# Some Error Estimates for the Discretization of Parabolic Equations on General Multidimensional Nonconforming Spatial Meshes

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# Aim

The aim of the present contribution is to deal with some error estimates of an implicit finite volume scheme of a non stationary heat equation on a general class of meshes has been recently used to approximate stationary equations.

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# Overview : References based on...


- Bradji, A.: Some simples error estimates for finite volume approximation of parabolic equations. Comptes Rendus de l'Académie de Sciences, Paris, 346/9-10, 571–574 (2008 ).
- Bradji, A. and Fuhrmann, J.: Some error estimates in finite volume method for parabolic equations.Proceedings of the 5th International Symposium on Finite Volume for Complex Applications/ eds. by Eymard and Hérard, Wiley, 233–240 (2008).


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# Overview : References based on...(Suite)


- (Handbook) Eymard, R., Gallouët T., Herbin, R.: Finite volume methods. Handbook of Numerical Analysis. P. G. Ciarlet and J. L. Lions (eds.), VII , 723–1020 (2000).
- Eymard, R., Gallouët, T., Herbin, R.: Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. IMA J. Numer. Anal., Advance Access published on June 16, 2009; doi: doi:10.1093/imanum/drn084.


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# Plan of the presentation

1. Motivation and statment of previous results
   1. Problem to be discretized
   2. Heat equation and its classical finite volume discretization, i.e. using the classical Admissible mesh.
   3. Error estimate of Handbook of Eymard, Gallouët, and Herbin, i.e. $L^\infty(L^2)$ error estimate
   4. Why we looked for other error estimate? (would say in other norms)
   5. Statment of our previous results
2. Our main result
   1. What advantages of our present contribution?
   2. Statement of the present result

# Non stationary Heat equation

$$u_t(x,t) - \Delta u(x,t) = f(x,t), \; (x,t) \in \Omega \times (0,T), \tag{1}$$

where, $\Omega$ is an open bounded polyhedral subset in $\mathbb{R}^d$, with $d \in \mathbb{N}^\star$, $T > 0$ , and $f$ is a given function.
An initial condition is given by:

$$u(x,0) = u^0(x), \; x \in \Omega. \tag{2}$$

A Dirichlet boundary condition is defined by

$$u(x,t) = 0, \; (x,t) \in \partial\Omega \times (0,T), \tag{3}$$

where, we denote by $\partial\Omega = \overline{\Omega} \setminus \Omega$ the boundary of $\Omega$.

# About Heat equation?

1. (some physics): Heat equation $u_t - \Delta\, u$ is typically used in different applications, such as *fluid mechanics*, *heat and mass transfer*,...
2. (existence and uniqueness): existence and uniqueness of a weak solution of heat equation, with (2) ( *initial condition*) and (3) ( *Dirichlet boundary condition*) can be formulated using Bochner spaces; see for instance Evans book of partial differential equation

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# Finite volume discretization, using classical *admissible mesh* in space, for heat equation

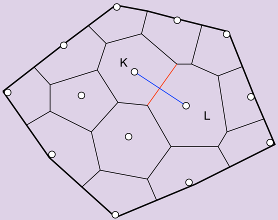
**Definition (Classical finite volume discretization, i.e. using Admissible mesh)**

There are two variables in
$u_t(x,t) - \Delta\, u(x,t) = f(x,t),\;\; x \in \Omega,\; t \in (0,T), \Omega \subset \mathbb{R}^d$

1. domain space $\Omega$ is discretized using admissible mesh, see next for the definition of admissible mesh.
2. uniform mesh on $(0,T)$ with constant step $k = T/(N+1)$.

## Definition (Admissible Mesh for $\Omega \subset \mathbb{R}^d$)

In the sense of Handbook EGH. $K \in \mathcal{T}$ are the control volumes and $\sigma$ are the edges of the control volumes $K$.




$T_{K,L} = m_{K,L}/d_{K,L}$

# Some properties of the classical admissible mesh

1. (orthogonality propert): $\sigma_{KL}$ othogonal to $\sigma$

# Definition of a classical two point finite volume scheme, using *admissible mesh*, for Heat equation

Principles of the classical scheme:

1. discretization of spatial $\Omega$: performed using admissible in the sense of EGH
2. discretization of time interval $(0, T)$: performed using constant step $k = T/(N+1)$, $N \in \mathbb{N}$
3. integration of $u_t - \Delta u$ on each $(t_n, t_{n+1}) \times K$, $K \in \mathcal{T}$ (control volume), $n = 0, \ldots, N$.

