An $L^\infty(H^1)$– error estimate for GSs applied to time fractional diffusion equations

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Problem to be solved

Time Fractional Diffusion Equation:

$$\partial_t^\alpha u(\boldsymbol{x},t) - \nabla\cdot(\kappa\nabla u)(\boldsymbol{x},t) = f(\boldsymbol{x},t), \qquad (\boldsymbol{x},t)\in\Omega\times(0,T), \tag{1}$$

where

- Ω is an open polyhedral bounded subset in $\mathbb{R}^d$,
- $T>0$, $0<\alpha<1$ are given.
- κ satisfies $\kappa\in\mathcal{C}^1(\overline{\Omega})$ and $\kappa(\boldsymbol{x})>\kappa_0>0$.
- f is a given smooth function.
- The operator ∂_t^α is the Caputo derivative defined by:

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u_t(s)ds. \tag{2}$$

- Initial condition is given by

$$u(\boldsymbol{x},0)=0, \qquad \boldsymbol{x}\in\Omega. \tag{3}$$

- Homogeneous Dirichlet boundary conditions are given by

$$u(\boldsymbol{x},t)=0, \qquad (\boldsymbol{x},t)\in\partial\Omega\times(0,T). \tag{4}$$

Application...

Fractional differential equations have been successfully used in theory and they appear in many areas of application, see [1].

Discretization in space: GDM framework introduced in [8]

Let Ω be an open domain of $\mathbb{R}^d$, with $d\in\mathbb{N}^\star$. An approximate gradient discretization is given by $\mathcal{D}=(X_{\mathcal{D},0},\Pi_\mathcal{D},\nabla_\mathcal{D})$, where

1. The set of discrete unknowns $X_{\mathcal{D},0}$ is a finite dimensional vector space on $\mathbb{R}$.

2. The linear mapping $\Pi_\mathcal{D}: X_{\mathcal{D},0}\to L^2(\Omega)$ is the reconstruction of the approximate function.

3. The gradient reconstruction $\nabla_\mathcal{D}: X_{\mathcal{D},0}\to L^2(\Omega)^d$ is a linear mapping which reconstructs, from an element of $X_{\mathcal{D},0}$, a "gradient" (vector-valued function) over Ω. The gradient reconstruction must be chosen such that $\|\nabla_\mathcal{D}\cdot\|_{L^2(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$. Let us define the bi-linear form $\langle\cdot,\cdot\rangle_{\mathcal{D},\kappa}$ given by

$$\langle u,v\rangle_{\mathcal{D},\kappa} = \int_\Omega \kappa(\boldsymbol{x})\nabla_\mathcal{D}u(\boldsymbol{x})\cdot\nabla_\mathcal{D}v(\boldsymbol{x})d\boldsymbol{x}, \quad \forall(u,v)\in X_{\mathcal{D},0}\times X_{\mathcal{D},0}. \tag{5}$$

In order to analyse the convergence of the gradient schemes, we consider the following parameters related to the mesh $\mathcal{D}$:

1. The coercivity of the discretization is measured via the constant $C_\mathcal{D}$ given by

$$C_\mathcal{D} = \max_{v\in X_{\mathcal{D},0}\setminus\{0\}}\frac{\|\Pi_\mathcal{D}v\|_{L^2(\Omega)}}{\|\nabla_\mathcal{D}v\|_{L^2(\Omega)^d}}.$$

This yields the Poincaré inequality: $\|\Pi_\mathcal{D}v\|_{L^2(\Omega)}\le C_\mathcal{D}\|\nabla_\mathcal{D}v\|_{L^2(\Omega)^d}$.

2. The consistency is measured through the interpolation error function $S_\mathcal{D}: H_0^1(\Omega)\to[0,+\infty)$ defined by, for all $\varphi\in H_0^1(\Omega)$

$$S_\mathcal{D}(\varphi) = \min_{v\in X_{\mathcal{D},0}}\left(\|\Pi_\mathcal{D}v-\varphi\|^2_{L^2(\Omega)} + \|\nabla_\mathcal{D}v-\nabla\varphi\|^2_{L^2(\Omega)^d}\right)^{\frac{1}{2}}.$$

3. The limit-conformity is measured through the conformity error function $W_\mathcal{D}: H_{\mathrm{div}}(\Omega)\to[0,+\infty)$ defined by, for all $\varphi\in H_{\mathrm{div}}(\Omega)$

$$W_\mathcal{D}(\varphi) = \max_{u\in X_{\mathcal{D},0}\setminus\{0\}}\frac{1}{\|\nabla_\mathcal{D}u\|_{L^2(\Omega)^d}}\left|\int_\Omega\left(\nabla_\mathcal{D}u(\boldsymbol{x})\cdot\varphi(\boldsymbol{x}) + \Pi_\mathcal{D}u(\boldsymbol{x})\mathrm{div}\varphi(\boldsymbol{x})\right)d\boldsymbol{x}\right|.$$

Additional assumption on $\|\Pi_\mathcal{D}\cdot\|_{L^2(\Omega)}$

We assume, in addition, that the generic mesh $\mathcal{D}=(X_{\mathcal{D},0},\Pi_\mathcal{D},\nabla_\mathcal{D})$ is chosen such that **$\|\Pi_\mathcal{D}\cdot\|_{L^2(\Omega)}$ is a norm on $X_{\mathcal{D},0}$**. This includes, for instance, Conforming FEMs, Cell-Centered SUSHI, and MFEMs.

