Convergence Analysis of a Finite Volume Scheme for a Distributed Order Diffusion Equation

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NMA 2022 (International Conference on Numerical Methods and Applications)

August 22–26, 2022, Borovets-Bulgaria

On-Line Presentation

Aim of the presentation

The aim of this talk is to establish a Finite Volume Scheme for a Distributed Order Diffusion Equation and prove its convergence.

Plan of this presentation

Some references

- A. Bradji and J. Fuhrmann. Convergence order of a finite volume scheme for the time-fractional diffusion equation. Numerical analysis and its applications, 33–45, Lecture Notes in Comput. Sci., 10187, Springer, Cham, 2017.
- K. Diethelm and N. Ford. Numerical analysis for distributed-order differential equations. J. Comput. Appl. Math. 225 (2009), no. 1, 96–104.
- R. Eymard, T. Gallouët, R. Herbin. Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. IMA J. Numer. Anal. 30/4, 1009–1043 (2010).
- G. H. Gao, H. W. Sun, and Z. Z. Sun. Some high-order difference schemes for the distributed-order differential equations. J. Comput. Phys. 298 (2015), 337–359.
- H. Ye, F. Liu, V. Anh, and I. Turner. Numerical analysis for the time distributed-order and Riesz space fractional diffusions on bounded domains. IMA J. Appl. Math. 80 (2015), no. 3, 825–838.

Problem to be solved

Equation

We consider the following time fractional diffusion equation:

$$\mathbb{D}_t^\alpha u(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega \times (0,T), \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is an open domain of $\mathbb{R}^d$, $T > 0$, and f is a given function with:

Distributed Caputo derivative

$$\mathbb{D}_t^\alpha \varphi(t) = \int_0^1 \omega(\alpha) \partial_t^\alpha \varphi(t) d\alpha \tag{2}$$

with $\partial_t^\alpha \varphi(t) = \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-s)^{-\alpha} \varphi' ds$ for $0 \leq \alpha < 1$ and $\partial_t^\alpha \varphi(t) = \varphi'(t)$ when $\alpha = 1$.

Problem to be solved: Suite

Hypotheses on the weight function ω

$$\omega(\alpha) \geq 0 \tag{3}$$

and

$$\int_0^1 \omega(\alpha) d\alpha = c_0 > 0. \tag{4}$$

Initial and boundary conditions

Initial condition

$$u(\boldsymbol{x},0) = u^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega.$$

Homogeneous Dirichlet boundary

$$u(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t) \in \partial\Omega \times (0,T).$$

What about time fractional diffusion equation?

Some physics

Fractional differential equations have been successfully used in the modeling of many different processes and systems. They are used, for instance, to describe anomalous transport in disordered semiconductors, penetration of light beam through a turbulent medium, transport of resonance radiation in plasma, blinking fluorescence of quantum dots, penetration and acceleration of cosmic ray in the Galaxy, and large-scale statistical Cosmography. We refer to the monograph Uchaikin (Fractional Derivatives for Physicists and Engineers, Springer-Verlag Heidelberg, 2013) where we find many details.

Introduction: Finite Volume from Admissible meshes to Nonconforming meshes

Finite volume methods are numerical methods approximating different types of Partial Differential Equations (PDEs). They are based on three principle ideas:

- Subdivision of the spatial domain into subsets called Control Volumes.
- Integration of the equation to be solved over the Control Volumes.
- Approximation of the derivatives appearing after integration.

[1] We mean here the "pure" finite volume methods and not finite volume-element methods

Introduction (suite): Finite Volume from Admissible meshes to Nonconforming meshes

Finite Volume methods passed by three steps:

First step

Finite Volume methods using Admissible meshes.

Second step

SUSHI method.

Introduction (suite): Finite Volume from Admissible meshes to Nonconforming meshes

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Introduction (suite): Finite Volume methods on admissible meshes

Definition

Let $\mathcal{T}$ be an Admissible Mesh in the sense of Eymard et al. (Handbook, 2000).

$K \in \mathcal{T}$ are the control volumes and σ are the edges of the control volumes K.

