Some discrete a priori estimates for a finite volume scheme appearing in the discretization of a time dependent Joule heating system problem

Abdallah Bradji

Department of Mathematics, University Of Annaba, Algeria

Origin of this study: Joule heating time dependent System

Let us consider the following nonlinear Joule heating system. We seek a couple of real valued functions (u, φ) defined on $\Omega \times [0, T]$ and satisfying

$$u_t(x,t) - \Delta u(x,t) = \kappa(u) \, |\nabla \varphi|^2 (x,t), \quad (x,t) \in \Omega \times (0,T), \tag{1}$$

$$-\nabla \cdot \left(\kappa(u) \nabla \varphi \right)(x,t) = 0, \quad (x,t) \in \Omega \times (0,T), \tag{2}$$

where Ω is an open bounded polyhedral subset in $\mathbb{R}^d$, with $d = 2$ or 3, $T > 0$, and κ is a given function.
An initial condition is given by:

$$u(x,0) = u^0(x), \ x \in \Omega, \tag{3}$$

and Dirichlet boundary conditions

$$u(x,t) = 0 \ \text{ and } \ \varphi(x,t) = g(x,t), \ (x,t) \in \partial\Omega \times (0,T), \tag{4}$$

Why Joule heating System ?

- The system (1)–(2) is interesting model for electric heating of a conducting body, where u is the temperature, φ is the electric potential.
- Since φ is the solution of a diffusion equation, it is reasonable to seek φ in the space $H^1(\Omega)$, so that $\nabla\varphi \cdot \nabla\varphi \in L^1$. Hence the non stationary heat equation has a right hand–side in L^1 (with of course some assumption on the function κ), and its analysis falls out of the usual variational framework.

As initiation for a full analysis for a finite volume method

We consider for the sake of simplicity simplicity, we first begin the non-stationary heat equation (1) with the particular case $\kappa \equiv 1$, i.e. we study the following model:

$$u_t(x,t) - \Delta u(x,t) = F(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{5}$$

where $F \in L^1((0,T) \times \Omega)$ (or also $F \in L^1(0,T; L^1(\Omega))$) and we assume that (5) is equipped with initial condition

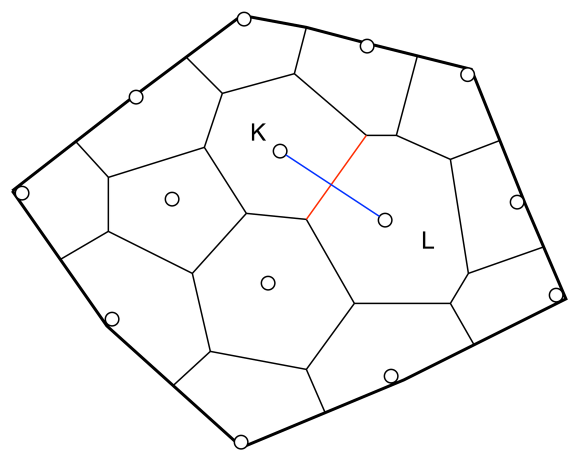
$$u(0) = u^0, \tag{6}$$

where u^0 is a given function defined on Ω.

Spatial discretization: admissible finite volume mesh

The finite volume mesh considered is the admissible mesh of [4].
Among the properties of this mesh, we quote

- Control volumes are finite family of open polygonal convex disjoint subsets of Ω.
- Orthogonality is required.
- The discrete unknowns are located at the centers of the control volumes.



$T_{K,L} = m_{K,L}/d_{K,L}$

Figure : Two adjacent control volumes in 2D

Time discretization

Let us consider as discretization for the interval $(0,T)$ the uniform mesh with a constant time step $k = \frac{T}{N+1}$, where $N \in \mathbb{N}^\star$, and we shall denote $t_n = nk$, for $n \in [\![0, N+1]\!]$

A generic finite volume scheme

For given η_K^0, $K \in \mathcal{T}$, we consider the implicit scheme, for $n \in [\![0, N]\!]$, find $\{\eta_K^n; K \in \mathcal{T}\}$

$$\mathrm{m}(K)\partial^1 \eta_K^{n+1} - \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma (\eta_L^{n+1} - \eta_K^{n+1}) = \mathrm{m}(K)\mathcal{S}_K^n, \ \forall K \in \mathcal{T}. \tag{7}$$

