SUSHI for a Time Fractional Diffusion Equation with Delay

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On-Line Presentation

Aim of the presentation

The aim of this talk is to establish a finite volume scheme along with a convergence analysis for a Time Fractional Diffusion Equation with a Time Delay. Such equation is a Semi-Linear

We combine here some of our works which dealt with numerical schemes for Time Fractional Diffusion Equations in part and other works concerned with the numerical approximation of Parabolic Equations with Delays

Plan of this presentation

1. Problem to be solved
2. Introduction: Finite Volume methods from Admissible to Nonconforming meshes (SUSHI scheme)
3. Finite Volume scheme for a Time Fractional Diffusion Equation with a Time Delay.
4. Convergence analysis for the numerical scheme
5. Conclusion and Perspectives

Some references

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- A. Bradji, T. Ghoudi. Some convergence results of a multidimensional finite volume scheme for a semilinear parabolic equation with a time delay. Numerical methods and applications, 351–359, Lecture Notes in Comput. Sci., 11189, Springer, Cham, 2019.
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Some references (Suite)

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Problem to be solved

Semi-Time Fractional Diffusion Equation with Delay

$$u_t^{\alpha}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t,u(\boldsymbol{x},t),u(\boldsymbol{x},t-\tau)), \quad (\boldsymbol{x},t) \in \Omega \times (0,T), \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is an open domain of $\mathbb{R}^d$, f is given function, and $T, \tau > 0$ are given. **The positive value τ called the delay.**

Fractional derivative is given in Caputo sense: $0 < \alpha < 1$

$$\partial_t^{\alpha}\varphi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\varphi'(s)ds \tag{2}$$

Problem to be solved: Suite

Initial conditions

$$u(\boldsymbol{x},t) = u^0(\boldsymbol{x},t), \quad \boldsymbol{x} \in \Omega, \ -\tau \le t \le 0. \tag{3}$$

Homogeneous Dirichlet boundary

$$u(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t) \in \partial\Omega \times (0,T). \tag{4}$$

What about Delay differential equations?

Some physics

Delay differential equations occur in many applications such as ecology and biology. They have long played important roles in the literature of theoretical population dynamics, and they have been continuing to serve as useful models.

Introduction: Finite Volume from Admissible meshes to Nonconforming meshes

Finite volume methods are numerical methods approximating different types of Partial Differential Equations (PDEs). They are based on three principle ideas:

- Subdivision of the spatial domain into subsets called Control Volumes.
- Integration of the equation to be solved over the Control Volumes.
- Approximation of the derivatives appearing after integration.

[1] We mean here the "pure" finite volume methods and not finite volume-element methods

Introduction (suite): Finite Volume from Admissible meshes to Nonconforming meshes

Finite Volume methods passed by two steps:

First step

Finite Volume methods using Admissible meshes.

Second step

SUSHI method.

Introduction (suite): Finite Volume from Admissible meshes to Nonconforming meshes

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Introduction (suite): Finite Volume methods on admissible meshes

Definition

Let $\mathcal{T}$ be an Admissible Mesh in the sense of Eymard et al. (Handbook, 2000).

$K \in \mathcal{T}$ are the control volumes and σ are the edges of the control volumes K.

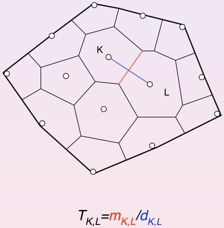

Figure: transmissivity between K and L: $\mathcal{T}_\sigma = \mathcal{T}_{K|L} = \frac{m_{K,L}}{d_{K,L}}$

Introduction (suite): Finite Volume methods on admissible meshes

Main properties of Admissible mesh:

1. Convexity of the Control Volumes.
2. The orthogonality property: the $(\boldsymbol{x}_K\boldsymbol{x}_L)$ is orthogonal to the common edge σ between the control volumes K and L.

Introduction (suite): Finite Volume methods on admissible meshes

Model to be solved:

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega \quad \text{and} \quad u(\boldsymbol{x}) = 0,\ \boldsymbol{x} \in \partial\Omega. \tag{5}$$

Principles of Finite Volume scheme:

1. Integration on each control volume K :$-\displaystyle\int_K \Delta u(\boldsymbol{x}) d\boldsymbol{x} = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$,
2. Integration by Parts gives :$-\displaystyle\int_{\partial K} \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$
3. Summing on the lines of K: $-\displaystyle\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$

Introduction (suite): Finite Volume methods on admissible meshes

Approximate Finite Volume Solution $u_{\mathcal{T}} = (u_K)_K$

$$-\sum_{\sigma\in\mathcal{E}_K}\frac{\mathrm{m}(\sigma)}{d_{K|L}}(u_L-u_K)=\int_K f(\boldsymbol{x})d\boldsymbol{x}. \qquad (6)$$

Matrix Form

$$\mathcal{A}^{\mathcal{T}} u_{\mathcal{T}} = f_{\mathcal{T}}.$$

Introduction (suite): Finite Volume methods on admissible meshes

Approximate Finite Volume Solution $u_{\mathcal{T}} = (u_K)_K$

$$-\sum_{\sigma \in \mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{d_{K|L}} (u_L - u_K) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}. \tag{6}$$

