Consistency in finite difference for fractional differential equations

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Abstract: The aim of this note is to prove some error estimate for the truncation error for approximation to fractional derivative. This estimate is useful in order to get a consistency for a finite difference scheme approximating fractional differential equations.

The aim is to prove the following statement

THEOREM 0.1 (cf. [TAD 04]) Let f be a smooth function defined on $(0,1)$ such that $f(0) = f(1) = 0$. Let $h = \frac{1}{N}$ and $\alpha \in]1,2[$. Then the following estimate holds:

$$\frac{\partial^\alpha f}{\partial x^\alpha} = \frac{1}{h^\alpha}\sum_{k=0}^{N} g_k f(x-(k-1)h) + O(h), \qquad [1]$$

where

$$\frac{\partial^\alpha f}{\partial x^\alpha} = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{x^2}\int_0^x \frac{f(\xi)}{(x-\xi)^{\alpha-1}}d\xi. \qquad [2]$$

To prove Theorem, we need some preliminary Lemmata.

LEMMA 0.2 (cf. [TAD 04]) Let $f \in \mathcal{C}^1(\mathbb{R})$ such that f and f' are belonging to $\mathbb{L}^1(\mathbb{R})$. Then the following estimate holds, for some constant C:

$$|\hat{f}(\xi)| \leq C(1+|\xi|)^{-1}, \qquad [3]$$

where $\hat{f}$ denotes the usual Fourier transform given by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\exp(-i\xi\, x)dx, \qquad [4]$$

and i is the complex number satisfying $i^2 = -1$.

To prove Lemma 0.2, we will use the following two Lemma which called Riemann–Lebesgue Lemma, see for example [ALL 90, Lemme 3, Page 476]

LEMMA 0.3 (RIEMANN–LEBESGUE LEMMA) Let $f \in \mathbb{L}^1(\mathbb{R})$. Then

$$\lim_{|\xi|\to+\infty}\hat{f}(\xi) = 0. \qquad [5]$$

Proof of Lemma 0.2 Let us consider $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ with support compact in $\mathbb{R}$ (it is denoted some time by $D(\mathbb{R})$). Using an integration by parts, we find Assume that

$$\int_{\mathbb{R}} \varphi(x) \exp\left(-i\xi\, x\right) dx = \frac{i}{\xi} f(x) \exp\left(-i\xi\, x\right)|_{-\infty}^{+\infty} - \frac{i}{\xi} \int_{\mathbb{R}} \varphi'(x) \exp\left(-i\xi\, x\right) dx. \qquad [6]$$

Using then the fact that φ vanishes on $-\infty$ and $+\infty$, [6] implies

$$\xi \int_{\mathbb{R}} \varphi(x) \exp\left(-i\xi\, x\right) dx = -i \int_{\mathbb{R}} \varphi'(x) \exp\left(-i\xi\, x\right) dx. \qquad [7]$$

This with the Riemann–Lebesgue Lemma with $f := \varphi'$ in [5], we get

$$\lim_{|\xi| \to +\infty} \xi \int_{\mathbb{R}} \varphi(x) \exp\left(-i\xi\, x\right) dx = 0. \qquad [8]$$

We also have thanks to Riemann–Lebesgue Lemma with $f := \varphi$ in [5]

$$\lim_{|\xi| \to +\infty} \int_{\mathbb{R}} \varphi(x) \exp\left(-i\xi\, x\right) dx = 0. \qquad [9]$$

Limits [8] and [9] imply that

$$\lim_{|\xi| \to +\infty} (1+\xi) \int_{\mathbb{R}} \varphi(x) \exp\left(-i\xi\, x\right) dx = 0. \qquad [10]$$

Let then a function $f \in \mathbb{L}^1(\mathbb{R})$. By density, there exists $\varphi_n \in D(\mathbb{R})$ such $\varphi_n \to f$ as $n \to \infty$. Using [10] yields that

$$\lim_{|\xi| \to +\infty} (1+\xi) \int_{\mathbb{R}} \varphi_n(x) \exp\left(-i\xi\, x\right) dx = 0. \qquad [11]$$

Which implies that, since $\varphi_n \to f$

$$\lim_{|x| \to +\infty} (1+\xi) \int_{\mathbb{R}} f(x) \exp\left(-i\xi\, x\right) dx = 0. \qquad [12]$$

This implies that$(1+\xi) \int_{\mathbb{R}} f(x) \exp\left(-i\xi\, x\right) dx$ is bounded. There exists a constant $C > 0$ such that

$$(1+\xi) \int_{\mathbb{R}} f(x) \exp\left(-i\xi\, x\right) dx \leq C. \qquad [13]$$

Which means that

$$(1+\xi)\hat{f}(\xi) \leq C, \qquad [14]$$

which completes the proof of Lemma 0.2.

References

[ALL 90] Allab, Kada: Elements d'Analyse. *Entreprise Nationale du Livre, Alger*, 1990.

[TAD 04] Tadjeran, Charles;Meerschaert, Mark M.; Scheffler, Hans-Peter: A second-order accurate numerical approximation for the fractional diffusion equation. *J. Comput. Phys.*, **213**, No. 1, 205–213 (2006).