# Definition of a classical two point finite volume scheme, using *admissible mesh*, for Heat equation (suite)

1. (integration by parts): integration by parts $\int_K (u(x,t_{n+1}) - u(x,t_n))dx - \int_{t_n}^{t_{n+1}} \int_{\partial K} \nabla u(x,t) \cdot \mathbf{n}_K dx, dt$, $\mathbf{n}_K$ normal to $\partial K$ outward to $K$
2. (summing on the edges of $K$):$\int_K (u(x,t_{n+1}) - u(x,t_n))dx - \int_{t_n}^{t_{n+1}} \sum_{\sigma\in\mathcal{E}_K} \int_\sigma \nabla u(x,t) \cdot \mathbf{n}_{K,\sigma} dx, dt$
3. (thanks to orthogonality property of admissible mesh and implicit choice ) $\mathrm{m}(K)(u(x_K,t_{n+1}) - u(x_K,t_n)) - \sum_{\sigma\in\mathcal{E}_K} (\mathrm{m}(\sigma)/(d_{K,\sigma})(u(x_L,t_{n+1}) - u(x_K,t_{n+1}))$

# Definition of a classical scheme, using *admissible mesh*, for Heat equation (suite)

1. The discretization then of $u_t - \Delta u = f$ is

$$\mathrm{m}(K)(u_K^{n+1} - u_K^n) - \sum_{\sigma \in \mathcal{E}_K} (\mathrm{m}(\sigma)/(d_{K,\sigma})(u_L^{n+1} - u_K^n) \qquad (4)$$

2. (discretization of initial condition used in EGH)

$$u_K^0 = u^0(x_K), \ \forall K \in \mathcal{T} \qquad (5)$$

3. (discretization of the homogeneous Dirichlet boundary condition): in (4), $u_L^{n+1} = 0$ if $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}}$.

# Known result of convergence given in EGH

Result given EGH: The scheme (4)–(5) given before is well posed (has a unique solution) and the following error estimate holds, for a constant $C$ only depending on $\Omega$, $T$, and $u$ :

$$\sum_{K \in \mathcal{T}} \mathrm{m}(K)(u(x_K, t_n) - u_K^n)^2 \leq C(h+k)^2, \forall\, n = 0, \ldots, N, \quad (6)$$

where $h$ is the mesh size of the space discretization, that is $h = \sup\{\mathrm{diam}(K),\ K \in \mathcal{T}\}$

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# Why we look for other error estimates?

So, it is obtained
$\sum_{K\in\mathcal{T}} \mathrm{m}(K)(u(x_K,t_n)-u_K^n)^2 \le C(h+k)^2, \forall\, n=0,\dots,N$.
Some comments on this estimate:

1. (approximation for spatial derivatives of the exact solution): the stated estimate allows only to approximate $u(x_K,t_n)$ (would say the solution itself only), $K\in\mathcal{T}$ and $n=0,\dots,N+1$; what about the approximation of the spatial derivatives. Indeed, it is some time interesting (in practice) to look for the approximation of the *flux* given by $\int_\sigma \nabla u\cdot \mathbf{n}_{K,\sigma}$.
2. (approximation for time derivative of exact solution?).

# An answer for our previous wishes!

The following two contributions, in finite volume, give an answer for previous wishes (would say a finite volume scheme allows us to approximate the exact solution and its first derivatives)

1. Bradji, A.: Some simples error estimates for finite volume approximation of parabolic equations. Comptes Rendus de l'Académie de Sciences, Paris, 346/9-10, 571–574 (2008 ).
2. Bradji, A. and Fuhrmann, J.: Some error estimates in finite volume method for parabolic equations. Proceedings of the 5th International Symposium on Finite Volume for Complex Applications/ Eymard and Hérard (ed.), Wiley, 233–240 (2008).

# How we could obtain previous wishes?

To obtain our wishes, i.e. finite volume scheme allows us to approximate the exact solution and its first derivatives, we have just modified the the approximation of the initial condition. This point will be detailed when we move to explain our NMA (present) contribution.

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# Some disadvantages of our previous works: mesh and space dimension

In our previous two contributions, we obtained an implicit finite volume scheme in which the solution and its first derivatives of the exact solution can be approximation, but,

1. (othogonality property in the mesh): there exists a family of points $(x_K)_{K\in\mathcal{T}}$, such that for a given edge $\sigma_{KL}$, the line segment $x_K x_L$ is orthogonal to this edge.

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# A new finite volume mesh: any space dimension and orthogonality property is not required

A new mesh recenly used in the following reference which deals with stationary equation:

1. Eymard, R., Gallouët, T., Herbin, R.: Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. IMA J. Numer. Anal., Advance Access published on June 16, 2009; doi: doi:10.1093/imanum/drn084.