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Introduction (suite): Finite Volume methods on admissible meshes

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$$\mathcal{A}^{\mathcal{T}}u_{\mathcal{T}}=f_{\mathcal{T}}.$$

Introduction (suite): Finite Volume methods on admissible meshes

Theorem

Let $\mathcal{X}(\mathcal{T})$: functions which are constant on each control volume K. Let $e_{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$ be defined by $e_K = u(\boldsymbol{x}_K) - u_K$ for any $K \in \mathcal{T}$. Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence results hold:

1. *H_0^1-error estimate*

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \leq Ch\|u\|_{2,\overline{\Omega}}, \tag{7}$$

where $\|\cdot\|_{1,\mathcal{T}}$ is the H_1^0-norm $\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 = \sum_{\sigma=K|L\in\mathcal{E}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(u_L - u_K)^2.$

2. *L^2-error estimate:*

$$\|e_{\mathcal{T}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{8}$$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Definition (New mesh of Eymard et *al.*, IMAJNA 2010)

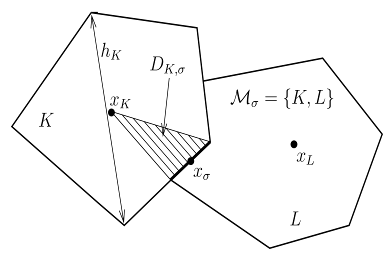

Figure: Notations for two neighbouring control volumes in $d = 2$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Main properties of this new mesh:

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (orthogonality property is not required): the orthogonality property is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convex, whereas the convexity property is not required in this new mesh.

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Principles of discretization for Poisson's equation:

1. Discrete unknowns: the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \{ \left((v_K)_{K\in\mathcal{M}} \,,\, (v_\sigma)_{\sigma\in\mathcal{E}} \right) \,,\;\; v_K, v_\sigma \in \mathbb{R},\;\; v_\sigma = 0,\, \forall\sigma \in \mathcal{E}_{\mathrm{ext}} \}$$

2. Discretization of the gradient: the discretization of ∇ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see Eymard et *al.* (IMAJNA, 2010):

    1. The discrete gradient $\nabla_{\mathcal{D}}$ is stable
    2. The discrete gradient $\nabla_{\mathcal{D}}$ is consistent.

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI

Weak formulation for Poisson's equation: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega). \tag{9}$$

SUSHI (Scheme Using stabilized Hybrid Interfaces) for Poisson's equation: Find $u_\mathcal{D} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\int_\Omega \nabla_\mathcal{D} u_\mathcal{D}(\boldsymbol{x}) \cdot \nabla_\mathcal{D} v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{10}$$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI

Theorem

Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence result hold:

1. *H_0^1-error estimate*

$$\|\nabla u - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{11}$$

2. *L^2-error estimate:*

$$\|u - \Pi_{\mathcal{M}} u_{\mathcal{D}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \tag{12}$$

Principles of the discretization

Definition of a discretization in time and its parameters

- The time discretization is performed with a constrained time step-size k such that $\frac{\tau}{k} \in \mathbb{N}$. We set then $k = \frac{\tau}{M}$, where $M \in \mathbb{N} \setminus \{0\}$. We denote by ∂^1 the discrete first time derivative given by $\partial^1 v^{j+1} = \frac{v^{j+1} - v^j}{k}$.
- Denote by N the integer part of $\frac{T}{k}$, i.e. $N = \left[\frac{T}{k}\right]$.
- We shall denote $t_n = nk$, for $n \in [\![-M, N]\!]$.
- As particular cases $t_{-M} = -\tau$, $t_0 = 0$, and $t_N \leq T$.

Advantages of this time discretization

The point $t = 0$ is a mesh point which is suitable since we have equation (1) defined for $t \in (0, T)$ and initial condition (3) defined for $t \in (-\tau, 0)$.

Principles of the discretization in Space

Discretization in space

We use SUSHI scheme

Principles of the discretization in Time

Discretization in time: $L1$ formula to approximate the Caputo derivative

$$\begin{aligned} \partial_t^{\alpha}\varphi(t_{n+1}) &= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\varphi'(s)ds \\ &\approx \sum_{j=0}^{n}k\lambda_j^{n+1,\alpha}\partial^1 u(t_{j+1}), \end{aligned} \tag{13}$$

where

$$\lambda_j^{n+1,\alpha} = \frac{(t_{n+1}-t_j)^{1-\alpha}-(t_{n+1}-t_{j+1})^{1-\alpha}}{k\Gamma(2-\alpha)} \tag{14}$$

Formulation of scheme

The finite volume scheme can then be defined as:

- Discretization of initial condition: For any $n \in [\![-M, 0]\!]$

$$(\nabla_{\mathcal{D}} u_{\mathcal{D}}^n, \nabla_{\mathcal{D}} v)_{(\mathbb{L}^2(\Omega))^d} = -\left(\Delta u^0(t_n), \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}, \qquad (15)$$