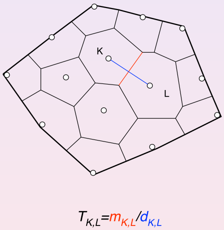

Figure: transmissivity between K and L: $\mathcal{T}_\sigma = \mathcal{T}_{K|L} = \frac{m_{K,L}}{d_{K,L}}$

Introduction (suite): Finite Volume methods on admissible meshes

Main properties of Admissible mesh:

1. Convexity of the Control Volumes.
2. The orthogonality property: the $(\boldsymbol{x}_K\boldsymbol{x}_L)$ is orthogonal to the common edge σ between the control volumes K and L.

Introduction (suite): Finite Volume methods on admissible meshes

Model to be solved:

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \quad \text{and} \quad u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega. \tag{5}$$

Principles of Finite Volume scheme:

1. Integration on each control volume K :$-\int_K \Delta u(\boldsymbol{x}) d\boldsymbol{x} = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$,
2. Integration by Parts gives :$-\int_{\partial K} \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$
3. Summing on the lines of K: $-\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$

Introduction (suite): Finite Volume methods on admissible meshes

Approximate Finite Volume Solution $u_{\mathcal{T}} = (u_K)_K$

$$-\sum_{\sigma \in \mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{d_{K|L}} (u_L - u_K) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}. \tag{6}$$

Matrix Form

$$\mathcal{A}^{\mathcal{T}} u_{\mathcal{T}} = f_{\mathcal{T}}.$$

Introduction (suite): Finite Volume methods on admissible meshes

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Introduction (suite): Finite Volume methods on admissible meshes

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Introduction (suite): Finite Volume methods on admissible meshes

Theorem

Let $\mathcal{X}(\mathcal{T})$: functions which are constant on each control volume K. Let $e_{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$ be defined by $e_K = u(\boldsymbol{x}_K) - u_K$ for any $K \in \mathcal{T}$. Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence results hold:

1. *H_0^1-error estimate*

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \leq Ch\|u\|_{2,\overline{\Omega}}, \tag{7}$$

where $\|\cdot\|_{1,\mathcal{T}}$ is the H_1^0-norm $\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 = \sum_{\sigma=K|L\in\mathcal{E}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(u_L - u_K)^2.$

2. *L^2-error estimate:*

$$\|e_{\mathcal{T}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{8}$$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Definition (New mesh of Eymard et *al.*, IMAJNA 2010)

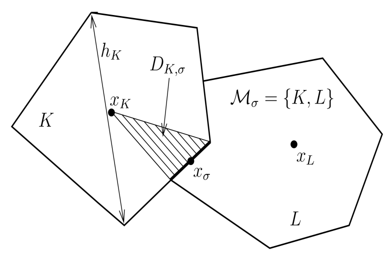

Figure: Notations for two neighbouring control volumes in $d = 2$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Main properties of this new mesh:

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (orthogonality property is not required): the orthogonality property is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convex, whereas the convexity property is not required in this new mesh.

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Principles of discretization for Poisson's equation:

1. Discrete unknowns: the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \{\left((v_K)_{K\in\mathcal{M}}\,,\,(v_\sigma)_{\sigma\in\mathcal{E}}\right),\;\; v_K, v_\sigma \in \mathbb{R},\;\; v_\sigma = 0,\, \forall\sigma \in \mathcal{E}_{\text{ext}}\}$$

2. Discretization of the gradient: the discretization of ∇ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see Eymard et *al.* (IMAJNA, 2010):

   1. The discrete gradient $\nabla_{\mathcal{D}}$ is stable
   2. The discrete gradient $\nabla_{\mathcal{D}}$ is consistent.

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI

Weak formulation for Poisson's equation: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega). \tag{9}$$

SUSHI (Scheme Using stabilized Hybrid Interfaces) for Poisson's equation: Find $u_\mathcal{D} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\int_\Omega \nabla_\mathcal{D} u_\mathcal{D}(\boldsymbol{x}) \cdot \nabla_\mathcal{D} v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{10}$$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI

Theorem

Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence result hold:

1. H_0^1*-error estimate*

$$\|\nabla u - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \leq Ch\|u\|_{2,\overline{\Omega}}. \quad (11)$$

2. L^2*-error estimate:*

$$\|u - \Pi_{\mathcal{M}} u_{\mathcal{D}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \quad (12)$$

Principles of the time- discretization

Step 1: Discretize the integral (2).

