On the Convergence Order in Sobolev Norms of COMSOL Solutions

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Abstract: The present work is a continuation of the work [1] which is published in the Proceedings of the COMSOL Conference 2007 held in Grenoble-France. In the work [1], we tested the convergence order of the COMSOL solutions in the so called average norm (which does not contain derivative), for several models. We observed that the convergence rate of COMSOL numerics is identical to the theoretically derived convergence rates. It is known that in practice, we need some times to compute an approximation for the gradient of the exact solution. Therefore, it is useful to test the convergence order of these approximations to the gradient of the exact solution. Which requires to test these approximation in some norms containing derivatives. Such norms could be the so called Sobolev norms of order one. In finite element method, there is a huge literature concerning the convergence order, in Sobolev norms of order one, of the finite element solutions towards the exact solution for several models. In the present work, we consider the same models of the work [1], we compute the convergence order of the COMSOL solutions in Sobolev norms of order one, and finally we compare the convergence order with that known in the finite element method. We observe that the convergence order, in Sobolev norms order one, of COMSOL solutions is identical to that known in finite element method.

Keywords: Finite Elements, convergence order, adaptive meshes, Poisson equation, potential equation

1. Introduction

The convergence of a numerical solution towards the analytical solution of a set-up of one or several partial differential equations generally depends on various characteristics of the problem and of the numerical algorithm. The convergence order is a measure for the improvement of the solution as a consequence of mesh refinement.

Some accuracy tests with COMSOL Multiphysics were already presented by Bradji & Holzbecher [1] and Holzbecher and Si [6]. Here we examine several test problems in 2D using classical partial differential equations (Poisson- and Laplace equations), and from different application fields, from electrostatics to fluid dynamics. We compare results from numerical experiments with theoretical results.

The norms in which we evaluate the errors are the so called energy norms which are a particular case of Sobolev norms. Indeed, in practice we need often not only the solution, but also its gradient. Since the goal of COMSOL Multiphysics is to obtain finite element approximations for partial differential equations, we evaluate by which order (rate) the gradient of the COMSOL solutions converges to the gradient of the exact solutions.

The convergence order is known theoretically in the finite element methods. The idea then is to show that the convergence order of the COMSOL finite element approximations is the same one as it is known from finite element theory. For illustration let us consider, for the sake of simplicity, the Laplace equation posed on a square with homogeneous boundary conditions. It is known, see [2], that the linear finite element approximate solution converges to the exact solution by order 'h' in the energy norm, where h denotes the mesh size, towards the exact solution. In test case 1, we show that the COMSOL solution converges by the same order, i.e. h, in the energy norm towards the exact solution.

We show this compatibility between the theoretical results concerning the convergence order of finite element in the energy norm and the convergence order of COMSOL solution in the energy norm, on simple examples.

2. Norms and Convergence Order

The convergence order ϑ is defined by the relationship

$$\|e\| = O\left(h^{\vartheta}\right) \tag{1}$$

where e denotes the error, $\|..\|$ a norm, and h the typical element size. In the following we use the maximum norm

$$\|e\|_{\infty} = \max\left(\left|u_{num} - u_{sol}\right|\right) \tag{2}$$

with the numerical solution u_{num} and the 'real' solution u_{sol}; and the average norm:

$$\|e\|_{2,0} = \sqrt{\int_{\Omega}\left(u_{num} - u_{sol}\right)^2} \tag{3}$$

where Ω denotes the model region. In our test cases the latter is mostly given by an analytical formula. The energy norm is given by

$$\|e\|_{2,1} = \sqrt{\int_{\Omega}\left(u_{num} - u_{sol}\right)^2 + \int_{\Omega}\left(\partial u_{num} - \partial u_{sol}\right)^2} \tag{4}$$

or, explicitly in 2D:

$$\|e\|_{2,1} = \sqrt{\int_{\Omega}\left(u_{num} - u_{sol}\right)^2 dxdy + \int_{\Omega}\left(\frac{\partial u_{num}}{\partial x} - \frac{\partial u_{sol}}{\partial x}\right)^2 dxdy + \int_{\Omega}\left(\frac{\partial u_{num}}{\partial y} - \frac{\partial u_{sol}}{\partial y}\right)^2 dxdy} \tag{5}$$

