A new analysis for a super-convergence result in the divergence norm for $\mathbb{RT}_0$–MFEs combined with the C.-N. method applied to 1D Parabolic Eqs

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Problem to be solved

One dimensional non-stationary heat equation:

$$u_t(\boldsymbol{x}, t) - u_{xx}(\boldsymbol{x}, t) = f(\boldsymbol{x}, t), \qquad (\boldsymbol{x}, t) \in \mathrm{I} \times (0, T), \tag{1}$$

where $\mathrm{I} = (0, 1)$, $T > 0$, and f is a given function defined on $\mathrm{I} \times (0, T)$. This equation is equipped with an initial condition given by:

$$u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathrm{I}, \tag{2}$$

where u^0 is a given function defined on I, and the homogeneous Dirichlet boundary conditions:

$$u(0, t) = u(1, t) = 0, \qquad t \in (0, T). \tag{3}$$

A mixed formulation and a MFE approximation

As a "formal" mixed formulation for (1)–(3) is (see for instance [7]), for each $t \in (0, T)$, find $(p(t), u(t)) \in H_{\mathrm{div}}(\mathrm{I}) \times L^2(\mathrm{I})$ such that, for all $(\varphi, \psi) \in L^2(\mathrm{I}) \times H_{\mathrm{div}}(\mathrm{I})$

$$(u_t(t), \varphi)_{L^2(\mathrm{I})} + (\varphi, \mathrm{div}\, p(t))_{L^2(\mathrm{I})} = (\varphi, f(t))_{L^2(\mathrm{I})}, \tag{4}$$

$$(\psi, p(t))_{L^2(\mathrm{I})} = (\mathrm{div}\, \psi, u(t))_{L^2(\mathrm{I})}, \tag{5}$$

and

$$u(0) = u^0. \tag{6}$$

The space $H_{\mathrm{div}}(\mathrm{I})$ in the case of one dimension is given by the Sobolev space $H_{\mathrm{div}}(\mathrm{I}) = H^1(\mathrm{I})$. The mesh points of $\mathrm{I} = (0, 1)$ are denoted by $0 = \boldsymbol{x}_0 < \boldsymbol{x}_1 \ldots < \boldsymbol{x}_{M+1} = 1$, with $M \in \mathbb{N} \setminus \{0\}$, and the constant step is given by $h = \boldsymbol{x}_{i+1} - \boldsymbol{x}_i = 1/(M + 1)$. We consider the sub-intervals $\mathrm{I}_i = (\boldsymbol{x}_i, \boldsymbol{x}_{i+1})$, for $i \in [\![0, M]\!]$. The discretization of the spaces $H_{\mathrm{div}}(\mathrm{I})$ and $L^2(\mathrm{I})$ is performed using the $\mathbb{RT}_0$-MFEs (see [8]):

$$V_h^{\mathrm{div}} = \{v \in H_{\mathrm{div}}(\mathrm{I}) : \quad v|_{\mathrm{I}_i} \in \mathbb{D}_0, \quad \forall i \in [\![0, M]\!]\} \tag{7}$$

and

$$W_h = \{u \in L^2(\mathrm{I}) : \quad u|_{\mathrm{I}_i} \in \mathbb{P}_0, \quad \forall i \in [\![0, M]\!]\}, \tag{8}$$

where $\mathbb{P}_0$ is the space of constant functions and

$$\mathbb{D}_0 = \mathbb{P}_0 \oplus \boldsymbol{x}\mathbb{P}_0.$$

The space V_h^{div} (resp. W_h) can be defined as the set of continuous functions (resp. functions of $L^2(\mathrm{I})$) which are linear (resp. constant) over each I_i, see [8]). The basis of V_h^{div} is the set of usual piece-wise linear shape functions.

