A convergence order for a finite volume scheme for a semilinear parabolic equation

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Aim

The aim of the present contribution is to provide a finite volume scheme as well as a convergence analysis for a semilinear parabolic equation.

Overview : References based on...

- Bradji, A. , Fuhrmann, J.: Some abstract error estimates of a finite volume scheme for a nonstationary heat equation on general nonconforming multidimensional spatial meshes. Applications of Mathematics, Praha, 58/1, 1–38, 2013.
- Eymard, R., Gallouët, T., Herbin, R.: Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes SUSHI. IMA J. Numer. Anal. 30/4, 1009–1043 (2010).

Overview : References based on...(Suite)

- R. Eymard, T. Gallouët and R. Herbin: Convergence of finite volume schemes for semilinear convection diffusion equations. Numer. Math. 82/1 (1999), 91–116.
- V. Thomée: Galerkin Finite Element Methods for Parabolic Problems. Springer-Verlag, Second Edition, Berlin (2006).
- M. Yang, C. Bi, and J. Liu: Postprocessing finite volume element method for semilinear parabolic problems. ESAIM: M2AN (Mathematical Modelling and Numerical Analysis) 43 (2009), 957–971.

Plan of the presentation

1. Motivation and statement of previous results
2. Problem to be discretized
3. Our main result
    1. Statement of the present result
    2. What advantages of our present contribution?
    3. Some idea on the proof of the present result
4. Conclusion and some perspectives

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Semilinear parabolic equation

Find a real valued function defined on $\Omega \times (0, T)$, where Ω is an open bounded polyhedral subset in $\mathbb{R}^d$, with $d \in \mathbb{N}^\star = \mathbb{N} \setminus \{0\}$ with

$$u_t(x,t) - \Delta u(x,t) = f(u)(x,t), \; (x,t) \in \Omega \times (0,T), \tag{1}$$

where $T > 0$, and f is a given function defined on $\mathbb{R}$ into $\mathbb{R}$.
An initial condition is given by:

$$u(x,0) = u^0(x), \; x \in \Omega, \tag{2}$$

and, for the sake of simplicity, we consider homogeneous Dirichlet boundary conditions, that is

$$u(x,t) = 0, \; (x,t) \in \partial\Omega \times (0,T). \tag{3}$$

About Semilinear parabolic equation?

1. (In real world): Problem (1)–(3) arises for instance in Combustion modeling, Epidemic phenomena, and Stochastics controls.
   C. Erdmann: Nonlinear Black-Scholes Equations. PhD Thesis, 2012, Universität Rostock.
2. (Existence and uniqueness): existence and uniqueness can be found for instance in the book of "L. C. Evans: Partial Differential Equations. Graduate Studies in Mathematics, American Mathematical Society, 19 (1998)." An assumption that the derivative of the source term is bounded is used, i.e. for some given κ, $|f'(s)| \leq \kappa$.

Some literature

- Finite element methods and finite volume element methods have been used to approximate the semilinear problem (1)–(3), see respectively Thomée (2006) and the stated article of Yang et al. (2009).
- The case of time independent semilinear equations is studied for instance in Eymard et al. (1999) where a finite volume scheme is presented and a convergence, under weak regularity assumption on the exact solution, of a family of approximate solutions is proved when the mesh size tends to zero.

Some facts to be taken in consideration to get a discretization scheme for the problem

- The finite volume mesh is nonconforming which is recently introduced.
- f is nonlinear.
- We aim not only to prove the convergence but to provide a convergence order.
- We hope to get not only a convergence towards the exact solution but also its first derivatives.

A new finite volume mesh: any space dimension and orthogonality property is not required

A new mesh recently used in the following reference which deals with stationary equation:

1. Eymard, R., Gallouët, T., Herbin, R.: Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. IMA J. Numer. Anal. 30/4, 1009–1043 (2010).

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Some advantages of the new mesh

Advantages of this new spatial mesh

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (othogonality property is not required): the othogonality property is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convexe, whereas the convexity property is not required in this new mesh.

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Figure for the mesh

Definition (New mesh of Eymard et *al.*(2010))

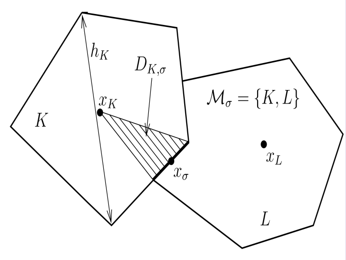

Figure : Notations for two neighbouring control volumes in $d = 2$

Discretization of the domain Ω and time interval $(0, T)$

Discretization is performed as:

1. Spatial domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is discretized using the new class of meshes.
2. The time interval $(0, T)$ constant step $k = T/(N+1)$, $N \in \mathbb{N}$.