In test cases in which the integrals can not be evaluated, for instance if u_{sol} is not known explicitly, we replace the evaluation of the integral by the evaluation of a sum of values on a given fixed mesh. We evaluate the average norm (2) using the an approximation of the expression on the right hand side of (2):

$$\|e\|_{2,0} = \sqrt{\sum\left(u_{num} - u_{sol}\right)^2 A_{\Omega}} \tag{6}$$

where A_{Ω} is the measure of the corresponding mesh element. If one uses meshes with elements of similar size, as another approximation, formula (6) can be simplified to:

$$\|e\|_{2,0} = \sqrt{\frac{A}{N}\sum\left(u_{num} - u_{sol}\right)^2} \tag{7}$$

with size of model region A and number of mesh nodes N. Thus, instead of equation (3) we use definition (7) and instead of equation (5) we compute the formula:

$$\|e\|_{2,1} = \sqrt{\frac{A}{N}\left[\sum\left(u_{num} - u_{sol}\right)^2 + \sum\left(\frac{\partial u_{num}}{\partial x} - \frac{\partial u_{sol}}{\partial x}\right)^2 + \sum\left(\frac{\partial u_{num}}{\partial y} - \frac{\partial u_{sol}}{\partial y}\right)^2\right]} \tag{8}$$

In order to determine the convergence order from numerical runs, the errors of runs with different refinement level have to be related. As it is difficult to evaluate the mean size of the elements h, one may alternatively use the degrees of freedom (DOF) for the determination of the convergence rate. From the property (1) and the relation

$$\mathrm{DOF} \sim h^{-d} \tag{9}$$

(where $d \in \{1,2,3\}$ denotes the dimension of the problem) one can easily derive the formula

$$\vartheta = -d\,\frac{\ln\left(\|e_1\|\right) - \ln\left(\|e_2\|\right)}{\ln\left(\mathrm{DOF}_1\right) - \ln\left(\mathrm{DOF}_2\right)} \tag{10}$$

where subscripts denote run number (see also: [7, 8]).

3. Poisson Equation with 1st Type Boundary Condition

3.1 Testcase 1

The first test problem concerns the Poisson equation

$$-\nabla^2 u = 1 \text{ within the unit circle} \tag{11}$$

with boundary condition of Dirichlet type: $u = 0$ at all positions of the circle. The analytical solution is given by:

$$u(x, y) = -(x^2 + y^2 - 1)/4 \tag{12}$$

In case of quadratic Lagrange elements, the COMSOL default setting, one obtains the exact solution in the interior (the solution is a quadratic function); only in the vicinity of the boundaries there are deviations, because the element shape at the boundary follows the geometry and the relationship between local and global coordinates is not linear (see also: COMSOL model library → benchmarks → Poisson unit disk).

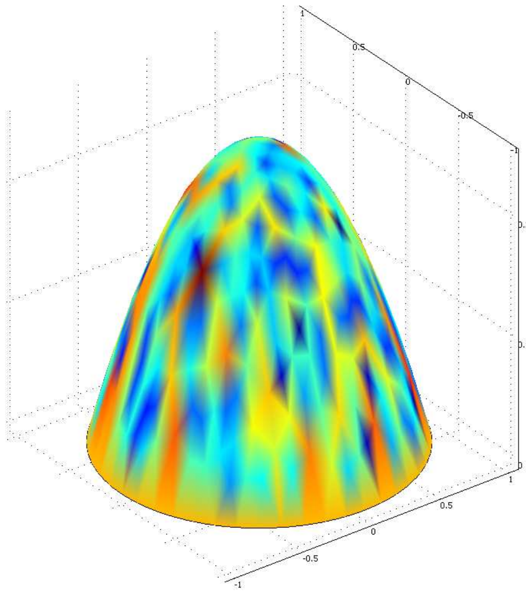

Figure 1. COMSOL numerical solution for the testcase 1 Poisson problem for linear elements and for the coarse grid; the solution is depicted as surface plot; the color represents the error u_{num}-u_{sol} (orange is zero)

