Note on the convergence of a finite volume scheme for a second order hyperbolic equation with a time delay in any space dimension

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Problem to be solved.

Second Order Hyperbolic Equation with a constant Time delay (see [6,7]):

$$u_{tt}(\mathbf{x},t) - \Delta u(\mathbf{x},t) + \alpha u_t(\mathbf{x},t) + \beta u_t(\mathbf{x},t-\tau) = f(\mathbf{x},t), \quad (\mathbf{x},t) \in \Omega \times (0,T), \quad (1)$$

where

- Ω is an open polyhedral bounded subset in $\mathbb{R}^d$,
- $T > 0$,
- **τ is the constant time delay**,
- α, β are given positive numbers,
- f is a given smooth function,
- Initial conditions are given by

$$u(\mathbf{x},0) = u^0(\mathbf{x}) \quad \text{and} \quad u_t(\mathbf{x},0) = u^1(\mathbf{x}), \qquad \mathbf{x} \in \Omega. \quad (2)$$

- Homogeneous Dirichlet boundary conditions are given by

$$u(\mathbf{x},t) = 0, \qquad (\mathbf{x},t) \in \partial\Omega \times (0,T). \quad (3)$$

Applications...

Delay differential equations occur in several applications such as ecology, biology, medicine, See References [1,5,6,7,8].

Finite volume mesh

The finite volume mesh considered is the one used in [3]. Among the properties of this mesh, we quote

- This new generic mesh is a generalization of the one introduced in [4].
- The control volumes are not necessary convex.
- No orthogonality is required.
- The discrete unknowns are located at the centers of the control volumes and at their interfaces.

Discrete Gradient and an Interpolation Operator.

- **A discrete space.** $X_{\mathcal{D}}$ as the set of all $\left((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}\right)$, where $v_K, v_\sigma \in \mathbb{R}$ for all $K \in \mathcal{M}$ and for all $\sigma \in \mathcal{E}$ such that $u_\sigma = 0$.
- **A discrete Gradient.** For $u = \left((u_K)_{K\in\mathcal{M}}, (u_\sigma)_{\sigma\in\mathcal{E}}\right) \in X_{\mathcal{D}}$, we define, for all $K \in \mathcal{M}$

$$\nabla_{\mathcal{D}} u(x) = \nabla_{K,\sigma} u, \text{ a. e. } x \in \mathcal{D}_{K,\sigma}, \quad (4)$$

where $\mathcal{D}_{K,\sigma}$ is the cone with vertex x_K and basis σ and

$$\nabla_{K,\sigma} u = \nabla_K u + \left(\frac{\sqrt{d}}{d_{K,\sigma}}\left(u_\sigma - u_K - \nabla_K u \cdot (x_\sigma - x_K)\right)\right)\mathbf{n}_{K,\sigma}, \quad (5)$$

where $\nabla_K u = \frac{1}{\mathrm{m}(K)} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\,(u_\sigma - u_K)\,\mathbf{n}_{K,\sigma}$.

- **An Interpolation operator.** For all $v \in X_{\mathcal{D}}$, we denote by $\Pi_{\mathcal{M}} v \in H_{\mathcal{M}}(\Omega)$ the function defined by $\Pi_{\mathcal{M}} v(\mathbf{x}) = v_K$, for a.e. $\mathbf{x} \in K$, for all $K \in \mathcal{M}$.
- **Definition of a bilinear form over $X_{\mathcal{D}} \times X_{\mathcal{D}}$:**

$$\langle u, v\rangle_F = \int_\Omega \nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x}.$$

Time discretization and discrete temporal derivative

- The time discretization is performed with a constrained time step-size k such that $\frac{\tau}{k} \in \mathbb{N}$.
- We set then $k = \frac{\tau}{M}$, where $M \in \mathbb{N} \setminus \{0\}$.
- Denote by N the integer part of $\frac{T}{k}$, i.e. $N = \left[\frac{T}{k}\right]$.
- We shall denote $t_n = nk$, for $n \in [\![-M, N]\!]$. As particular cases $t_{-M} = -\tau$, $t_0 = 0$, and $t_N \le T$.
- One of the advantages of this time discretization is that the point $t = 0$ is a mesh point which is suitable since we have equation (1) defined for $t \in (0,T)$ and the second initial condition in (2) is defined for $t \in (-\tau, 0)$.
- We denote by ∂^1 the discrete first time derivative given by

$$\partial^1 v^{j+1} = \frac{v^{j+1} - v^j}{k}.$$

- We define the discrete second time derivative ∂^2 as

$$\partial^2 v^{j+1} = \partial^1(\partial^1 v^{j+1}).$$

Formulation of a Finite Volume scheme

We now set a formulation of an implicit finite volume scheme for problem (1)–(3). The unknowns of this scheme are the set $\{u_{\mathcal{D}}^n; n \in [\![-M, N]\!]\}$ which are expected to approximate the set of the unknowns

$$\{u(t_n); n \in [\![-M, N]\!]\}.$$

- **Approximation of initial conditions (2).** The discretization of initial conditions (2) can be performed as:
  - Find $u_{\mathcal{D}}^0 \in X_{\mathcal{D},0}$ such that for all $v \in X_{\mathcal{D},0}$

$$\langle u_{\mathcal{D}}^0, v\rangle_F = -\left(\Delta u^0, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}. \quad (6)$$

  - Find $u_{\mathcal{D}}^n$ for $n \in [\![-M, -1]\!]$ such that for all $v \in X_{\mathcal{D},0}$

$$\langle \partial^1 u_{\mathcal{D}}^n, v\rangle_F = -\left(\Delta u^1(t_n), \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}. \quad (7)$$

- **Approximation of (1) and (3).** For any $n \in [\![0, N-1]\!]$, find $u_{\mathcal{D}}^{n+1} \in X_{\mathcal{D},0}$ such that, for all $v \in X_{\mathcal{D},0}$

$$\left(\partial^2 \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} + \langle u_{\mathcal{D}}^{n+1}, v\rangle_F + \alpha\left(\partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} + \beta\left(\partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1-M}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} = \left(f(t_{n+1}), \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}. \quad (8)$$

New Error estimate for the Scheme (6)–(8)

- $\mathbb{L}^\infty(L^2)$ and $\mathbb{L}^\infty(H_0^1)$ error estimates.

$$\max_{n=0}^{n=N} \|u(t_n) - \Pi_{\mathcal{M}} u_{\mathcal{D}}^n\|_{\mathbb{L}^2(\Omega)} + \max_{n=0}^{n=N} \|\nabla u(t_n) - \nabla_{\mathcal{D}} u_{\mathcal{D}}^n\|_{\mathbb{L}^2(\Omega)} \le C(k + h_{\mathcal{D}}). \quad (9)$$

- $W^{1,\infty}(\mathbb{L}^2)$–estimate.

$$\max_{n=-M+1}^{n=N} \left\|u_t(t_n) - \Pi_{\mathcal{M}} \partial^1 u_{\mathcal{D}}^n\right\|_{\mathbb{L}^2(\Omega)} \le C(k + h_{\mathcal{D}}). \quad (10)$$

Main idea on the proof

- A comparison with an optimal scheme.
- Well-developed discrete a priori estimates.

In Progress

- Extension to GDM.
- Extension to Second order time accurate schemes.
- Consider Time Fractional Delay Equations

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Presented at FVCA9, Bergen–Norway, June 15–19, 2020 (held ONLINE)

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