Note on a new high order piecewise linear finite element approximation for the wave equation in one dimensional space

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Problem to be solved

We consider the following one dimensional wave problem:

$$u_{tt}(x,t) - u_{xx}(x,t) = f(x,t),\ (x,t) \in \mathbf{I} \times (0,T), \tag{1}$$

where $\mathbf{I} = (0,1)$, $T > 0$, and f is a given function.
Initial conditions are defined by, for given functions u^0 and u^1:

$$u(x,0) = u^0(x) \ \text{ and } \ u_t(x,0) = u^1(x),\ x \in \mathbf{I}, \tag{2}$$

Homogeneous Dirichlet boundary conditions are given by

$$u(0,t) = u(1,t) = 0,\ t \in (0,T). \tag{3}$$

Definition of the meshes

- **Time discretization**: The time discretization is performed with a constant time step $k = \frac{T}{M+1}$, where $M \in \mathbb{N} \setminus \{0\}$, and we shall denote $t_n = nk$, for $n \in \llbracket 0, M+1 \rrbracket$.
- **Space discretization**: The space discretization is performed using a *non-uniform* mesh with the points $0 < x_0 < ... < x_{N+1} = 1$. We define $h_i = x_{i+1} - x_i$, for all $i \in \llbracket 0, N \rrbracket$. We then set

$$h = \max_{i=0}^{N} h_i. \tag{4}$$

Let $\mathcal{V}^h$ be the piecewise linear finite element space, i.e.

$$\mathcal{V}^h = \{v \in \mathcal{C}(\mathrm{I}), \mathrm{v}|_{\mathrm{I_i}} \in \mathcal{P}_1,\ \forall \mathrm{i} \in \llbracket 0, \mathrm{N} \rrbracket\}, \tag{5}$$

and we denote by

$$\mathcal{V}_0^h = \{v \in \mathcal{C}(\mathrm{I}), \mathrm{v}|_{\mathrm{I_i}} \in \mathcal{P}_1,\ \forall \mathrm{i} \in \llbracket 0, \mathrm{N} \rrbracket\} \cap H_0^1(\mathrm{I}), \tag{6}$$

where $\mathrm{I_i}$ is the subinterval $[x_i, x_{i+1}]$ and $\mathcal{P}_1$ denotes the space of affine polynomials.

Basic scheme

The basic scheme is based on a Newmark's method as discretization in time.
Find $(u_h^n)_{n=0}^{M+1} \in (\mathcal{V}_0^h)^{M+2}$ such that

- **Approximation of initial conditions (2).**

$$\mathbf{a}(u_h^0, v) = -(\Delta u^0, v)_{\mathbb{L}^2(\Omega)} = \mathbf{a}(u^0, v),\ \forall v \in \mathcal{V}_0^h, \tag{7}$$

and

$$\mathbf{a}(\partial^1 u_h^1, v) = -(\Delta \bar{u}^1,\ v)_{\mathbb{L}^2(\Omega)} = \mathbf{a}(\bar{u}^1, v),\ \forall v \in \mathcal{V}_0^h, \tag{8}$$

- **Approximation of the wave equation (1)**: For any $n \in \llbracket 1, M \rrbracket$, find $u_h^{n+1} \in \mathcal{V}_0^h$ such that, for all $v \in \mathcal{V}_0^h$

$$(\partial^2 u_h^{n+1}, v)_{\mathbb{L}^2(\Omega)} + \frac{1}{2}\mathbf{a}(u_h^{n+1} + u_h^{n-1}, v) = \frac{1}{2}(f(t_{n+1}) + f(t_{n-1}), v)_{\mathbb{L}^2(\Omega)}, \tag{9}$$

where

$$\bar{u}^1 = u^1 + \frac{k}{2}(\Delta u^0 + f(0)), \tag{10}$$

and $(\cdot,\cdot)_{\mathbb{L}^2(\Omega)}$ denotes the $\mathbb{L}^2$ inner product and

$$\mathbf{a}(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) dx.$$

$$\partial^1 v^n = \frac{v^n - v^{n-1}}{k},\ \forall n \in \llbracket 1, M+1 \rrbracket, \tag{11}$$

and

$$\partial^2 v^n = \frac{v^n - 2v^{n-1} + v^{n-2}}{k^2},\ \forall n \in \llbracket 2, M+1 \rrbracket. \tag{12}$$

Convergence order of basic scheme (7)–(10): First main result

- Discrete $\mathbb{L}^\infty(0,T; H_0^1(\mathbf{I}))$–estimate: for all $n \in \llbracket 0, M+1 \rrbracket$

$$|u_h^n - \pi u^n|_{1,\mathrm{I}} \le C(h+k)^2. \tag{13}$$

- Discrete $\mathbb{L}^\infty(0,T; \mathbb{L}^\infty(\mathbf{I}))$–estimate: for all $n \in \llbracket 0, M+1 \rrbracket$

$$\| u_h^n - \pi u^n \|_{\mathcal{C}(\mathbf{I})} \le C(h+k)^2. \tag{14}$$

- Discrete $\mathcal{W}^{1,\infty}(0,T; \mathbb{L}^2(\mathbf{I}))$–estimate: for all $n \in \llbracket 1, M+1 \rrbracket$

$$\| \partial^1 (u_h^n - \pi u(t_n)) \|_{\mathbb{L}^2(\mathbf{I})} \le C(h+k)^2. \tag{15}$$

The operator π is the linear interpolant operator.