We discretize $[0,1]$ with uniform mesh $\{\xi = i/q; \quad i = 0, \dots, q\}$ with constant time step $1/q$ where $q \in \mathbb{N} \setminus \{0\}$.

$$\mathbb{D}_t^{\alpha}\varphi(t) \approx \frac{1}{q}\sum_{i=0}^{q-1}\lambda_i\omega(\alpha_i)\partial_t^{\alpha_i}\varphi(t), \tag{13}$$

with $\lambda_i > 0$ and $\alpha_0 < \alpha_1 < \ldots < \alpha_{q-1}$.

Nice remark

The RHS of (13) is a Multi-Term Fractional Operator.

Examples of quadratures

Examples of quadratures

- Composite Mid point Formula
- Composite Trapezoid Formula
- Composite Simpson Formula

Principles of the time-discretization: Suite

Step 2: Discretize Caputo derivatives.

We define $k = T/(M+1)$ and $t_n = nk$, $n = 0, \ldots, M+1$.

$$\partial_t^{\alpha_i}\varphi(t_{n+1}) = \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{n} d_{j,\alpha_i}\overline{\partial}^{\alpha_i}\varphi(t_{n-j+1}) + \mathbb{T}_{\alpha_i}^{n+1}, \tag{14}$$

where $\overline{\partial}^{\alpha_i}$ is the following discrete derivative of order α

$$\overline{\partial}^{\alpha_i} v^{j+1} = \frac{v^{j+1} - v^j}{k^{\alpha_i}} \quad \text{and} \quad d_{j,\alpha_i} = (j+1)^{1-\alpha_i} - j^{1-\alpha_i} \tag{15}$$

and

$$\begin{aligned} |\mathbb{T}_{\alpha_i}^{n+1}| &\leq Ck^{2-\alpha_i} \\ &\leq Ck^{2-\alpha_{q-1}}, \quad \text{for} \quad k \leq 1. \end{aligned} \tag{16}$$

Principles of the discretization (suite)

Discretization in space

We use SUSHI scheme

Formulation of scheme

The finite volume scheme can then be defined as:

- Discretization of initial condition: Find $u_{\mathcal{D}}^0 \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^0, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\left(\Delta u^0, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}, \tag{17}$$

- Discretization of the factional heat equation: and for any $n \in [\![0, M]\!]$, find $u_{\mathcal{D}}^{n+1} \in \mathcal{X}_{\mathcal{D},0}$ such that, for all $v \in \mathcal{X}_{\mathcal{D},0}$

$$\frac{1}{q\Gamma(2-\alpha)} \sum_{i=0}^{q-1} \lambda_i \omega(\alpha_i) \sum_{j=0}^{n} d_{j,\alpha_i} \left(\overline{\partial}^{\alpha_i} \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} + \left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = \left(f(t_{n+1}), \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}. \tag{18}$$

Convergence result

Theorem (An $L^\infty(L^2)$-error estimate)

For smooth solution u and weight function ω

$$\|u(t_n) - \Pi_{\mathcal{M}} u_{\mathcal{D}}^n\|_{\mathbb{L}^2(\Omega)} \le C\left(h_{\mathcal{D}} + k^{2-\alpha_{q-1}} + \mathbb{T}(q)\right), \tag{19}$$

where $\mathbb{T}(q)$ is quadrature error in the approximation:

$$\int_0^1 \psi(\alpha)d\alpha \approx \frac{1}{q}\sum_{i=0}^{q-1} \lambda_i \psi(\alpha_i).$$

Convergence result in a particular case: Mid Point formula

Corollary (An $L^\infty(L^2)$-error estimate)

For Composite Mid Point formula and smooth solution u and weight function ω

$$\|u(t_n) - \Pi_{\mathcal{M}} u_{\mathcal{D}}^n\|_{\mathbb{L}^2(\Omega)} \le C\left(h_{\mathcal{D}} + k^{1+1/(2q)} + q^{-2}\right). \quad (20)$$

Perspective

First perspective

Convergence analysis in other discrete norms