Note on a new piecewise linear finite element approximation of order four for one dimensional second order elliptic problems on general meshes

Abdallah Bradji

Department of Mathematics, University Of Annaba, Algeria

Problem to be solved

We consider the following second order elliptic problem, with homogeneous boundary conditions:

$$-\left(p(x)u_x(x)\right)_x + q(x)u(x) = f(x), \; x \in \mathbf{I} = (0,1), \tag{1}$$

and

$$u(0) = u(1) = 0, \tag{2}$$

Why problem (1)–(2)?

- It is a simple model.
- Problem (1)–(2) is a classical Sturm–Liouville problem.
- It represents a number of different of physical processes, i.e. the distribution of temperature along the rod or the displacement of a rotating string, see [2].

Weak formulation

$$\mathbf{a}(u,v) = \mathbf{F}(v), \; \forall v \in \mathcal{H}_0^1(\mathbf{I}), \tag{3}$$

where

$$\mathbf{a}(u,v) = \int_{\mathbf{I}} \left(p(x)u_x(x)v_x(x) + q(x)u(x)v(x)\right) dx, \tag{4}$$

and

$$\mathbf{F}(v) = \int_{\mathbf{I}} f(x)v(x)dx. \tag{5}$$

Meshes and standard scheme

Let us introduce a *non- uniform* mesh with the points $0 < x_0 < \ldots < x_{N+1} = 1$. We define $h_i = x_{i+1} - x_i$, for all $i \in [\![0, N]\!]$. We then set $h = \max_{i=0}^{N} h_i$.

Let $\mathcal{V}^h$ be the piecewise linear finite element space, i.e.

$$\mathcal{V}^h = \{v \in \mathcal{C}(\mathbf{I}), v|_{\mathbf{I}_i} \in \mathcal{P}_1, \; \forall i \in [\![0, N]\!]\} \cap \mathcal{H}_0^1(\mathbf{I}), \tag{6}$$

where $\mathbf{I}_i$ is the subinterval $[x_i, x_{i+1}]$ and $\mathcal{P}_1$ denotes the space of affine polynomials.

The approximate finite element solution $u^h \in \mathcal{V}^h$ is defined by

$$\mathbf{a}(u^h, v^h) = \mathbf{F}(v^h), \; \forall v^h \in \mathcal{V}^h. \tag{7}$$

Advantages and disadvantages

- Advantage: The matrix used in the finite element scheme (7) is sparse (tridiagonal). Computational cost is not much.
- Disadvantage: Convergence order is only two.

$$\|u^h - \pi u\|_{\mathcal{H}^1(\mathbf{I})} \le Ch^2 \|f\|_{\mathbb{L}^2(\mathbf{I})}. \tag{8}$$

Two approaches to increase the Convergence:

- Usual approach: We increase the degree of the polynomials used in the finite element spaces. This leads to matrices with many non zero entries. Which increases the Computational Cost.
- Our approach: We use the same scheme (7) by changing only the right hand side. This means that the same original sparse (tridiagonal) matrix will be used.

An auxiliary helpful term

Let us consider the expression

$$d_i(v^h) = -\int_{\mathbf{I}_i} p_x(x)\left(\alpha_i^1(x)\Sigma_i + \alpha_i^2(x)\Xi_i\right) v_x^h(x)dx \tag{9}$$

$$+ \int_{\mathbf{I}_i} q(x)\left(\alpha_i^1(x)\Sigma_i + \alpha_i^2(x)\Xi_i\right) v^h(x)dx, \tag{10}$$

where

- Σ_i is a second order approximation for $u_{xx}(x_i)$
- Ξ_i is a first order approximation for $u_{xxx}(x_i)$
- α_i^1 and α_i^2 are some given coefficients stem from an application of a convenient Taylor expansion.

An auxiliary solution

Let us consider the following correction:

$$\mathbf{a}(w_1^h, v^h) = -\sum_{i=0}^{N} \int_{\mathbf{I}_i} p_x(x)\left(\alpha_i^1(x)\Sigma_i + \alpha_i^2(x)\Xi_i\right) v_x^h(x)dx$$

$$+ \sum_{i=0}^{N} \int_{\mathbf{I}_i} q(x)\left(\alpha_i^1(x)\Sigma_i + \alpha_i^2(x)\Xi_i\right) v^h(x)dx, \tag{11}$$

Formulation of a new fourth order piecewise linear approximation

$$u_1^h = u^h - w_1^h. \tag{12}$$

Statement of the main result

Assume that the exact solution u is smooth, i.e. $u \in \mathcal{H}^4(\mathbf{I})$

$$\|u_1^h - \pi u\|_{\mathcal{H}^1(\mathbf{I})} \le Ch^4 \|u\|_{\mathcal{H}^4(\mathbf{I})}. \tag{13}$$

Comments: Advantages

- u_1^h is of order four.
- u_1^h can be computed using the same simple tridiagonal matrix used to compute the solution u^h of scheme (7), i.e. without make appeal to schemes of high order which lead in general to matrices of many non zero entries.
- The mesh used here is arbitrary, i.e. without any restricted condition on the mesh.
- The manner by which u_1^h is computed allows us to compute successively piecewise linear approximate finite element solutions u_n^h of order h^{2n+2}.
- In addition to the feature that u_n^h are of high order, i.e. h^{2n+2}, the piecewise linear approximate finite element solutions u_n^h can be computed using the same matrix simple tridiagonal matrix used to compute the solution u^h of scheme (7).

Perspectives

- Extending the results for partial differential equations in several space dimension.
- We obtained a high order approximate solution u_1^h under smooth regularity on the exact solution. Is it possible to increase the order for some non-smooth exact solutions?.

Numerical tests

The mesh is non uniform and satisfying $h_i = h$ when i is even and $h_i = \frac{h}{2}$ when i is odd. The problem considered in these computational results is the equation $-u_{xx} + u = f$, where $u(x) = \sin(\pi x)$ on $\mathbf{I} = (0,1)$.

The convergence order of $\| u^h - \pi u\|_{\mathcal{H}^1(\mathbf{I})}$ and $\| u_1^h - \pi u\|_{\mathcal{H}^1(\mathbf{I})}$ in $\mathcal{H}_0^1$–norm on nonuniform mesh.

h	Order of u_1^h	Order of u^h
2/33	-	-
1/39	3.9690	1.9961
2/153	3.9771	1.9974
2/303	3.9822	1.9982
2/453	3.9831	1.9984
2/603	4.1031	1.9986

References

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Contact: abdallah.bradji@univ-annaba.dz, bradji@cmi.univ-mrs.fr