A note on a new second order approximation based on a low–order finite volume scheme for the wave equation in one space dimension

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Problem to be solved

We consider the following one dimensional wave problem:

$$u_{tt}(x,t) - u_{xx}(x,t) = f(x,t), \ (x,t) \in \mathbf{I} \times (0,T), \tag{1}$$

where $\mathbf{I} = (0,1)$, $T > 0$, and f is a given function.
Initial conditions are defined by, for given functions u^0 and u^1:

$$u(x,0) = u^0(x) \ \text{and} \ u_t(x,0) = u^1(x), \ x \in \mathbf{I}, \tag{2}$$

Homogeneous Dirichlet boundary conditions are given by

$$u(0,t) = u(1,t) = 0, \ t \in (0,T). \tag{3}$$

Definition of the meshes

- Time discretization: The time discretization is performed with a constant time step $k = \frac{T}{M+1}$, where $M \in \mathbb{N} \setminus \{0\}$, and we shall denote $t_n = nk$, for $n \in [\![0, M+1]\!]$.
- Space discretization: The spatial domain $\mathbf{I}$ is discretized using the admissible one–dimensional mesh of [2] which we recall here for the sake of completeness. An admissible mesh $\mathcal{T}$ of $\mathbf{I} = (0,1)$ is given by a family $\{K_i;\ i \in [\![1,N]\!]\}$, $N \in \mathbb{N}^\star$ of control volumes, such that $K_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ and a family $\{x_i;\ i \in [\![0,N+1]\!]\}$ such that

$$x_0 = x_{\frac{1}{2}} = 0 < x_1 < x_{\frac{3}{2}} < \cdots < x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}} < \cdots < x_N < x_{N+\frac{1}{2}} = x_{N+1} = 1$$

and, for $i \in [\![1,N]\!]$:

$$h_i = \mathrm{m}(K_i) = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \quad h_i^- = x_i - x_{i-\frac{1}{2}} \ \text{and} \ h_i^+ = x_{i+\frac{1}{2}} - x_i$$

We set $h_{i+\frac{1}{2}} = x_{i+1} - x_i$, for all $i \in [\![0,N]\!]$, and $h = \max_{i \in [\![1,N]\!]} h_i$.

Notations and definitions

- Finite volume space: Define $X(\mathcal{T})$ as the set of functions from $\mathbf{I}$ to $\mathbb{R}$ which are constant on each control volume K_i, $i \in [\![1,N]\!]$, of the mesh. We shall sometime identify $X(\mathcal{T})$ with $\mathbb{R}^N$. For each $u \in X(\mathcal{T})$, we define the discrete H_0^1–norm by

$$\| u \|_{1,\mathcal{T}} = \Big(\sum_{i=1}^{N-1} \frac{(u_{i+1}-u_i)^2}{h_{i+\frac{1}{2}}} + \frac{(u_1)^2}{h_{\frac{1}{2}}} + \frac{(u_N)^2}{h_{N+\frac{1}{2}}}\Big)^{\frac{1}{2}},$$

where u_i denotes the value taken by $u \in X(\mathcal{T})$ on the control volume $K_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$.

- Interpolation operator: For all function $\varphi \in \mathcal{C}(\bar{\mathbf{I}})$, we denote by $\Pi_{\mathcal{T}}\varphi \in X(\mathcal{T})$ the function defined by $\Pi_{\mathcal{T}}\varphi(x) = \varphi(x_i)$, for a.e. $x \in K_i$, for all $i \in [\![1,N]\!]$.

- Numerical flux: For $u = (u_i)_{i=1}^N \in X(\mathcal{T})$, we define :

$$\mathbb{F}_{i+\frac{1}{2}}(u) = \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} \ \text{and} \ \mathbb{D}_{i+\frac{1}{2}}(u) = \mathbb{F}_{i+\frac{1}{2}}(u) - \mathbb{F}_{i-\frac{1}{2}}(u). \tag{4}$$

A first order scheme: old scheme of [1]

