Note on the convergence of a finite volume scheme using a general nonconforming mesh for an oblique derivative boundary value problem

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Model of oblique derivative boundary value problem

Consider the following problem with oblique boundary condition

\[ -\Delta u(x) = f(x), \quad x \in \Omega, \]  

(1)

where \( \Omega \) is an open bounded polygonal connected subset of \( \mathbb{R}^2 \) and \( \omega_1(x) + \omega_2(x) = 0, \quad x \in \partial\Omega. \)

(2)

where \( x = (x,y) \) is the current point of \( \mathbb{R}^2 \), \( \omega_1 = \nabla u \cdot n \) and \( \omega_2 = \nabla u \cdot t, \) with \( n = (n_1, n_2)^T \) (resp. \( t = (t_1, t_2)^T \)) is the normal vector to the boundary \( \partial\Omega \) and outward to \( \Omega \) (resp. is a tangential derivative), and \( \omega_1 \) is a given constant. We will assume that \( \omega_2 > 0. \)

We add the following condition to get the uniqueness for (1)-(2)

\[ \int_{\partial\Omega} u(x)dx = 0. \]

(3)

Why oblique derivative boundary value problem ?

- Unusual boundary condition
- It appears in the modeling of some mechanical problems, but perhaps not directly under the form (1)-(2), see [4] and references therein.
- Extension of the previous work [1].

Finite volume mesh

The finite volume mesh considered is the one used in [3]. Among the properties of this mesh, we quote:

- This new generic mesh is a generalization of the one introduced in [2].
- The control volumes are not necessary convexes.
- No orthogonality is required.
- The discrete unknowns are located at the centers of the control volumes and at their interfaces.

\[ \begin{array}{c}
\text{Figure: Two adjacent control volumes in 2D} \\
\end{array} \]

Discrete Gradient

We define the space \( \mathcal{V}_h \) as the set of all \( \left\{ (v_h)_K \in C^0(X_K) \right\}_{K \in \mathcal{M}} \), where \( v_h \in \mathbb{R}^K \) for all \( K \in \mathcal{M} \) and for all \( e \in \mathcal{E}. \)

For \( \mathbb{V} \left( \omega_1, (\omega_2, t) \right) \in \Omega \), we define, for all \( K \in \mathcal{M} \)

\[ \mathcal{V}_h(k) = \mathcal{V}_h(u, a, e) \in \mathcal{D}_h, \]

(4)

\[ \mathcal{D}_h = \{ \phi \in C^0(X_h) \} \]

where \( X_h = \text{vertex} \ x_h \text{ and basis} \ a \)

\[ \mathcal{V}_h = \{ v_h \in \mathbb{R}^K \} \]

\[ \mathcal{V}_h(k) = \mathcal{V}_h(u, a, e) \in \mathcal{D}_h, \]

(5)

\[ \text{where} \ v_h = \int_{X_h} p(\sigma) \left\{ \omega_1 - \omega_2 \right\} \text{d}x, \text{and} \ d \text{ is the space dimension}. \]

Useful notations

The following definition will help us to define the finite volume scheme we shall present and to prove its convergence:

Let \( a \in \mathbb{R}^2 \) and \( n \) be the normal vector to \( \partial\Omega, \) outward to \( \Omega. \) Recall that \( t = (-n_1, n_2)^T \) where \( n = (n_1, n_2)^T, \) then

\[ a = (a_1, a_2), \quad a_1 = (1 - s) a, \quad a_2 = s(0, 1)^T \] where \( a, b \) are chosen such that

\[ b - at \approx b - a. \]

We denote by \( a \) (resp. \( a^r \)) the element of \( \mathbb{R}^2 \) such that \( a \) is in the closure of \( e \) (resp. \( b \) is in the closure of \( a^r \)) and \( a^r \neq a. \)

We also set \( a_h = b \) and \( a^r_h = a \) (so that \( |a_h| = |a^r_h| = |a - b| \)).

Finite volume scheme for (1)-(3)

We define the finite volume approximation for (1)-(3) as

\[ u_0 \in \{ (u)_K \}_{K \in \mathcal{M}} \in \mathbb{K}_h \text{ such that} \]

\[ \left\{ \begin{array}{l}
\int_{X_h} (\omega_1 + \omega_2) u_0 d x = 0, \\
\int_{X_h} \sum_{K \in \mathcal{M}} m(K) u_0 d x = 0,
\end{array} \right. \]

(6)

and

\[ \int_{X_h} \sum_{K \in \mathcal{M}} m(K) u_0 d x = 0. \]

(7)

Convergence order of volume scheme (6)-(7)

The following convergence result is proved:

\[ \| \nabla D u - \nabla D u_h \|_{L^2(\Omega)} \leq C \sqrt{h}, \]

(8)

where \( h \) is the mesh size.

A comparison with our previous work [1]

The convergence result (8) provides an error estimate for the approximation of the gradient, whereas the convergence result of [1] is only provided in a discrete \( H^1 \)-norm.

Some results of error estimate (8)

- The convergence result (8) yields a discrete \( H^1 \) estimate (thanks to a result in [3]).
- Thanks to the techniques of [2], the stated discrete \( H^1 \) estimate in the previous item yields an \( L^2 \) estimate.

Stability and Consistency results from [3] for the Discrete Gradient

- Stability result:

\[ C_0 \| v_h \|_{L^2(\Omega)} \leq \left\| \nabla D u \right\|_{L^2(\Omega)} \leq C_0 \| v_h \|_{L^2(\Omega)} \forall v_h \in \mathcal{V}_h. \]

(9)

\[ |v_h| = \sum_{K \in \mathcal{M}} \frac{m(K)}{2} \left( \right) (v_h)^2 \]

\[ \| \nabla D u - \nabla D u_h \|_{L^2(\Omega)} \leq C \sqrt{h} \| u \|_{L^2(\Omega)} \]

(10)

Idea on the Proof of error estimate (8)

Using the techniques of [1] combined with [3] leads to:

\[ \sum_{K \in \mathcal{M}} m(K) \left( v_h - v_h \right) \]

(11)

\[ \| \nabla D u_h \|_{L^2(\Omega)} \]

(12)

and

\[ \| u_h \|_{L^2(\Omega)} \]

(13)

Taking \( v = v_h \) in (11), using the stability result (9) and consistency result (10) yields the desired estimate (8).

In Progress

We consider the same model of [4]: non-stationary Heat equation with non-linear oblique boundary condition.

References


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