University Habilitation

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Plan

- **Aim of this presentation**
- **Curriculum vitae**
- **Research interest and some achievements**
  - **Improved convergence order of numerical solutions**
  - **Numerical approximation of problems with irregular data**
  - **Uses of COMSOL Multiphysics**
- **Some works under preparation**
- **Some perspectives**

Aim of this presentation

Aim of this presentation

The aim of this presentation is to give some idea about :

- *my career as a teacher*
- *my career as researcher*
- *some of my works under preparation*
- *some of my perspectives*

Curriculum vitae : 1–Education

- November 14th 2005 : Ph.D Thesis in Applied Mathematics, University of Marseille, France. Thesis Advisor : Prof. T. Gallouët.
  Title of Thesis : Improved Convergence Order in Finite Volume and Finite Elements Methods.
- Nov. 2002–Nov. 2005 : University of Marseille, France, Ph.D in Applied Mathematics
- July 1996 : Magister in Applied Mathematics, Annaba, Algeria
- Sep. 1994–July 1996 : University of Annaba, Algeria, Master in Applied Mathematics

Curriculum vitae : 1–Education (suite)

- Oct. 1993–July 1994 : DEA in Applied Mathematics, Annaba
- July 1993 : DES in Mathematics, Annaba, Algeria.

Curriculum vitae : 2–Positions

- Since 1999 : "Enseignant Chercheur" at the University of Annaba, Algeria.
- Mars 1 st 06–Mai 31 st 07 : Postdoc in WIAS : Weierstrass Institute for Applied Analysis and Stochastics, Berlin, Germany
- Sept. 04–Aug. 05 : (ATER) Teaching Assistant, University of Marseille, France

Curriculum vitae : 2–Positions (suite)

- Sep. 97-Mars 99 : Military Service ; Mars 15th 98–Mars 15th 99 : Teacher of Mathematics in "Académie Militaire Interarmes" (AMIA) de Cherchell, Algérie
- Oct. 93-Jan.97 : (Assistant Technique) Lecturer in Mathematics, University of Annaba, Algeria

Curriculum vitae : 3–Courses Taught

- Undergraduate (Tronc Commun) : Analysis, Statistics, Numerical Analysis, Algebra, Undergraduate.
- Graduation (DES) : Numerical Analysis.
- Postgraduation (Master) : Numerical Analysis.
- Supervision (Encadrement) of two students of Master, in Marseille in a project entitled : Coupled System with Irregular Data.

Curriculum vitae : 4–Research projects

I'm the chef of a project, B01120090113, accepted starting from January 2010 "L'Analyse mathématique et numérique de la récuperation assistée des hydrocarbures"

Curriculum vitae : 5–Other activities

- Reviewer for Mathematical Reviews of American Mathematical Society (AMS) since March 31, 2008
- Reviewer for Zentralblatt MATH since April 23, 2008

Research interest and some achievements : 1–Improved convergence order of numerical solutions

Introduction : basic knowledge

Finite differences, elements, volumes are among the more popular methods to approximate differential or partial differential equations.
We consider Ω :

$$\mathcal{L}\, u(x) = f(x), \; x \in \Omega \qquad [1]$$

Principle of Finite Differences methods : The basic idea of the finite difference methods is to choose some points belonging to Ω, called *mesh points* and we approximate the derivatives which appear in $\mathcal{L}$ by *difference quotients*.

Example of a finite difference scheme

Let us consider

$$-u_{xx}(x) = f(x),\ x \in (0,1), \qquad [2]$$

with Dirichlet boundary conditions $u(0) = u(1) = 0$ We consider the *uniform mesh* $x_i = ih$, $i \in \{0, \ldots, N\}$, with $N \in \mathbb{N} \setminus \{0\}$. Replacing x by x_i in [2], we get

$$-u_{xx}(x_i) = f(x_i),\ i \in \{1, \ldots, N-1\}, \qquad [3]$$

Example of a finite difference scheme (suite)

Recall

$$-u_{xx}(x_i) = f(x_i),\ i \in \{1, \ldots, N-1\}, \qquad [4]$$

We approximate $u_{xx}(x_i)$ by the *difference quotient* $\frac{u(x_{i+1})-2u(x_i)+u(x_{i-1})}{h^2}$.

The finite difference schme then could be given as

$$-\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} = f(x_i),\ i \in \{1, \ldots, N-1\}, \qquad [5]$$

with, thanks to the homogeneous boundary condition, $u_0 = u_N = 0$.