Figure 1 depicts the solution and the average error distribution for the testcase. The results in case of linear Lagrange elements, i.e. degrees of freedom (DOF), the error in the average norm and the resulting values for the convergence rate are given in Table 1. The error is evaluated directly with the COMSOL graphical user interface and, after export to the scipt program, on a triangular mesh of 1468 nodes. Both values turn out to be almost identical. We also report the execution time in seconds on a common PC.

Table 1: Results for the Poisson testcase 1 with Dirichlet- boundary conditions (1^{st} order elements, average norm)

DOF	**Time** [s]	$\|e\|_{2,0}\cdot 10^3$	ϑ	
403	0.03-0.094	1.7	2.02	
1561	0.063	0.4391		1.98
6145	0.297	0.1107	2.04	
24385	1.06-1.16	0.0277		1.98
97153	5.81-7.86	0.0069		

For the convergence rate the theoretical value of 2 (see: [2]) for that situation is confirmed by the numerical simulations.
The results in case of linear Lagrange elements, i.e. degrees of freedom (DOF), the error in the energy norm and the resulting values for the convergence rate are given in Table 2. The convergence order shown in Table 1 is 2 in the average norm, which corresponds to theoretical findings for linear finite element approximate solutions [2].

Table 2: Results for the Poisson testcase 1 with Dirichlet-type boundary conditions (1^{st} order elements, energy norm)

DOF	$\|e\|_{2,1}$	ϑ	
403	0.0283	0.98	
1561	0.0146		1.01
6145	0.0073	0.98	
24385	0.0037		1.00
97153	0.0019		

Table 2 shows that the convergence order of the COMSOL solution is 1, as it is known for linear finite element approximate solutions.

Table 3: Results for the Poisson testcase 1 with Dirichlet- boundary conditions (1^{st} order elements, maximum norm)

DOF	$\|e\|_{\infty}\cdot 10^6$	ϑ	
403	360	0.44	
1561	267.5		1.66
6145	85.89	5.07	
24385	2.614		1.77
97153	0.7688		

Results of Table 3 do not provide a clear conclusion concerning the convergence rate. The reason is still unclear.
The results in case of quadratic Lagrange elements, i.e. degrees of freedom (DOF), the error in the average norm and the resulting values for the convergence rate are given in Tables 4 and 5.

Table 4: Results for the Poisson testcase 1 with Dirichlet- boundary conditions (2nd order elements, average norm)

<table>
<tr><th>DOF</th><th>time</th><th>$\|e\|_{2,0} \cdot 10^6$</th><th colspan="2">ϑ</th></tr>
<tr><td>1561</td><td>0.05-0.09</td><td>3.712</td><td rowspan="2">3.58</td><td></td></tr>
<tr><td>6145</td><td>0.250</td><td>0.3192</td><td rowspan="2">3.14</td></tr>
<tr><td>24383</td><td>1.094</td><td>0.0368</td><td rowspan="2">3.15</td></tr>
<tr><td>97163</td><td>5.438</td><td>0.00417</td><td rowspan="2">2.83</td></tr>
<tr><td>387841</td><td>98.0</td><td>0.00059</td><td></td></tr>
</table>

Table 5: Results for the Poisson testcase 1 with Dirichlet- boundary conditions (2nd order elements, energy norm)

<table>
<tr><th>DOF</th><th>$\|e\|_{2,1} \cdot 10^6$</th><th colspan="2">ϑ</th></tr>
<tr><td>1561</td><td>0.3871</td><td rowspan="2">2.17</td><td></td></tr>
<tr><td>6145</td><td>0.0876</td><td rowspan="2">2.04</td></tr>
<tr><td>24383</td><td>0.0215</td><td rowspan="2">2.00</td></tr>
<tr><td>97153</td><td>0.0054</td><td rowspan="2">2.15</td></tr>
<tr><td>387841</td><td>0.0012</td><td></td></tr>
</table>