Some discrete norms

Define $X(\mathcal{T})$ as the set of functions from Ω to $\mathbb{R}$ which are constant on each control volume K, $K \in \mathcal{T}$, of the mesh. For each $u = (u_K)_{K \in \mathcal{T}}$ and $q \in [1, \infty)$, we define the discrete $W_0^{1,q}$–norm by

$$\| u \|_{1,q,\mathcal{T}}^q = \sum_{\sigma \in \mathcal{E}} \mathrm{m}(\sigma) d_\sigma \left(\frac{u_L - u_K}{d_\sigma} \right)^q. \tag{8}$$

Some known results, see [6]

The following *a priori estimates* are given in [6] but without details on the right hand side by which the $L^\infty(L^1)$–estimate is bounded above. Full and detailed proof will be provided in extended paper. Assume that there exists a solution $\{\eta_K^{n+1}; K \in \mathcal{T}\}$ for (7) for all $n \in [\![0, N+1]\!]$ and let us denote

$$\mathcal{S} = \sum_{n=0}^{N+1} \sum_{K \in \mathcal{T}} k \mathrm{m}(K) \, |\mathcal{S}_K^n| \, . \tag{9}$$

Then the following $L^\infty(0,T; L^1(\Omega))$ and $L^q(0,T; W_0^{1,q}(\Omega))$–estimates holds, for a positive constants C_1 only depending on Ω, d, and T, for all $n \in [\![0, N+1]\!]$

$$\|\eta_\mathcal{T}^n\|_{L^1(\Omega)} \leq C_1 \left(\|\eta_\mathcal{T}^0\|_{L^1(\Omega)} + \mathcal{S} \right), \tag{10}$$

and

$$\left(\sum_{n=0}^{N+1} k \| \eta_\mathcal{T}^n \|_{1,q,\mathcal{T}}^q \right)^{\frac{1}{q}} \leq C_2 \left(\|\eta_\mathcal{T}^0\|_{L^1(\Omega)} + \mathcal{S} \right). \tag{11}$$

Some new and useful a priori estimate results

Assume that there exists a solution $\{\eta_K^{n+1}; K \in \mathcal{T}\}$ for (7) for all $n \in [\![0, N+1]\!]$. Let us consider

$$\mathcal{S}_1 = \sum_{n=1}^{N+1} \sum_{K \in \mathcal{T}} k \mathrm{m}(K) \, |\partial^1 \mathcal{S}_K^n| \quad \text{and} \quad \hat{\mathcal{S}} = \max_{n=0}^{N+1} \left\{ \sum_{K \in \mathcal{T}} \mathrm{m}(K) |\mathcal{S}_K^n| \right\}. \tag{12}$$

Then the following error estimates hold:

- Discrete $W^{1,\infty}(0,T; L^1(\Omega))$–estimate: for all $n \in [\![1, N+1]\!]$

$$\|\partial^1 \eta_\mathcal{T}^n\|_{L^1(\Omega)} \leq C_1 \left(\|\partial^1 \eta_\mathcal{T}^1\|_{L^1(\Omega)} + \mathcal{S}_1 \right). \tag{13}$$

- $L^\infty(0,T; W_0^{1,q}(\Omega))$–estimate: For all $1 \leq q < \frac{d}{d-1}$, for all $n \in [\![0, N+1]\!]$

$$\| \eta_\mathcal{T}^n \|_{1,q,\mathcal{T}} \leq C_3 \left(\hat{\mathcal{S}} + \mathcal{S}_1 + \|\partial^1 \eta_\mathcal{T}^1\|_{L^1(\Omega)} + \| \eta_\mathcal{T}^0 \|_{1,q,\mathcal{T}} \right). \tag{14}$$

Comments on the new estimates (13) and (14)

Estimates (13)–(14) contain estimate for the spatial and temporal derivatives of the discrete solution of scheme (7). This will help us to get a convergence order in discrete norms involving derivatives of the error.

Perspectives

Use of these *a priori estimates* to obtain a convergence order for a full finite volume discretization for the Joule Heating System (1)–(2).

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Contact: abdallah.bradji@univ-annaba.dz, bradji@cmi.univ-mrs.fr