Principle of finite element methods

Recall

$$\mathcal{L}\, u(x) = f(x),\ x \in \Omega \qquad [6]$$

Principle of Finite element methods : Finite element method are first based on *weak formulation* for the problem (6) : Find $u \in \mathcal{H}$

$$a(u, v) = \mathcal{F}(v),\ \forall v \in \mathcal{V}, \qquad [7]$$

where $a(\cdot, \cdot)$ (resp. $\mathcal{F}(\cdot)$) is a bilinear form (resp. linear form). In general the sets $\mathcal{H}$ and $\mathcal{W}$ are given in terms of the so called Sobolev spaces.
The finite element method is based, in general, on the approximation of the sets $\mathcal{H}$ and $\mathcal{W}$ by *piecewise polynomes*

Example of a finite element scheme

Let us consider

$$-u_{xx}(x) = f(x),\ x \in (0,1), \qquad [8]$$

with Dirichlet boundary conditions $u(0) = u(1) = 0$.
A *weak formulation* for [8] could be given as : Find $u \in H_0^1(0,1)$

$$\int_0^1 u_x(x) v_x(x) dx = \int_0^1 f(x) v(x) dx,\ \forall\, v \in H_0^1(0,1). \qquad [9]$$

We consider the *uniform mesh* $x_i = ih$, $i \in \{0, \ldots, N\}$, with $N \in \mathbb{N} \setminus \{0\}$.
The space $H_0^1(0,1)$ could be approximated, for instance, by *piecewise linear polynomes* $\mathcal{V}^h$, that is

$$\mathcal{V}^h = \{v \in \mathcal{C}(0,1),\ v|_{(x_i, x_{+1})} \in \mathcal{P}_1\}. \qquad [10]$$

Example of a finite element scheme (suite)

Recall

$$-u_{xx}(x) = f(x),\ x \in (0,1), \qquad [11]$$

with $u(0) = u(1) = 0$.

Weak formulation : Find $u \in H_0^1(0,1)$

$$\int_0^1 u_x(x)v_x(x)dx = \int_0^1 f(x)v(x)dx,\ \forall\, v \in H_0^1(0,1). \qquad [12]$$

Finite element space :

$$\mathcal{V}^h = \{v \in \mathcal{C}(0,1),\ v|_{(x_i,x_{+1})} \in \mathcal{P}_1\}. \qquad [13]$$

Finite element scheme : Find $u^h \in \mathcal{V}^h$

$$\int_0^1 u_x^h(x)v_x^h(x)dx = \int_0^1 f(x)v^h(x)dx,\ \forall\, v^h \in \mathcal{V}^h. \qquad [14]$$

Principle of finite volume methods

Definition

Let $\mathcal{T}$ be a finite volume mesh

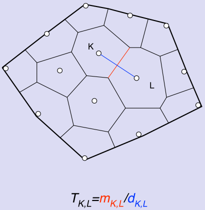


FIG.: transmissivity between K and L : $T_{K,L} = \frac{m_{K,L}}{d_{K,L}}$

Principle of finite volume methods (suite)

The sub polygonal domains called control volumes of the finite volumes mesh $\mathcal{T}$.

The finite volume scheme is based on the :

- integration of the equation to be solved on each control volume
- integration by parts to transfer the integration on the control volumes to integration on the interfaces across the control volumes.
- we use the transmissibility property to approximate the derivatives which appear after integration

First example of a FV scheme : 1D example

$$-u_{xx}(x) = f(x),\ x \in (0,1) \text{ and } u(0) = u(1) = 0. \qquad [15]$$

Definition

Finite volume mesh $\mathcal{T}$ is given by

- *the control volumes $K_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, $i \in \{1, \ldots, N\}$, for a given $N \in \mathbb{N} \setminus \{0\}$, with $x_{\frac{1}{2}} = 0$ and $x_{N+\frac{1}{2}} = 1$,*
- *given points $\{x_i,\ i = 0 \ldots N+1\}$ such that $x_0 = 0$, $x_{N+1} = 1$ and $x_i \in K_i$, for all $i \in \{1, \ldots, N\}$.*

First example of a finite volume methods scheme : one dimension example (suite)

Recall

$$-u_{xx}(x) = f(x),\ x \in (0,1) \text{ and } u(0) = u(1) = 0. \qquad [16]$$

Integrating the differential equation of [16] on each control volume $K_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ yields

$$u_x(x_{i-\frac{1}{2}}) - u_x(x_{i+\frac{1}{2}}) = \int_{K_i} f(x)dx,\ \forall\ i \in \{1, \ldots, N\} \qquad [17]$$

$u_x(x_{i-\frac{1}{2}}) \simeq \frac{u(x_i)-u(x_{i-1})}{h_{i-\frac{1}{2}}}$, where $h_{i-\frac{1}{2}} = x_i - x_{i-1}$.