Table 6: Results for the Poisson testcase 1 with Dirichlet- boundary conditions (2nd order elements, maximum norm)

<table>
<tr><th>DOF</th><th>$\|e\|_{\infty} \cdot 10^6$</th><th colspan="2">$\vartheta$</th></tr>
<tr><td>1561</td><td>13.82</td><td rowspan="2">3.03</td><td></td></tr>
<tr><td>6145</td><td>1.735</td><td rowspan="2">3.01</td></tr>
<tr><td>24385</td><td>0.2174</td><td rowspan="2">3.00</td></tr>
<tr><td>97153</td><td>0.0272</td><td rowspan="2">3.00</td></tr>
<tr><td>387841</td><td>0.0034</td><td></td></tr>
</table>

It has to be noted that there are errors of 2nd order elements only on the boundary, as the boundary line is not approximated exactly by the elements. In the interior the second order approximation fits exactly with the second order solution of the test problem.

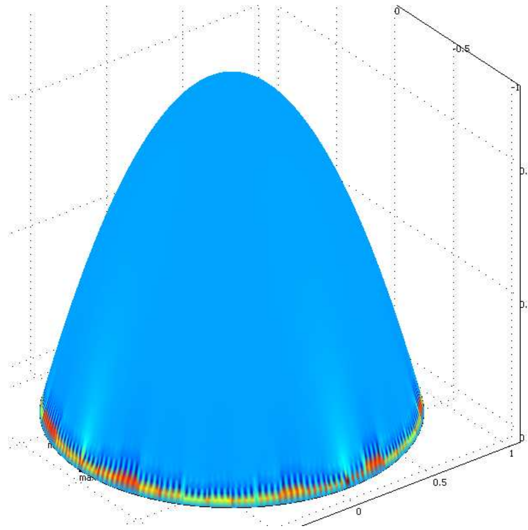

Figure 2. COMSOL numerical solution for the testcase 1 Poisson problem for quadratic elements and for the 2 x refined mesh; the solution is depicted as surface plot; the color represents the error u_{num}-u_{sol} (blue is zero)

3.2 Testcase 2

Another testcase, tackled in the literature [9], concerns the function

$$u(x,y) = \sin(xy)\sin((1-x)(1-y)) \tag{13}$$

in the unit square, which is depicted in Figure 3. It is a solution of the Poisson equation:

$$-\nabla^2 u = 2\left(x(1-x) + y(1-y)\right) \tag{14}$$

The results for linear and quadratic elements are given in the following tables. The execution time (in seconds) for linear elements was: 0.062, 0.11, 0.39, 1.735, 10.2, 271 for the respective runs; and for quadratic elements: 0.125, 0.344, 1.61, 7.6, 279. Default UMFPACK was chosen as a solver in all runs, except for the run for the finest mesh, for which GMRES was used (because of memory problems). The iterative solver obviously converges much slower than UMFPACK.

In maximum norm as well as in the average norm the convergence order for linear elements is 2, the convergence order for quadratic elements is 3. In the energy norm the convergence order is reduced to 1 for linear elements and to 2 for quadratic elements.

Table 7: Results for the Poisson testcase 2 with Dirichlet- boundary conditions for the maximum norm $\|e\|_\infty$; norm superscripts and convergence order ϑ subscripts for element order

DOF	$\|e\|_\infty^1$	ϑ_1		$\|e\|_\infty^2$	ϑ_2	
520	4.1 10^{-4}	1.81				
2017	1.2 10^{-4}		1.97	9.4 10^{-6}		3.00
7945	3.1 10^{-5}	1.97		1.2 10^{-6}	2.98	
31537	8.0 10^{-6}		2.01	1.5 10^{-7}		3.11
1.25 10^5	2.0 10^{-6}	2.00		1.8 10^{-8}	2.97	
5 10^5	5.0 10^{-7}			2.3 10^{-9}		