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# Some advantages of the new mesh

Advantages of this new spatial mesh

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (othogonality property is not required): the othogonality property is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convexe, whereas the convexity property is not required in this new mesh.

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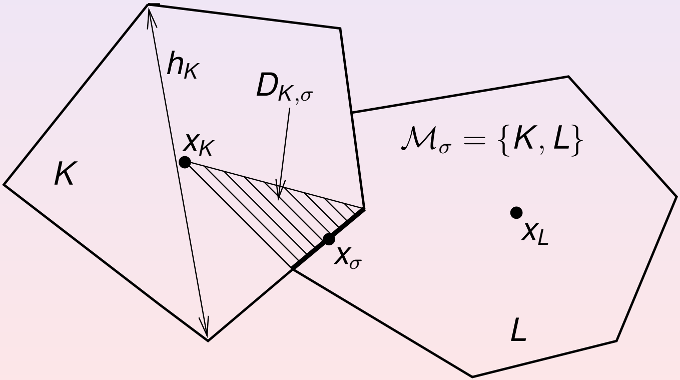
# Figure for the mesh



# Discretization of the domain $\Omega$ and time interval $(0, T)$

Discretization is performed as:

1. Spatial domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is discretized using the new class of meshhes.
2. The time interval $(0, T)$ constant step $k = T/(N+1)$, $N \in \mathbb{N}$.

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# Principles of our scheme

Principles of our scheme

1. discretization of heat equation: the discretization of $u_t - \Delta u = f$ stems from weak formulation (like in finite element method)
$\int_\Omega u_t(x,t)v(x)dx - \int_\Omega \nabla u(x,t) \cdot \nabla v(x)dx = \int_\Omega f(x,t)v(x)dx$, for all $v \in H_0^1(\Omega)$
2. (discretization of initial condition $u(x,0) = u^0(x)$): will be given later
3. (discretization of boundary condition $u(x,t) = 0$, $x \in \partial\Omega$ and $t \in (0,T)$): will be given later

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# Discretization of the equation $u_t - \Delta u = f$

recall
$\int_\Omega u_t(x,t)v(x)dx - \int_\Omega \nabla u(x,t) \cdot \nabla v(x)dx = \int_\Omega f(x,t)v(x)dx$,
for all $v \in H_0^1(\Omega)$

1. (Discrete unknowns): the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \{((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}), \ v_\sigma = 0, \ \forall\sigma \in \mathcal{E}_{\mathrm{ext}}\}$$

2. (discretization of the gradient): the discretization of $\nabla$ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see EGH (IMAJNA).

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# Formulation for the discretization of the equation $u_t - \Delta u = f$

recall
$\int_\Omega u_t(x,t)v(x)dx - \int_\Omega \nabla u(x,t) \cdot \nabla v(x)dx = \int_\Omega f(x,t)v(x)dx$,
for all $v \in H_0^1(\Omega)$.
So a discretization can be done as : For any $n \in [\![ 0, N ]\!]$, find $u_{\mathcal{D}}^n \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\partial^1 \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)} + \left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}} v\right)_{(\mathbb{L}^2(\Omega))^d} = \sum_{K\in\mathcal{M}} \mathrm{m}(K) f_K^n v_K, \ \forall v \in \mathcal{X}_{\mathcal{D},0} \quad (7)$$

# Discretization boundary condition and initial condition

1. (discretization of initial condition $u(x,0) = u^0(x)$): as follows (it is discrete projection for $u^0$): find $u^0_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\nabla_{\mathcal{D}} u^0_{\mathcal{D}}, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^2(\Omega)\right)^d} = -\left(\Delta u^0, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}, \ \forall\, v \in \mathcal{X}_{\mathcal{D},0}, \tag{8}$$

where $\partial^1 v^n = \frac{v^n - v^{n-1}}{k}$, $f^n_K = \dfrac{1}{k\mathrm{m}(K)} \displaystyle\int_{t_n}^{t_{n+1}} \int_K f(x,t)\,dx\,dt$

2. (discretization of boundary condition $u(x,t) = 0$, $x \in \partial\Omega$ and $t \in (0,T)$): is included by the fact that $u^n_\sigma = 0$, for all edges on the boundary $\partial\Omega$

# Remark on the discretization of the initial condition

1. The discretization of initial condition $u(x, 0) = u^0(x)$ can be also performed as

$$u_K^0 = u^0(x_K), \ \forall K, \tag{9}$$

but this choice seems not optimal in the point of wiew of the error estimate.