First example of a finite volume methods scheme : one dimension example (suite)

Recall, after integration on each control volume $K_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, we got

$$u_x(x_{i-\frac{1}{2}}) - u_x(x_{i+\frac{1}{2}}) = \int_{K_i} f(x)dx, \ \forall\, i \in \{1, \ldots, N\} \qquad [18]$$

$u_x(x_{i-\frac{1}{2}}) \simeq \frac{u(x_i)-u(x_{i-1})}{h_{i-\frac{1}{2}}}$, where $h_{i-\frac{1}{2}} = x_i - x_{i-1}$.

Finite volume schemes :

$$\frac{u_i - u_{i-1}}{h_{i-\frac{1}{2}}} - \frac{u_{i+1} - u_i}{h_{i+\frac{1}{2}}} = \int_{K_i} f(x)dx, \ \forall\, i \in \{1, \ldots, N\}, \qquad [19]$$

with $u_0 = u_{N+1} = 0$.

Second example of FV scheme : 2D example

Recall

$$-\Delta u(x) = f(x),\ x \in \Omega \text{ and } u(x) = 0\ \forall x \in \partial\Omega. \qquad [20]$$

where Ω is a bounded polygonal connected subset of $\mathbb{R}^2$.
Consider finite volume mesh $\mathcal{T}$, as it is stated before ; see before.
Integrating the partial differential equation of [20] on each control volume K yields

$$-\int_K \Delta u(x)dx = \int_K f(x)dx,\ \forall\ K \in \mathcal{T}. \qquad [21]$$

Integration by part gives

$$-\int_{\partial K} \nabla u(x) \cdot \mathbf{n}_K(x)dx = \int_K f(x)dx,\ \forall\ K \in \mathcal{T}. \qquad [22]$$

Second example of a FV scheme : 2D example (suite)

Recall

$$-\int_{\partial K} \nabla u(x) \cdot \mathbf{n}_K(x) dx = \int_K f(x) dx, \ \forall\, K \in \mathcal{T}, \qquad [23]$$

where $\mathbf{n}_K$ is the normal vector to ∂K outward to K.
Since K is a polygone, we then sum [23] on the edges of K

$$-\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla u(x) \cdot \mathbf{n}_K(x) dx = \int_K f(x) dx. \qquad [24]$$

Transmissiblity property (othogonality), see figure

$$\int_\sigma \nabla u(x) \cdot \mathbf{n}_K(x) \simeq \mathrm{m}(\sigma) \frac{u(x_L) - u(x_K)}{d_{K,L}}$$

Second example of FV scheme : 2D example (suite)

Recall

Equation to be solved

$$-\Delta u(x) = f(x),\ x \in \Omega \text{ and } u(x) = 0\ \forall x \in \partial\Omega. \qquad [25]$$

Principle of the finite volume approximation

$$\int_\sigma \nabla u(x) \cdot \mathbf{n}_K(x) \simeq \mathrm{m}(\sigma)\frac{u(x_L) - u(x_K)}{d_{K,L}}$$

Finite volume scheme could be suggested then as

$$-\sum_{\sigma\in\mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{d_{K,L}}(u_L - u_K) = \int_K f(x)dx. \qquad [26]$$

Remarks on finite volume and finite difference methods

- finite difference method is based on the approximation of the derivatives, which appear in the differential or partial equation, whereas the finite volume method is based on the approximation of these derivatives after their integration. This means that we gain a derivative in finite volume method with respect to finite difference method.