An existing MFE scheme, see [2]

Find $(p_h^n, u_h^n) \in V_h^{\mathrm{div}} \times W_h$ such that:

- For any $n \in [\![0, N]\!]$ and for all $\varphi \in W_h$:

$$\left(\partial^1 u_h^{n+1}, \varphi\right)_{L^2(\mathrm{I})} + \left(\left(p_h^{n+\frac{1}{2}}\right)_{\boldsymbol{x}}, \varphi\right)_{L^2(\mathrm{I})} = \left(f(t_{n+\frac{1}{2}}), \varphi\right)_{L^2(\mathrm{I})}, \tag{9}$$

- For any $n \in [\![0, N+1]\!]$:

$$(p_h^n, \psi)_{L^2(\mathrm{I})} = (u_h^n, \psi_{\boldsymbol{x}})_{L^2(\mathrm{I})}, \qquad \forall \psi \in V_h^{\mathrm{div}}, \tag{10}$$

where

$$p_h^0 = -\Pi_h(u^0)_{\boldsymbol{x}}. \tag{11}$$

Known convergence result, see [2]

$$\max_{n=0}^{N} \|u_x(t_{n+\frac{1}{2}}) + p_h^{n+\frac{1}{2}}\|_{H^1(\mathrm{I})} \le C\left(h + k^2\right). \tag{12}$$

Definition of interpolation operators, see [9]

We shall use the following interpolation operators over the spaces V_h^{div} and W_h, see [9]:

- The usual linear interpolation operator Π_h over V_h^{div}.
- The interpolation operator J_h over W_h given by $J_h u(\boldsymbol{x}) = J_i$, for $\boldsymbol{x} \in \mathrm{I}_i$ with J_i is given by the mean value over I_i

$$J_i = \frac{1}{h}\int_{\mathrm{I}_i} u(\boldsymbol{x})d\boldsymbol{x}. \tag{13}$$

Our aim: Super-convergence phenomenon

The aim is bi-fold:

- We improve the order in space (which is only h) in (12) to order h^2 by comparing the discrete solution $p_h^{n+\frac{1}{2}}$ with the linear interpolation Π_h of $p = -u_{\boldsymbol{x}}$, i.e. that is a super-convergence for the MFE scheme (9)–(11) (see [9]).
- Prove the order two in space stated in the previous item in the divergence norm (in space), i.e. H^1-norm. More precise, we shall prove this super-convergence result in $L^2(H^1)$–norm.

Main result: Super-convergence result

The following $L^2(H_{\mathrm{div}}(\mathrm{I}))$–error estimate holds:

$$\left(\sum_{n=0}^{N} k \left\|\Pi_h u_x(t_{n+\frac{1}{2}}) + p_h^{n+\frac{1}{2}}\right\|_{1,\mathrm{I}}^2\right)^{\frac{1}{2}} \le C(h + k)^2. \tag{14}$$

Super-convergence for elliptic equations is not stated explicitly

Let us consider the following second order elliptic equation in 1D:

$$-\omega_{\boldsymbol{xx}}(\boldsymbol{x}) = F(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathrm{I} = (0, 1) \quad \text{and} \quad u(0) = u(1) = 0. \tag{15}$$

The MFE scheme for the problem (15) is: Find $(p_h, \omega_h) \in V_h^{\mathrm{div}} \times W_h$ such that, for all $(\varphi, \psi) \in W_h \times V_h^{\mathrm{div}}$

$$((p_h)_{\boldsymbol{x}}, \varphi)_{L^2(\mathrm{I})} = (F, \varphi)_{L^2(\mathrm{I})} \quad \text{and} \quad (p_h, \psi)_{L^2(\mathrm{I})} = (\omega_h, \psi_{\boldsymbol{x}})_{L^2(\mathrm{I})}. \tag{16}$$

Using the error estimates [8, Page 237] yields the following first order estimate for the MFE scheme (16)

$$\|p_h + u_x\|_{1,\mathrm{I}} + \|\omega_h - \omega\|_{L^2(\mathrm{I})} \le Ch. \tag{17}$$

We are able to prove the following super-convergence result::

$$\|p_h + \Pi_h u_x\|_{1,\mathrm{I}} + \|\omega_h - J_h \omega\|_{L^2(\mathrm{I})} \le Ch^2. \tag{18}$$

Main idea behind in the proof

Developed a new discrete a priori estime

In Progress

- Extension to multi-dimensional Parabolic equations.
- Extension to Non-Linear Parabolic equations.
- Extension to Evolutionary Navier Stokes equation.

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