Time discretization and discrete temporal derivative

The discretization of $[0,T]$ is performed with a constant time step

$$k=\frac{T}{N+1},$$

where $N\in\mathbb{N}^\star$. Mesh points in time: $t_n=nk, \forall n\in[\![0,N+1]\!]$. The discrete temporal derivative given by $\partial^1 v^n = \dfrac{v^n-v^{n-1}}{k}$.

Approximation of the Caputo derivative, see [1]

$$\partial_t^\alpha u(t_{n+\sigma}) = \sum_{j=0}^n k\lambda_j^{n+1}\partial^1 u(t_{j+1}) + \mathbb{T}_1^{n+1}(u), \tag{6}$$

where $\sigma=1-\alpha/2$, λ_j^{n+1} are developed in [1], and

$$|\mathbb{T}_1^{n+1}(u)|\le Ck^{3-\alpha}. \tag{7}$$

Properties of the approximation of the Caputo derivative

- Decreasing property:

$$\lambda_{j+1}^{n+1}>\lambda_j^{n+1}>\lambda_0=\frac{1}{2T^\alpha\Gamma(1-\alpha)} \quad\text{and}\quad \sum_{j=0}^n k\lambda_j^{n+1}\le\frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \tag{8}$$

- Stability: For all $\left(\beta^j\right)_{j=0}^{N+1}\in\mathbb{R}^{N+2}$ and for all $n\in[\![0,N]\!]$:

$$\left(\sigma\beta^{n+1}+(1-\sigma)\beta^n\right)\sum_{j=0}^n\lambda_j^{n+1}(\beta^{j+1}-\beta^j)\ge\frac{1}{2}\sum_{j=0}^n\lambda_j^{n+1}\left((\beta^{j+1})^2-(\beta^j)^2\right). \tag{9}$$

Formulation of a Gradient Scheme

$$\sum_{j=0}^n k\lambda_j^{n+1}\left(\partial^1 u_\mathcal{D}^{j+1},v\right)_{L^2(\Omega)} + \left(\nabla_\mathcal{D}u_\mathcal{D}^{n+\sigma},\nabla_\mathcal{D}v\right)_{L^2(\Omega)} = \left(f(t_{n+\sigma}),v\right)_{L^2(\Omega)}, \tag{10}$$

where $u_\mathcal{D}^0$ is defined using a discrete projection.

New $\mathbb{L}^\infty(H_0^1)$–Error estimate for the Scheme (10)

- Error estimate in the discrete norm of $\mathbb{L}^\infty(H_0^1)$

$$\max_{n=0}^{n=N+1}\|\nabla u(t_n)-\nabla_\mathcal{D}u_\mathcal{D}^n\|_{\mathbb{L}^2(\Omega)}\le C(k^2+\mathbb{E}_\mathcal{D}^k(u)), \tag{11}$$

where $\mathbb{E}_\mathcal{D}^k(u)=\max_{j\in\{0,1\}}\max_{n\in[\![j,N+1]\!]}\mathbb{E}_\mathcal{D}(\partial^j u(t_n))$ and

$$\mathbb{E}_\mathcal{D}(\overline{u})=\max\left(C_\mathcal{D}W_\mathcal{D}(\kappa\nabla\overline{u})+(C_\mathcal{D}+1)S_\mathcal{D}(\overline{u}), W_\mathcal{D}(\kappa\nabla\overline{u})+2S_\mathcal{D}(\overline{u})\right). \tag{12}$$

- This results improves the one of [5] which dealt with only $\mathbb{L}^\infty(L^2)$–Error estimate.
- The convergence order in time of the GS (10) is better than the one of [3] which dealt with a SUSHI scheme of only order $k^{2-\alpha}$ in time .

Main ideas on the proof

- We introduce an approximation for the operator $-\nabla(\kappa\nabla\cdot)$ similar to the discrete Laplace operator introduced in [8].
- A well-developed discrete a priori estimate..

In Progress

Extension to GDM without the stated assumption on $\|\Pi_\mathcal{D}\cdot\|_{L^2(\Omega)}$. This work is the subject of [2].

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