- we can prove that the finite volume schemes are not consistent in the sense of finite difference method, e.g.

$$\frac{1}{h_i}\left(\frac{u(x_i)-u(x_{i-1})}{h_{i-\frac{1}{2}}}-\frac{u(x_{i+1})-u(x_i)}{h_{i+\frac{1}{2}}}\right)\not\to 0. \qquad [27]$$

What does it mean improved convergence order of numerical solutions

Problem to be solved

$$\mathcal{L}\, u(x) = f(x),\ x \in \Omega. \qquad [28]$$

Numerical approximation leads to algebraic system to be solved

$$\mathcal{A}^{\mathcal{T}}\, u_{\mathcal{T}} = f_{\mathcal{T}}, \qquad [29]$$

where $\mathcal{A}^{\mathcal{T}}$ is matrix, $u_{\mathcal{T}}$ the numerical approximation, may be a finite difference, element, or volume solution.

What does it mean improved convergence order of numerical solutions

To get a higher order approximation $u_{\mathcal{T}}$, we either have

- finite difference or finite volume methods : we choose difference quotients of higher order. This oblige us to increase the number of points used in these difference quotients

- finite element methods : we have to increase the degree of the finite element spaces

These issues lead to complex algebraic system to be solved which increases the cost of the computation.

Sense of improving convergence order

Issue : we use a scheme of lower order, the matrix is nice in many cases, e.g tridiagonal, bloc tridiagonal,...

$$\mathcal{A}^{\mathcal{T}} u_{\mathcal{T}} = f_{\mathcal{T}}. \qquad [30]$$

- computation of the truncation error

$$\mathcal{A}^{\mathcal{T}} u = f_{\mathcal{T}} + \delta_1 + \delta_2, \qquad [31]$$

  δ_2 is of higher order
- we resolve the system :

$$\mathcal{A}^{\mathcal{T}} e = \delta_1 \qquad [32]$$

Sense of improving convergence order of numerical solutions

- we correct u by e to obtain the solution $u_1^{\mathcal{T}} = u_{\mathcal{T}} - e_{\mathcal{T}}$; we expect that u_1 is of higher order since

$$\mathcal{A}^{\mathcal{T}}\left(u - u_1^{\mathcal{T}}\right) = \delta_2. \qquad [33]$$

Definition of improving convergence order of numerical solutions

Definition (Improving convergence order)

- *lower order scheme in which the matrix is nice (tridiagonal, bloch tridiagonal, sparse)*

$$\mathcal{A}^{\mathcal{T}} u_{\mathcal{T}} = f_{\mathcal{T}}. \qquad [34]$$

- *we correct the solution* $u_{\mathcal{T}}$, *by a defect* $\mathcal{D}$, *to produce new approximation* $u_1^{\mathcal{T}} = u_{\mathcal{T}} + \mathcal{D}$ *of higher order*

$$\mathcal{A}^{\mathcal{T}} u_1^{\mathcal{T}} = f_{\mathcal{T}}^1. \qquad [35]$$

Some of my contributions in improving convergence order of numerical solutions

- Towards an approach to improve convergence order in finite volume and finite element methods.
  Proceedings of ICNAAM "International Conference in Numerical Analysis and Applied Mathematics", 1162–1165, 2009.

- Some simples error estimates for finite volume approximation of parabolic equations.
  Comptes Rendus de l'Académie de Sciences, Paris, 346/9-10 pp. 571-574, 2008.

Some of my contributions in improving convergence order of numerical solutions (suite)

- Some error estimates in finite volume methods for parabolic equations.
  With J. Fuhrmann. Finite Volumes for Complex Applications V, Proceedings of the 5th International Symposium on Finite Volume for Complex Applications.

- Optimal defect corrections on composite nonmatching finite element meshes.
  With A.-S. Chibi. IMA Journal of Numerical Analysis, **27** (4), 765– 780, 2007

Some of my contributions in improving convergence order of numerical solutions (suite)

- Error Estimate for Finite Volume Approximate Solutions of Some Oblique Derivative Boundary Problems.
  With T. Gallouët. International Journal on Finite Volumes. **3** (2), 35 pages (electronic), 2006

- Impropved Convergence Order for Finite Volume Solutions. Part I : 1D Problems.
  With B. Atfeh. Arab Journal of Mathematical Sciences. **11** (1), 1–30, 2005. With B. Atfeh. Arab Journal of Mathematical Sciences. **11** (2), 1–53, 2005.