Table 8: Results for the Poisson testcase 2 with Dirichlet- boundary conditions for the average norm $\|e\|_{2,0}$; norm superscripts and convergence order ϑ subscripts for element order

DOF	$\|e\|_{2,0}^1$	ϑ_1		$\|e\|_{2,0}^2$	ϑ_2	
520	1.3 10^{-4}	2.09				
2017	3.16 10^{-5}		2.0	1.33 10^{-6}		3.03
7945	7.99 10^{-6}	2.0		1.67 10^{-7}	3.02	
31537	2.00 10^{-6}		2.01	2.09 10^{-8}		3.02
1.25 10^5	5.02 10^{-7}	2.01		2.62 10^{-9}	2.99	
5 10^5	1.25 10^{-7}			3.3 10^{-10}		

For coarse meshes the error for quadratic elements is much less than for linear elements, even if compared for the same DOF.

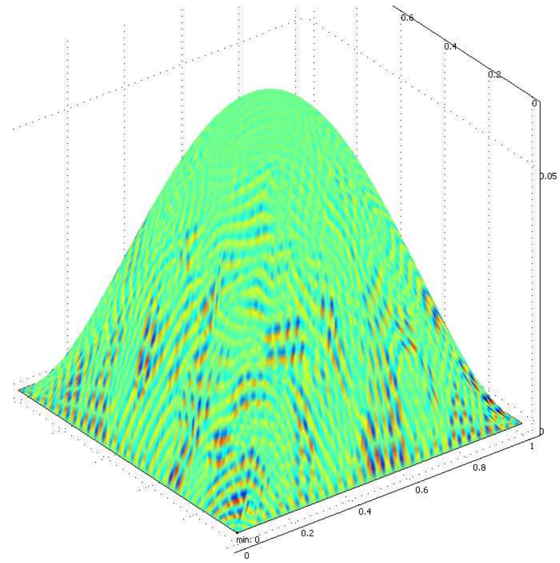

Figure 3. COMSOL numerical solution for the testcase 2 Poisson problem for quadratic elements and for the 2 x refined mesh; the solution is depicted as surface plot; the color represents the error u_{num}-u_{sol} (green is zero, red and blue show positive and negative deviations in the numerical solution)

Table 9: Results for the Poisson testcase 2 with Dirichlet- boundary conditions for the energy norm $\|e\|_{2,1}$; norm superscripts and convergence order ϑ subscripts for element order

DOF	$\|e\|_{2,1}^1$	ϑ_1		$\|e\|_{2,1}^2$	ϑ_2	
520	8.6 10^{-3}	1.0				
2017	4.3 10^{-3}		1.01	2.29 10^{-4}		2.05
7945	2.2 10^{-3}	1.0		5.60 10^{-5}	1.96	
31537	1.1 10^{-3}		1.0	1.45 10^{-5}		2.01
1.25 10^5	5.5 10^{-4}	1.01		3.63 10^{-6}	2.00	
5 10^5	2.73 10^{-4}			9.06 10^{-7}		

4. Poisson Equation with Dirac Right Hand Side

The third test problem concerns the differential equation

$$-\nabla^2 u = \delta(0) \text{ within the unit circle} \tag{15}$$

with Dirac's δ-function on the right hand side. The boundary conditions are again of Dirichlet type: $u = 0$ on all positions of the unit circle. The analytical solution is given by the logarithmic function:

$$u(x,y) = -\ln(r)/2\pi = -\ln(r^2)/4\pi \tag{16}$$

(see also: 'Implementing a point source' in the COMSOL Users Guide and the point-source benchmark example). The solution is visualized in Figure 4.