Principles of our scheme

Principles of our scheme

1. Discretization of nonlinear Heat equation: the discretization of $u_t - \Delta u = f(u)$ stems from weak formulation (like in finite element method), for all $v \in H_0^1(\Omega)$

$$\int_\Omega u_t(x,t)v(x)dx + \int_\Omega \nabla u(x,t) \cdot \nabla v(x)dx = \int_\Omega f(u(x,t))v(x)dx. \quad (4)$$

2. (Discretization of initial condition $u(x,0) = u^0(x)$): as usual, i.e. Point-wise approximation.

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Discretization of the equation $u_t - \Delta u = f(u)$

1. (Discrete unknowns): the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \{\left((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}\right), \ v_\sigma = 0, \ \forall\sigma\in\mathcal{E}_{\mathrm{ext}}\}$$

2. (discretization of the gradient): the discretization of ∇ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see EGH (IMAJNA, 2010).

Discretization of $u_t - \Delta u = f(u)$ (suite)

So a discretization can be done as : For any $n \in [\![0, N]\!]$, find $u_{\mathcal{D}}^n \in \mathcal{X}_{\mathcal{D},0}$ such that, for all $v \in \mathcal{X}_{\mathcal{D},0}$

$$\left(\partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v \right)_{\mathbb{L}^2} + \left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}} v \right)_{(\mathbb{L}^2)^d}$$

$$= \left(f(\Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}), \Pi_{\mathcal{M}} v \right)_{\mathbb{L}^2(\Omega)},$$

where $\Pi_{\mathcal{M}}$ is constant on each control volume K.

Discretization boundary condition and initial condition

$$u_K^0 = u^0(x_K), \ \forall K, \tag{5}$$

and

$$u_\sigma^0 = u^0(x_\sigma), \ \forall \sigma, \tag{6}$$

A main result: error estimates

Theorem

- *Existence and uniqueness when the mesh size k of time discretization is small.*

A main result: error estimates (Suite)

Theorem

- $\mathcal{W}^{1,2}(0,T;\mathbb{L}^2(\Omega))$*–estimate: for all* $n \in [\![1, N+1]\!]$

$$\left(\sum_{n=1}^{N+1} k \| \, \partial^1 \left(\mathcal{P}_{\mathcal{M}} \, u(t_n) - \Pi_{\mathcal{M}} \, u_{\mathcal{D}}^n \right) \|_{\mathbb{L}^2(\Omega)}^2 \right)^{\frac{1}{2}}$$
$$\leq \quad C(k + h_{\mathcal{D}}) \| \, u \|_{\mathcal{C}^2([0,T];\; \mathcal{C}^2(\overline{\Omega}))}. \qquad (7)$$

- *Error estimate in the gradient approximation:*

$$\| \nabla_{\mathcal{D}} \, u_{\mathcal{D}}^n - \nabla \, u(\cdot, t_n) \|_{\left(\mathbb{L}^2(\Omega)\right)^d} \leq C(h_{\mathcal{D}} + k) \| \, u \|_{\mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))}. \qquad (8)$$

Theorem is useful

1. Estimate (7) allows us to approximate (the time derivative) $u_t(x_K, t_n)$, all K and n
2. Estimate (8) allows us to approximate the first spatial derivatives $\frac{\partial u}{\partial x_i}(x_K, t_n)$, all K and n.

An idea on the proof

- Contractive mapping technique to prove the existence and uniqueness.
- Comparison with the scheme: for any $n \in [\![0, N+1]\!]$, find $\bar{u}_{\mathcal{D}}^n \in \mathcal{X}_{\mathcal{D},0}$ such that, for all $v \in \mathcal{X}_{\mathcal{D},0}$

$$\left(\nabla_{\mathcal{D}} \bar{u}_{\mathcal{D}}^n, \nabla_{\mathcal{D}} v \right)_{(\mathbb{L}^2)^d} = \left(-\Delta u(t_n), \Pi_{\mathcal{M}} v \right)_{\mathbb{L}^2(\Omega)} . \qquad (9)$$

Conclusion

We considered a semilinear parabolic equation. A convergence analysis of a new finite volume scheme is provided in several discrete norms. We plan to consider:

- The Crank-Nicolson method in order to improve the order in time.
- High order finite volume approximations for semilinear parabolic equations. In progress