Note on the convergence of a finite volume scheme using a general nonconforming mesh for an oblique derivative boundary value problem

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Model of oblique derivative boundary value problem

Consider the following problem with oblique boundary condition

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}), \mathbf{x} \in \Omega, \quad (1)$$

where Ω is an open bounded polygonal connected subset of $\mathbb{R}^2$ and

$$u_n(\mathbf{x}) + \alpha u_t(\mathbf{x}) = 0, \ \mathbf{x} \in \partial\Omega, \quad (2)$$

where $\mathbf{x} = (x, y)$ is the current point of $\mathbb{R}^2$, $u_n = \nabla u \cdot \mathbf{n}$ and $u_t = \nabla u \cdot \mathbf{t}$, with $\mathbf{n} = (\mathbf{n}_x, \mathbf{n}_y)^t$ (resp. $\mathbf{t} = (-\mathbf{n}_y, \mathbf{n}_x)^t$) is the normal vector to the boundary $\partial\Omega$ and outward to Ω (resp. is a tangential derivative), and α is a given constant. We will assume that $\alpha > 0$.
We add the following condition to get the uniqueness for (1)–(2)

$$\int_\Omega u(\mathbf{x})d\mathbf{x} = 0. \quad (3)$$

Why oblique derivative boundary value problem ?

- Unusual boundary condition
- It appears in the modeling of some mechanical problems, but perhaps not directly under the form (1)–(2), see [4] and references therein.
- Extension of the previous work [1].

Finite volume mesh

The finite volume mesh considered is the one used in [3]. Among the properties of this mesh, we quote

- This new generic mesh is a generalization of the one introduced in [2].
- The control volumes are not necessary convexes.
- No orthogonality is required.
- The discrete unknowns are located at the centers of the control volumes and at their interfaces.

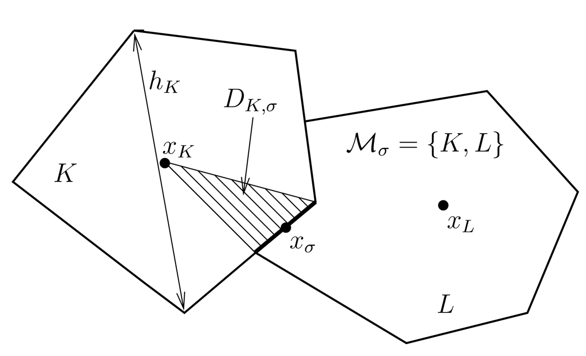

Figure: Two adjacent control volumes in 2D

Discrete Gradient

We define the space $X_\mathcal{D}$ as the set of all $\left((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}\right)$, where $v_K, v_\sigma \in \mathbb{R}$ for all $K \in \mathcal{M}$ and for all $\sigma \in \mathcal{E}$.
For $u = \left((u_K)_{K\in\mathcal{M}}, (u_\sigma)_{\sigma\in\mathcal{E}}\right) \in X_\mathcal{D}$, we define, for all $K \in \mathcal{M}$

$$\nabla_\mathcal{D} u(x) = \nabla_{K,\sigma} u, \ \text{a. e. } x \in \mathcal{D}_{K,\sigma}, \quad (4)$$

where $\mathcal{D}_{K,\sigma}$ is the cone with vertex x_K and basis σ and

$$\nabla_{K,\sigma} u = \nabla_K u + \left(\frac{\sqrt{d}}{d_{K,\sigma}}\left(u_\sigma - u_K - \nabla_K u \cdot (x_\sigma - x_K)\right)\right)\mathbf{n}_{K,\sigma}, \quad (5)$$

where $\nabla_K u = \dfrac{1}{\mathrm{m}(K)} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma)\left(u_\sigma - u_K\right)\mathbf{n}_{K,\sigma}$ and d is the space dimension.

Useful notations

The following definition will help us to define the finite volume scheme we shall present and to prove its convergence:
Let $\sigma \in \mathcal{E}_{\text{ext}}$ and $\mathbf{n}$ be the normal vector to σ, outward to Ω. Recall that $\mathbf{t} = (-\mathbf{n}_y, \mathbf{n}_x)^t$ where $\mathbf{n} = (\mathbf{n}_x, \mathbf{n}_y)^t$, then $\sigma = (a, b) = \{sa + (1-s)b, s \in [0, 1]\}$ where a, b are chosen such that $|b - a|\mathbf{t} = b - a$. We denote by σ^- (resp. σ^+) the element of $\mathcal{E}_{\text{ext}}$ such that a is in the closure of σ^- (resp. b is in the closure of σ^+) and $\sigma^- \neq \sigma$ (resp. $\sigma^+ \neq \sigma$). We also set $\sigma_e = b$ and $\sigma_b = a$ (so that $|\sigma_e - \sigma_b|\mathbf{t} = \sigma_e - \sigma_b$).