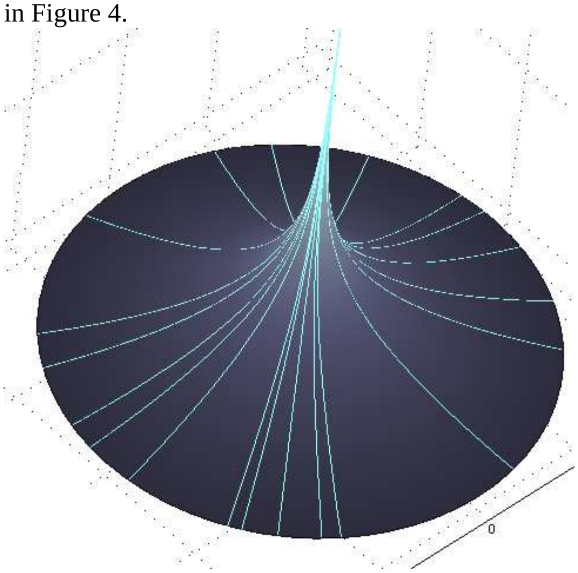

Figure 4: COMSOL numerical solution for the Poisson problem with Dirac right hand side; the figure shows the solution as surface, error as grey scale (dark: small; light: high), selected streamlines

As the right hand side of the equation is has a singularity, the classical results for regular right hand sides are not valid and a lower convergence rate can be expected. Bradji & Holzbecher [1] already showed that COMSOL solutions show the expected behavior with a convergence rate reduced to 1 (for quadratic elements and average norm).

Here we chose the linear and quadratic elements, in the average norm. Table 10 lists the results. It is again confirmed that the cubic convergence, which is valid for the smooth right hand side, can obviously not be reached for the singular right hand side. The theoretical result of first order convergence for 2nd order elements [3] is confirmed.

For 1st order elements the convergence rate is obviously better, but it is strongly decreasing. Comparison of the results for the same DOF shows the better performance of the higher order elements, which are also better with respect to the execution time.

Table 10: Results for the Poisson problem with Dirac right hand side (1st and 2nd order elements, average norm)

First order elements				
DOF	**time**	$\|e\|_{2,0} \cdot 10^3$	ϑ_1	
777	0.046-0.06	0.9784	1.97	
3041	0.125	0.2557		1.90
12033	0.5-0.547	0.0692	1.69	
47873	2.344	0.0214		1.37
190977	18.11	0.0083		
Second order elements				
DOF	**time**	$\|e\|_{2,0} \cdot 10^3$	ϑ_2	
3041	0.141	0.0601	1.13	
12033	0.515	0.0277		1.01
47873	2.563	0.0138	1.00	
190977	13.95	0.0069		

5. Conclusions and Outlook

For two test cases of the classical Poisson equation with regular right hand side we found convergence rates of 2 (for linear elements) and 3 (for quadratic elements) for average norm. For the Sobolev norm, in which also the derivatives are considered, the convergence rate is reduced to 1 in case of linear, and 2 in case of quadratic elements.

In addition to the simulations in the energy norms (Sobolev norms of order one), we also have attempted to compute the convergence order of the COMSOL solutions in the maximum norm. Simulation results showed that convergence order is three in the maximum norm when we use quadratic elements (2nd order elements), see Table 6 of Test case 1 and Tables 6 and 7 of Test case 2. This is the same order as in the theory of finite element methods when we use quadratic elements.

Whereas, we did not observe the known theoretical order $h^2\left|\log(h)\right|$ for linear finite element (1st order elements) , see Table 3 of Test case 1 and Table 7 in the Test case 2.

This perhaps could be a nice path to be followed in the future.

We also examined the situation with a degenerated right hand side and confirmed that that also leads to a significant reduction of the convergence rate.

This present work and its previous work [1] are intended as an introduction to study the behavior of COMSOL Multiphysics finite element solutions in the light of theoretical results.

In the future work we will focus on the convergence order of the COMSOL solution of more complicated models in the higher order Sobolev norms. We will focus for instance on the convergence order of the COMSOL approximations of both the steady state and non steady state of Navier Stokes equations.

Another topic to study is at which level the maximum principle is satisfied by the COMSOL finite element solutions.

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