2. The choice used in our contribution is based on the obvious equation that $u^0$ is the solution of $-\Delta w = g$, where $g = -\Delta u^0$. We will see why this choice is useful

# A main result: error estimates

## Theorem

*For each $n \in [\![ 0, N+1 ]\!]$, let us define the error $e^n_{\mathcal{M}} \in H_{\mathcal{M}}(\Omega)$ by:*

$$e^n_{\mathcal{M}} = \mathcal{P}_{\mathcal{M}}\, u(\cdot, t_n) - \Pi_{\mathcal{M}}\, u^n_{\mathcal{D}}. \tag{10}$$

*Then, the following error estimates hold*

- *discrete $\mathbb{L}^\infty(0, T; H^1_0(\Omega))$–estimate: for all $n \in [\![ 0, N+1 ]\!]$*

$$\| e^n_{\mathcal{M}} \|_{1,2,\mathcal{M}} \leq C(h_{\mathcal{D}} + k) \| u \|_{\mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))}. \tag{11}$$

# A main result: error estimates (Suite)

## Theorem

- $\mathcal{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega))$*–estimate: for all* $n \in [\![ 1, N+1 ]\!]$

$$\| \partial^1 e^n_{\mathcal{M}} \|_{\mathbb{L}^2(\Omega)} \le C(h_{\mathcal{D}} + k) \| u \|_{\mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))}. \tag{12}$$

*where* $\partial^1 v_n = \frac{v_n - v_{n-1}}{k}$*.*

- *error estimate in the gradient approximation: for all* $n \in [\![ 0, N+1 ]\!]$

$$\| \nabla_{\mathcal{D}} u^n_{\mathcal{D}} - \nabla u(\cdot, t_n) \|_{\left(\mathbb{L}^2(\Omega)\right)^d} \le C(h_{\mathcal{D}} + k) \| u \|_{\mathcal{C}^2([0,T];\mathcal{C}^2(\overline{\Omega}))}. \tag{13}$$

# Theorem is useful

1. Estimate (12) allows us to approximate (the time derivative) $u_t(x_K, t_n)$, all $K$ and $n$
2. Estimate (13) allows us to approximate the first spatial derivatives $\frac{\partial u}{\partial x_i}(x_K, t_n)$, all $K$ and $n$.

# An idea on the proof

Comparson with the scheme: for any $n \in [\![ 0, N+1 ]\!]$, find $\bar{u}^n_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left( \nabla_{\mathcal{D}} \bar{u}^n_{\mathcal{D}}, \nabla_{\mathcal{D}} v \right)_{(\mathbb{L}^2(\Omega))^d} = - \sum_{K \in \mathcal{M}} v_K \int_K \Delta u(x, t_n) dx, \ \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{14}$$

We will use mainly the two facts: $(\bar{u}^n_{\mathcal{D}})_{n \in [\![ 0, N+1 ]\!]}$ is an usual approximation for $u$ (for each n, it is like an approximation of elliptic equation) and $u^0_{\mathcal{D}} = \bar{u}^0_{\mathcal{D}}$.

# Some remarks

- (maximum priciple): the classical two finite volume scheme satisfies the maximum principle, whereas this property (maximum principle) is not satified by the schemes generating by the general class of meshes, see Handbook EGH (Proposition 9.4, Page 769).
- (approximation of the gradient): in finite element methods the approximation of the gradient is straigtforward which is not the case of finite volume methods.

# Some remarks (suite)

- (the new mesh and fe element mesh): the general class of meshes of EGH (IMAJNA) allowed the case that the interface between two control volumes can be two straight lines which is not the case in finite element meshes; so for this reason that the meshes of EGH (IMAJNA) are considered as nonconforming meshes.
- (orthogonality property): in the new general class of meshes, the orthogonality property is not required but addtional discrete unknowns are required, i.e. the addtional discrete unknowns are those located on the interfaces of the control volumes.

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# Some remarks (suite)

- (discontinuous matrix diffusion coefficients): the mesh in the classical discretization of elliptic problems discontinuous matrix diffusion coefficients should be adapted to fit the discontinuities of the data, see Hanbook EGH, pages 815–817. This adaptation of the mesh for such problems is not required when we use the general class meshes.

# Conclusion

We considered the heat equation (as model), with initial and homogeneous boundary conditions in any space dimension. An implicit finite volume scheme using a general class of meshes is provided. Thanks to special attention in the discretization of the initial condition, we obtain error estimates of order $h + k$ in several norms which allow us to get approximations for the first derivatives of order $h + k$

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