- Discretization of the delayed equation: For any $n \in [\![0, N-1]\!]$, find $u_{\mathcal{D}}^n \in \mathcal{X}_{\mathcal{D},0}$ such that, for all $v \in \mathcal{X}_{\mathcal{D},0}$

$$\begin{aligned}\sum_{j=0}^{n} k\lambda_j^{n+1,\alpha}\left(\partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^{j+1}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} &+ \left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}} v\right)_{(\mathbb{L}^2(\Omega))^d} \\ = \quad &\left(f(t_{n+1}, \Pi_{\mathcal{M}} u_{\mathcal{D}}^n, \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1-M}), \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}, \qquad (16)\end{aligned}$$

where $f(t_{n+1}, \Pi_{\mathcal{M}} u_{\mathcal{D}}^n, \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1-M})$ denotes the function

$$\boldsymbol{x} \mapsto f(\boldsymbol{x}, t_{n+1}, \Pi_{\mathcal{M}} u_{\mathcal{D}}^n(\boldsymbol{x}), \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1-M}).$$

Useful Assumption

To deal with the convergence analysis, we need the following assumption on the function f:

Assumption (Assumption on f)

We assume that the function $f(\boldsymbol{x}, t, s, r)$ *is Lipschitz continuous with respect to* (s, r) *with constant* κ*, i.e. for all* $(\boldsymbol{x}, t, s, r), (\boldsymbol{x}, t, s', r') \in \Omega \times \mathbb{R}^3$

$$|f(\boldsymbol{x}, t, s, r) - f(\boldsymbol{x}, t, s', r')| \leq \kappa\left(|s - s'| + |r - r'|\right).$$

Statement of the convergence results

Theorem (Error estimates)

We assume that u is sufficiently smooth.

- $\mathbb{L}^\infty(H_0^1)$*–estimate. For all* $n \in [\![-M, N]\!]$

$$\|\nabla_{\mathcal{D}} u_{\mathcal{D}}^n - \nabla u(t_n)\|_{\mathbb{L}^2(\Omega)} \le C(k^{2-\alpha} + h_{\mathcal{D}}). \tag{17}$$

- $W^{1,2}(\mathbb{L}^2)$*–estimate.*

$$\left(\sum_{n=-M+1}^{N} k \left\| u_t(t_n) - \Pi_{\mathcal{M}} \partial^1 u_{\mathcal{D}}^n \right\|_{\mathbb{L}^2(\Omega)}^2 \right)^{\frac{1}{2}} \le C(k^{2-\alpha} + h_{\mathcal{D}}). \tag{18}$$

Idea on the proof

The proof is manly based on two facts:

- Comparison with an optimal scheme : for any $n \in [\![0, N+1]\!]$, find $\bar{u}^n_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\nabla_{\mathcal{D}} \bar{u}^n_{\mathcal{D}}, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\sum_{K \in \mathcal{M}} v_K \int_K \Delta u(x, t_n) dx, \ \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{19}$$

- A convenient a priori estimate.

Conclusion and Perspectives

Conclusion

We considered the convergence of an implicit finite volume scheme, in any space dimension, for a simple semi-linear Time Fractional Diffusion equation with Delay. The disretization in space is performed using SUSHI method whereas the discretization in time is given by the application of $L1$-formula on uniform temporal mesh.

The order is proved to be one in space and $2-\alpha$ in time.

First perspective

The use of $L2-1_\sigma$ method in order to improve the order in time.

Conclusion and Perspectives

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First perspective

The use of $L2 - 1_\sigma$ method in order to improve the order in time.

Perspectives

Second perspective

Extension to the the case when the right hand side involves the exact solution and its gradient

$$\partial_t^\alpha u(\boldsymbol{x},t)-\Delta u(\boldsymbol{x},t)=f\left(\boldsymbol{x},t,u(\boldsymbol{x},t),u(\boldsymbol{x},t-\tau),\nabla u(\boldsymbol{x},t),\nabla u(\boldsymbol{x},t-\tau)\right).$$

Third perspective

Several delays.

Fourth perspective

Delays are not numbers but functions depending on t.

Perspectives

Second perspective

Extension to the the case when the right hand side involves the exact solution and its gradient

$$\partial_t^\alpha u(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = f\left(\boldsymbol{x}, t, u(\boldsymbol{x}, t), u(\boldsymbol{x}, t-\tau), \nabla u(\boldsymbol{x}, t), \nabla u(\boldsymbol{x}, t-\tau)\right).$$

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Third perspective

Several delays.

Fourth perspective

Delays are not numbers but functions depending on t.

Perspectives (Suite)

Fifth perspective

Extension to Time-Fractional Diffusion-Wave equation.

Sixth perspective

Consider Non-Uniform temporal mesh.

Perspectives (Suite)

Fifth perspective

Extension to Time-Fractional Diffusion-Wave equation.

Sixth perspective

Consider Non-Uniform temporal mesh.