An auxiliary helpful element in $\mathcal{V}_0^h$

For each $n \in \llbracket 0, M+1 \rrbracket$, let $s_h^n \in \mathcal{V}_0^h$ be the solution of the following scheme

$$s_h^0 = 0, \tag{16}$$

$$\mathbf{a}(\partial^1 s_h^1, v) = \frac{k^2}{6}(\Delta(f_t(0) + \Delta u^1), v)_{\mathbb{L}^2(\Omega)},\ \forall v \in \mathcal{V}_0^h, \tag{17}$$

and for all $n \in \llbracket 1, M \rrbracket$

$$(\partial^2 \bar{s}_h^{n+1}, v)_{\mathbb{L}^2(\mathrm{I})} + \frac{1}{2}\mathbf{a}(s_h^{n+1} + s_h^{n-1},\ v) = \sum_{i=0}^{N} \int_{\mathrm{I_i}} \alpha_i(x) w_h^n(x_i) v(x) dx + \frac{5k^2}{12}(z_h^n\ v)_{\mathbb{L}^2(\mathrm{I})}. \tag{18}$$

where

- w_i^n is an approximation for u_{ttxx}
- z_i^n is an approximation for u_{tttt}

How to compute w_i^n and z_i^n involved in (18)?: Second main resul

- **w_i^n: approximation for u_{ttxx}**. Differentiating (1) twice with respect to x and twice with respect to t imply that $w = u_{ttxx}$ is satisfying

$$w_{tt}(x,t) - w_{xx}(x,t) = F(x,t),\ (x,t) \in \mathbf{I} \times (0,T), \tag{19}$$

where

$$F = f_{ttxx}. \tag{20}$$

Also, $w = u_{ttxx}$ satisfies

- First initial condition:

$$w(x,0) = G^0(x),\ x \in \mathbf{I}, \tag{21}$$

where

$$G^0(x) = (u^0)_{xxxx}(x) + f_{xx}(x,0),\ x \in \mathbf{I} \tag{22}$$

- Second initial condition:

$$w_t(x,0) = G^1(x),\ x \in \mathbf{I}, \tag{23}$$

with

$$G^1(x) = (u^1)_{xxxx}(x) + f_{xxt}(x,0),\ x \in \mathbf{I}. \tag{24}$$

- Boundary conditions:

$$w(0,t) = w(1,t) = 0,\ t \in (0,T). \tag{25}$$

The problem (19)–(25) is similar to the problem (1)–(3) satisfied by u. Hence $w = u_{ttxx}$ can be approximated using the same scheme (7)–(10).

- **z_i^n: approximation for u_{tttt}**. Thanks to equation (1), we have (recall that $w = u_{ttxx}$)

$$u_{tttt}(x,t) = w(x,t) + f_{tt}(x,t),\ (x,t) \in \mathbf{I} \times (0,T). \tag{26}$$

As an approximation z^n for the unknown function $z = u_{tttt}(t_n)$, we suggest

$$z_h^n = w_h^n + f_{tt}(t_n). \tag{27}$$

Definition of a new third order approximation: Third main resul

We define the new approximation $u^{n,1} = (u_i^{n,1})_{i \in \llbracket 1, N \rrbracket}$: For each $n \in \llbracket 0, M+1 \rrbracket$, let $u_h^{n,1} \in \mathcal{V}_0^h$ be the new approximation given by

$$u_h^{n,1} = u_h^n - s_h^n. \tag{28}$$

Statement of Convergence results

- Discrete $\mathbb{L}^\infty(0,T; H_0^1(\mathbf{I}))$–estimate: for all $n \in \llbracket 0, M+1 \rrbracket$

$$|u_h^{n,1} - \pi u^n|_{1,\mathrm{I}} \le C(h+k)^3. \tag{29}$$

- Discrete $\mathbb{L}^\infty(0,T; \mathbb{L}^\infty(\mathbf{I}))$–estimate: for all $n \in \llbracket 0, M+1 \rrbracket$

$$\|u_h^{n,1} - \pi u^n\|_{\mathcal{C}(\mathbf{I})} \le C(h+k)^3. \tag{30}$$

- Discrete $\mathcal{W}^{1,\infty}(0,T; \mathbb{L}^2(\mathbf{I}))$–estimate: for all$n \in \llbracket 1, M+1 \rrbracket$

$$\|\partial^1 \left(u_h^{n,1} - \pi u(t_n)\right)\|_{\mathbb{L}^2(\mathbf{I})} \le C(h+k)^3. \tag{31}$$

Idea on the Proof of Statement of (29)–(31)

- Convergence of w^n of u_{ttxx}
- An a priori estimate for the discrete problem

References

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