Finite volume scheme for (1)–(3)

We define the finite volume approximation for (1)–(3) as $u_\mathcal{D} = \left((u_K)_{K\in\mathcal{M}}, (u_\sigma)_{\sigma\in\mathcal{E}}\right) \in X_\mathcal{D}$ such that

$$\langle u_\mathcal{D}, v\rangle_F + \alpha \sum_{\sigma\in\mathcal{E}_{\text{ext}}} (u_\sigma - u_{\sigma^-})v_\sigma = \left(f, \Pi_\mathcal{M} v\right)_{\mathbb{L}^2(\Omega)}, \quad \forall v \in X_\mathcal{D} \quad (6)$$

and

$$\sum_{K\in\mathcal{M}} \mathrm{m}(K)u_K = 0, \quad (7)$$

where $\langle u, v\rangle_F = \int_\Omega \nabla_\mathcal{D} u(x) \cdot \nabla_\mathcal{D} v(x)dx$.

Convergence order of volume scheme (6)–(7)

The following convergence result is proved:

$$\|\nabla_\mathcal{D} u_\mathcal{D} - \nabla u\|_{\mathbb{L}^2(\Omega)} \leq C_1\sqrt{h_\mathcal{D}}\|u\|_{\mathcal{C}^2(\overline{\Omega})}. \quad (8)$$

where the size $h_\mathcal{D}$ is the mesh size.

A comparison with our previous work [1]

The convergence result (8) provides an error estimate for the the approximation of the gradient, whereas the convergence result of [1] is only provided in a discrete H_0^1-norm.

Some results of error estimate (8)

- The convergence result (8) yields a discrete H_0^1-estimate (thanks to a result in [3]).
- Thanks to the techniques of [2], the stated discrete H_0^1-estimate in the previous item yields an $\mathbb{L}^2$-estimate

Stability and Consistency results from [3] for the Discrete Gradient

- Stability result:

$$C_2|v|_X \leq \|\nabla_\mathcal{D} v\|_{\mathbb{L}^2(\Omega)} \leq C_3|v|_X, \ \forall v \in X_\mathcal{D}, \quad (9)$$

where

$$|v|_X^2 = \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{d_{K,\sigma}}(v_\sigma - v_K)^2.$$

- Consistency result:

$$\|\nabla_\mathcal{D} u - \nabla u\|_{\mathbb{L}^2(\Omega)} \leq C_4 h_\mathcal{D}\|u\|_{\mathcal{C}^2(\overline{\Omega})}. \quad (10)$$

Idea on the Proof of error estimate (8)

Using the techniques of [1] combined with [3] leads to

$$\langle e_\mathcal{D}, v\rangle_F + \alpha\sum_{\sigma\in\mathcal{E}_{\text{ext}}} v_\sigma\left(e_\sigma - e_{\sigma^-}\right) = \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}(v_K - v_\sigma)\mathcal{R}_{K,\sigma}(u) - \alpha\sum_{\sigma\in\mathcal{E}_{\text{ext}}} r_\sigma\left(v_{\sigma^+} - v_\sigma\right), \quad (11)$$

where

$$e_\mathcal{D} = \mathcal{P}_\mathcal{D} u - u_\mathcal{D}, \quad (12)$$

$\mathcal{P}_\mathcal{D} u = \left((u(\mathbf{x}_K))_{K\in\mathcal{M}}, (u(\mathbf{x}_\sigma))_{\sigma\in\mathcal{E}}\right)$,

$$\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}\frac{d_{K,\sigma}}{\mathrm{m}(\sigma)}\left(\mathcal{R}_{K,\sigma}(u)\right)^2\right)^{\frac{1}{2}} \leq C_5 h_\mathcal{D}\|u\|_{\mathcal{C}^2(\overline{\Omega})}, \quad (13)$$

and

$$|r_\sigma| \leq C_6\mathrm{m}(\sigma)\|u\|_{\mathcal{C}^1(\overline{\Omega})}. \quad (14)$$

Taking $v = e_\mathcal{D}$ in (11), using the stability result (9) and consistency result (10) yields the desired estimate (8). ■

In Progress

We consider the same model of [4]: non-stationary Heat equation with non-linear oblique boundary condition.

References

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