- **Discretization of equation (1):** For all $n \in [\![1,M]\!]$, find $u^{n+1} = (u_i^n)_{i=1}^N \in X(\mathcal{T})$ such that

$$h_i \partial^2 u_i^{n+1} - \mathbb{D}_{i+\frac{1}{2}}(u^n) = \frac{1}{k}\int_{nk}^{(n+1)k} \int_{K_i} f(x,t)\,dxdt, \quad \forall i \in [\![1,N]\!], \tag{5}$$

with $u_0^n = u_{N+1}^n = 0$, for all $n \in [\![0,M+1]\!]$

- **Discretization of initial conditions (2):** For all $i \in [\![1,N]\!]$

$$-\mathbb{D}_{i+\frac{1}{2}}(u^0) = -\int_{K_i} (u^0)_{xx}(x)dx \ \text{and} \ -\mathbb{D}_{i+\frac{1}{2}}(\partial^1 u^1) = -\int_{K_i} (u^1)_{xx}(x)dx. \tag{6}$$

where ∂^1 denotes the discrete temporal derivative:

$$\partial^1 v^n = \frac{v^n - v^{n-1}}{k},$$

and ∂^2 denotes the discrete second temporal derivative

$$\partial^2 v^n = \partial^1(\partial^1 v^n).$$

It is proved in [1] that scheme (5)–(6) is of first order. This contribution deals with a new second order finite volume approximation for problem (1)–(3). This second order approximation can be computed using the same scheme (5)–(6).

An auxiliary helpful term

Let us consider the expression

$$d_i^n = -h_i \frac{h_i^+ - h_i^-}{2} l_i^n - k\frac{h_i}{2} z_i^n - k\frac{h_i}{2} s_i^n - \frac{h_{i+1}^- - h_i^+}{2}\alpha_{i+1}^{n+1} + \frac{h_i^- - h_{i-1}^+}{2}\alpha_i^{n+1}, \tag{7}$$

where

- α_i^n is an approximation for u_{xx}
- z_i^n is an approximation for u_{xxt}
- l_i^n is an approximation for u_{ttx}
- s_i^n is an approximation for u_{ttt}

How to compute d_i^n given by (7)?

- α_i^n: approximation for u_{xx}. Since u_{xx} is a second time integration of u_{xxtt}, we are able to derive an approximation for u_{xx} using a *discrete second time integration* of u_{xxtt}.
  - Approximation of u_{xxtt}. We have thanks (1) $u_{xxtt} = f_{xx} + u_{xxxx}$. So, an approximation for u_{xxxx} yields an approximation for u_{xxtt}. To derive an approximation for u_{xxxx}, we remark that $\varphi = u_{xxxx}$ is satisfying

$$\varphi_{tt} - \varphi_{xx} = f_{xxxx}, \tag{8}$$

$$\varphi(x,0) = (u^0)_{xxxx}(x) \ \text{and} \ \varphi_t(x,0) = (u^1)_{xxxx}(x), \ x \in \mathbf{I}, \tag{9}$$

and

$$\varphi(0,t) = -f_{xx}(0,t) - f_{tt}(0,t) \ \text{and} \ \varphi(1,t) = -f_{xx}(1,t) - f_{tt}(1,t). \tag{10}$$

The problem (8)–(10) is similar to the problem (1)–(3) satisfied by u. Hence $\varphi = u_{xxxx}$ can be approximated using the same scheme (5)–(6). Let $\varphi^{n+1} = \left(\varphi_i^{n+1}\right)_{i=1}^N \in X(\mathcal{T})$ denote this application.
An approximation $\psi^{n+1} = \left(\psi_i^{n+1}\right)_{i=1}^N \in X(\mathcal{T})$ for u_{xxtt} can be defined as $\psi_i^n = f_{xx}(x_i,t_n) + \varphi_i^n$.

  - A convenient approximation for $v = u_{xx}$. One remarks that the unknown function $v = u_{xx}$ is a second integration in time of $\psi = u_{xxtt}$, one can attempt to look for an approximation for $v = u_{xx}$ using a *second numerical integration in time* for $\psi^n = (\psi_i^n)_{i \in [\![1,N]\!]}$ which is an approximation for $\psi = u_{xxtt}$. Let $(\alpha^n)_{n \in [\![0,M+1]\!]} \in (X(\mathcal{T}))^{M+2}$ be defined as $\partial^2\alpha^n = \psi^n$, for all $n \in [\![2,M+1]\!]$. Some computations lead to, for $l \in [\![2,M+1]\!]$

$$\alpha^l = k^2 \sum_{j=2}^{l}\sum_{n=2}^{j} \psi^n + t_l\partial^1\alpha^1 + \alpha^0, \tag{11}$$

where we choose $\alpha_i^0 = (u^0)_{xx}(x_i)$ and $\alpha_i^1 = k(u^1)_{xx}(x_i) + (u^0)_{xx}(x_i)$, for all $i \in [\![1,N]\!]$.

