On the convergence order of the COMSOL solutions in the Sobolev norms

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Motivation

Let Ω be an open bounded polygonal connected subset of $\mathbb{R}^d$, $d \in \{1, 2, 3\}$, and consider (with some convenient boundary conditions for the well–posedness) :

$$-\mathcal{L}u(\mathbf{x}) = f(\mathbf{x}), \mathbf{x} \in \Omega. \tag{1}$$

Consider a finite element mesh $\mathcal{T}$, with mesh size h.
The discretization of (1) leads to an algebraic system to be solved:

$$\mathcal{A}u_h = f^h. \tag{2}$$

Convergence order of u^h, in some norm $\| \cdot \|$, towards the exact solution u is some positive real θ defined by

$$\|u - u^h\| \leq Ch^\theta. \tag{3}$$

Thanks to (3), the convergence, then, measures the accuracy of the convergence of $u^h \to u$

Convergence order of finite element solution and COMSOL solution

Some useful remarks:

- There is a huge literature concerning the convergence order of finite solutions
- Some of these results, perhaps, have not justified theroretically or numerically yet (some times, the optimility of some theoretical results is good question!!
- Since the COMSOL solutions are finite element approximations, one could speak about:
  - a comparison between the convergence order of finite element solution on hand and COMSOL solutions on the other hand
  - COMSOL simulations are useful paths in order to confirm or not some optimal results, conjectures concerning convergence order of finite element solutions

Useful rule to compute convergence order

Let us denote by e the error in the finite element approximation, that is

$$e = u - u^h, \quad (4)$$

where u is the exact solution and u^h is the finite element solution.
To determine the convergence order θ, we use the following formula (recall that $d \in \{1, 2, 3\}$):

$$\theta = -d \frac{\ln(\|e_1\|) - \ln(\|e_2\|)}{\ln(\mathrm{DOF}_1\|) - \ln(\mathrm{DOF}_2)}, \quad (5)$$

where DOF_i is the degree of freedom corresponding to the mesh where it is computed the error e_i, for $i \in \{1, 2\}$.

Norms we are interested with

- Maximum norm

$$\|e\|_\infty = \max(|u - u_h|), \tag{6}$$

- average norm

$$\|e\|_{2,0} = \sqrt{\int_\Omega (u - u_h)^2}, \tag{7}$$

- energy norm

$$\|e\|_{2,1} = \sqrt{\int_\Omega (u - u_h)^2 + \sum_{i=1}^{d} \int_\Omega \left(\partial_i (u - u_h)\right)^2}, \tag{8}$$

References

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Poisson equation with regular data

Consider equation:

$$-\Delta u = 1, \ \Omega, \tag{9}$$

Ω is the unit disk.
The exact solution is

$$u(x, y) = -\frac{x^2 + y^2 - 1}{4}, \tag{10}$$

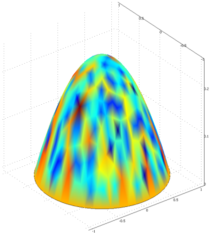

Figure: Linear COMSOL solution of the solution

Some results of tests in the average norm, with linear elements

DOF	**Times** [s]	$\|e\|_{2,0}.10^3$	θ
403	0.03–0.094	1.7	2.02
1561	0.063	0.4391	2.02
6145	1.297	0.1101	2.04
24385	1.06–1.16	0.0277	2.04
97153	5.81–7.86	0.0069	2.04

From these tests, we deduce that:

The convergence order, in the average norm, of the Poisson's equation, with homogeneous boundary conditions and right–hand side =1, and with linear finite elements is equal $\theta = 2$. This confirms well the known theoretical results in finite element methods

Some results of tests in the energy norm (that is a Sobolev norm of order one), with linear elements

DOF	$\|e\|_{2,1}$	θ
403	0.0283	0.98
1561	0.0146	0.98
6145	0.0073	0.98
24385	0.0037	0.98
97153	0.0019	0.98

From these tests, we deduce that:

The convergence order, in the energy norm (that is a Sobolev norm of order one), of the Poisson's equation, with homogeneous boundary conditions and right–hand side =1, is equal $\theta = 1$. This confirms well the known theoretical results in finite element methods

Poisson equation with irregular data

Consider equation:

$$-\Delta u = \delta(0), \ \Omega, \quad (11)$$

Ω is the unit disk $\delta(0)$ is the Dirac's function.
The exact solution is

$$u(x, y) = -\frac{\ln(r)}{2\pi}. \quad (12)$$

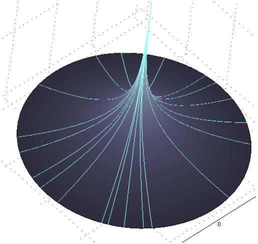

Figure: COMSOL solution of equation

Some results of tests in the average norm, with linear elements

DOF	**Times** [s]	$\|e\|_{2,0}.10^3$	θ
777	0.046–0.06	0.9784	1.97
3041	0.125	0.2557	1.97
12033	0.5–0.547	0.0692	1.69
47873	2.344	0.0277	0.0214
1.69			
190977	18.11	0.0083	1.69

From these tests, we deduce that:

The convergence order, in the average norm, of the Poisson's equation, with homogeneous boundary conditions and right–hand side =Dirac, is equal $\theta < 2$. This confirms well the known theoretical results in finite element method, see for instance Clain, Scott

Some results of tests in average norm, with quadratic elements

DOF	**Times** [s]	$\|\|e\|\|_{2,0}.10^3$	θ
3041	0.141	0.0601	1.13
12033	0.1515	0.0277	1.13
47873	2.563	0.0138	1.00
190977	13.95	0.0069	1.00

From these tests, we deduce that:

The convergence order, in the average norm, of the Poisson's equation, with homogeneous boundary conditions and right–hand side =Dirac, is equal $\theta = 1$. This confirms well the known theoretical results in finite element methods, see Scott

Some prospects

Extend our computations to more complex equations like Stokes and Navier Stokes equations.

Extend our work to some finite element discretizations in which the convergence order is not confirmed.