\mathbb{D}_0 = (\mathbb{P}_0)^d \oplus \boldsymbol{x}\mathbb{P}_0$.

## Time discretization

We consider constant time step $k = \dfrac{T}{N+1}$, where $N \in \mathbb{N}^\star$. The mesh points are denoted by $t_n = nk$, for $n \in [\![0, N+1]\!]$.

- **Discrete temporal derivative**

$$\partial^1 v^{n+1} = \frac{v^{n+1} - v^n}{k}.$$

- **Arithmetic mean value**:

$$v^{n+\frac{1}{2}} = \frac{v^{n+1} + v^n}{2}.$$

# Formulation of a MFE scheme using Crank-Nicolson method

The unknowns of this scheme are the set of the couples

$$\left\{(p_h^n, u_h^n) \in V_h^{\mathrm{div}} \times W_h; n \in [\![0, N+1]\!]\right\}.$$

These unknowns are expected to approximate the set of the unknowns

$$\{(-\nabla u(t_n), u(t_n)); \quad n \in [\![0, N+1]\!]\}.$$

# Formulation of a MFE scheme using Crank-Nicolson method (Suite)

- For any $n \in [\![0, N]\!]$ and for all $\varphi \in W_h$ :

$$\left(\partial^1 u_h^{n+1}, \varphi\right)_{L^2(\Omega)} + \left(\nabla \cdot p_h^{n+\frac{1}{2}}, \varphi\right)_{L^2(\Omega)} = \left(\frac{f(t_{n+1}) + f(t_n)}{2}, \varphi\right)_{L^2(\Omega)}, \tag{6}$$

- For any $n \in [\![0, N+1]\!]$:

$$(p_h^n, \psi)_{L^2(\Omega)^d} = (u_h^n, \nabla \cdot \psi)_{L^2(\Omega)}, \qquad \forall \psi \in V_h^{\text{div}}, \tag{7}$$

where

$$\left(\nabla \cdot p_h^0, \varphi\right)_{L^2(\Omega)} = \left(-\Delta u^0, \varphi\right)_{L^2(\Omega)}, \qquad \forall \varphi \in W_h. \tag{8}$$

# Some references on the subject

- P. G. Ciarlet, The Finite Element Method for Elliptic Problems. Classics in Applied Mathematics, 40. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (2002).
- C. Johnson and V. Thomee, Error estimates for some mixed finite element methods for parabolic type problems. RAIRO Anal. Numér. 15/1, 41–78 (1981).
- A. Quarteroni and A. Valli, Numerical Approximation of Partial Differential Equations. Springer Series in Computational Mathematics 23. Berlin: Springer. (2008)

# Known error estimates

## Johnson and Thomee (1981): $L^\infty\left(\left(L^2\right)^d\right)$–error estimate

$$\max_{n=0}^{N+1} \|p_h^n + \nabla u(t_n)\|_{L^2(\Omega)^d} \leq C(k^2 + h). \tag{9}$$

## Our aim

Our aim is to improve the error estimate (9) in the sense that the norm $L^\infty\left(\left(L^2\right)^d\right)$ is replaced by $L^2(H_{\mathrm{div}})$.

# Statement of the main result

**Theorem (New error estimate for scheme (6)–(8))**

*The following $L^2(H_{\mathrm{div}})$–error estimate holds:*

$$\left(\sum_{n=0}^{N+1} k\|\mathrm{div} p_h^n + \Delta u(t_n)\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \leq C(k^2 + h). \tag{10}$$

# Idea on the proof

## Lemma (New a priori estimate)

*Assume that* $\left((\eta_{\mathcal{D}}^n)_{n=0}^{N+1}, (\overline{\eta}_{\mathcal{D}}^n)_{n=0}^{N+1}\right) \in \left(V_h^{\mathrm{div}}\right)^{N+2} \times W_h^{N+2}$ *such that* $\overline{\eta}_h^0 = 0$

- *For any* $n \in [\![0, N]\!]$, *for all* $\varphi \in W_h$:

$$\left(\partial^1 \overline{\eta}_h^{n+1}, \varphi\right)_{L^2(\Omega)} + \left(\mathrm{div}\eta_h^{n+\frac{1}{2}}, \varphi\right)_{L^2(\Omega)} = \left(\mathcal{S}^{n+1}, \varphi\right)_{L^2(\Omega)}, \tag{11}$$

- *For any* $n \in [\![0, N+1]\!]$:

$$(\eta_h^n, \psi)_{L^2(\Omega)^d} = (\overline{\eta}_h^n, \mathrm{div}\psi)_{L^2(\Omega)}, \qquad \forall \psi \in V_h^{\mathrm{div}}. \tag{12}$$

*Then, the following* $L^2(H_{\mathrm{div}})$*–a priori estimate holds:*

$$\left(\sum_{n=0}^{N+1} k \|\mathrm{div}\eta_h^n\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \le C \max_{n=0}^{N} \|\mathcal{S}^{n+1}\|_{L^2(\Omega)}. \tag{13}$$

# Conclusion

## First result obtained

We established a numerical scheme using Lowest Order Raviart-Thomas Mixed Finite Element methods as discretization in space and Crank-Nicolson finite difference method as discretization in time.

## Second result obtained

We proved a new convergence result towards $-\nabla u(t)$ in the norm of $L^2(H_{\text{div}})$. The order in time is two and is one in space.

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# Perspectives

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One of the main perspectives is to extend this result to a large class of mixed finite elements. In progress.

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