- z_i^n: approximation for u_{xxt}. The unknown u_{xxt} is the time derivative of u_{xx}. We are able then to derive an approximation for u_{xxt} using the discrete time derivative ∂^1 of the approximation α^n of u_{xx}. $z_i^n = \partial^1\alpha_i^{n+1}$

The approximations l_i^n and s_i^n can be computed the same techniques used to compute the approximation α^n of u_{xx}.

Definition of a new second order approximation

We define the new approximation $u^{n,1} = (u_i^{n,1})_{i \in [\![1,N]\!]}$, for any $n \in [\![0,M+1]\!]$ as

$$h_i\,\partial^2 u_i^{n+1,1} - \mathbb{D}_{i+\frac{1}{2}}\left(u^{n+1,1}\right) = \frac{1}{k}\int_{nk}^{(n+1)k}\int_{K_i} f(x,t)dx\,dt + d_i^n, \ \ \forall\, i \in [\![1,N]\!] \tag{12}$$

with $u_0^{n,1} = u_{N+1}^{n,1} = 0$ for all $n \in [\![0,M+1]\!]$ and, for any $i \in [\![1,N]\!]$

$$-\mathbb{D}_{i+\frac{1}{2}}\left(u^{0,1}\right) = -\int_{K_i}(u^0)_{xx}(x)dx - \frac{1}{2}\delta_1\left((h_{i+1}^- - h_i^+)\alpha_{i+1}^0\right), \tag{13}$$

$$-\mathbb{D}_{i+\frac{1}{2}}\left(u^{1,1}\right) = -\int_{K_i}(u^0 + k\bar{u}^1)_{xx}(x)dx - \frac{1}{2}\delta_1\left((h_{i+1}^- - h_i^+)\alpha_{i+1}^1\right), \tag{14}$$

where $\bar{u}^1 = u^1 + \frac{k}{2}(f(0) + (u^0)_{xx})$ and $\delta_1 v_i = v_i - v_{i-1}$.

Statement of Convergence results

Le us the error by $e_{\mathcal{T}}^{n,1}(x) = u(x_i,t_n) - u_i^{n,1}$, for a.e. $x \in K_i$, for all $K_i \in \mathcal{T}$.

- Discrete $\mathbb{L}^\infty(0,T;H_0^1(\mathbf{I}))$–estimate: for all $n \in [\![0,M+1]\!]$

$$\| e_{\mathcal{T}}^{n,1} \|_{1,\mathcal{T}} \le C(h+k)^2 \| u \|_{\mathcal{C}^4([0,T];\,\mathcal{C}^6(\bar{\mathbf{I}}))}. \tag{15}$$

- $\mathcal{W}^{1,\infty}(0,T;\mathbb{L}^2(\mathbf{I}))$–estimate: for all $n \in [\![1,M+1]\!]$

$$\| \partial^1 e_{\mathcal{T}}^{n,1} \|_{\mathbb{L}^2(\mathbf{I})} \le C(h+k)^2 \| u \|_{\mathcal{C}^4([0,T];\,\mathcal{C}^6(\bar{\mathbf{I}}))}. \tag{16}$$

Idea on the Proof of Statement of (15)–(16)

- Convergence of α^n of u_{xx}
- An a priori estimate for the discrete problem

Numerical tests

We $u(x,t) = \sin(\pi x)\cos(\pi t)$, $(x,t) \in (0,1) \times (0,1)$. We set $h_i = h$ for even i, $h_i = h/2$ for odd i, and $x_i = (x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}})/2$, for $i \in [\![1,N]\!]$. The step is taken as $k = h/2$. The following table shows that the order of scheme (12)–(14) is two:

h	Error in $W^{1,\infty}(L^2)$		Error in $L^\infty(H_0^1)$	
	Error	Order	Error	Order
1/225	0.0000946	——	0.0000770	——
1/300	0.0000533	1.9942888	0.0000434	1.9929847
1/375	0.0000341	2.0015768	0.0000278	1.9961295
1/450	0.0000237	1.9954982	0.0000193	2.0015786

References

[1] Bradji, A.: A theoretical analysis of a new finite volume scheme for second order hyperbolic equations on general nonconforming multidimensional spatial meshes. Numer. Methods Partial Differ. Eq., 29/1, 1–39, 2013.

[2] Eymard R., Gallouët T. and Herbin R.: Finite Volume Methods, Handbook for Numerical Analysis, Ph. Ciarlet J.L. Lions (Eds.), North Holland, 2000, vol. VII pp. 715-1022.

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