Simulation of ohmic losses

- Let Ω domain of $\mathbb{R}^d$, $1 \leq d \leq 3$ made up of a Thermally and Electrically conducting material.
- u (resp. ϕ) denotes the temperature (resp. Electrical potential).
- κ (resp. λ) Electrical Conductivity (resp. Thermal conductivity).
- Diffusion of Electricity in a resistive medium induces some heating, knows as **Ohmic Losses**.
- The **Ohmic Losses** may be written as $\kappa|\nabla\phi|^2$

A model subproblem

$$-\nabla(\kappa(x,u(x))\nabla\phi(x)) = f(x,u(x)),\ x \in \Omega$$

$$\phi(x) = 0,\ x \in \partial\Omega,$$

$$-\nabla(\lambda(x,u(x))\nabla u(x)) = \kappa(x,u(x))|\nabla\phi|^2(x),\ x \in \Omega,$$

$$u(x) = 0,\ x \in \partial\Omega,$$

κ, λ, f bounded functions from $\Omega \times \mathbb{R}$ to $\mathbb{R}$,
continuous with respect to $y \in \mathbb{R}$ for a.e. $x \in \Omega$, and
measurable with respect to $x \in \Omega$ for any $y \in \Omega$

$$\exists\alpha > 0;\ \alpha \le \kappa(x,y) \text{ and } \alpha \le \lambda(x,y),\ \forall y \in \mathbb{R},\ \text{for a.e. } x \in \Omega.$$

Some difficulties

Recall that :

$$(\textbf{Problem}) \begin{cases} -\nabla(\kappa(x,u(x))\nabla\phi(x)) = f(x,u(x)), \ x \in \Omega \\ \phi(x) = 0, \ x \in \partial\Omega, \\ -\nabla(\lambda(x,u(x))\nabla u(x)) = \kappa(x,u(x))|\nabla\phi|^2(x), \ \Omega \\ u(x) = 0, \ x \in \partial\Omega, \end{cases}$$

- The Problem is coupled ;
- Each Equation is Nonlinear.
- If we assume that $\phi \in H^1(\Omega)$, $\kappa(\cdot,u(\cdot))|\nabla\phi|^2 \in L^1(\Omega)$ (in which the classical tools do not work, for instance, Lax Milgram lemma).

Weak form

$$(WF)\begin{cases}(\phi, u) \in H_0^1(\Omega) \times \bigcap_{p<\frac{d}{d-1}} W_0^{1,p}(\Omega),\\ \int_\Omega \kappa(\cdot, u)\nabla\phi\cdot\nabla\psi\,dx = \int_\Omega f(\cdot, u)\psi\,dx,\ \forall\psi \in H_0^1(\Omega)\\ \int_\Omega \lambda(\cdot, u)\nabla u\cdot\nabla v\,dx = \int_\Omega \kappa(\cdot, u)|\nabla\phi|^2 v\,dx\\ \qquad\qquad \forall v \in \bigcup_{r>d} W_0^{1,r}(\Omega),\end{cases}$$

- Gallouët-Herbin. 94 : Existence of a solution to (WF). Adaptation of techniques of Boccardo-Gallouët for elliptic problems with L^1 and measure data, see for instance (and References therein),
- Result also obtained here as a by-product of the convergence of the schemes.

Some remarks from the existence of Gallouët-Herbin. 94

- $\phi \in H_0^1(\Omega)$, we expect (but we should wait u) to find a numerical solution converges to ϕ (using classical finite element and finite volume methods).
- $u \in \bigcap_{p<2} W_0^{1,p}(\Omega),\ d = 2$ (resp. $\bigcap_{p<\frac{3}{2}} W_0^{1,p}(\Omega)$, $d = 3$). Which means that $u \notin H^1(\Omega)$. **The classical tools of numerical analysis do not work**

Principles of finite volume method, Examples

We assume that that the finite element methods is well known.
Consider the Diffusion Equation :

$$(Ex.)\begin{cases} -\Delta u = g, \ \Omega \\ u = 0, \ \partial\Omega \end{cases}$$

We assume that : $g \in L^2(\Omega)$ (means that $\int_\Omega g^2(x)dx < \infty$).
Weak formulation for (Ex.) :

$$(WF\,Ex.)\begin{cases} \phi \in H_0^1(\Omega), \\ \int_\Omega \nabla\phi \cdot \nabla\psi \, dx = \int_\Omega g\psi \, dx, \ \forall \psi \in H_0^1(\Omega). \end{cases}$$

Finite Volume scheme for (Ex.)

Recall that

$$(Ex.)\begin{cases} -\Delta u = g, \ \Omega \\ u = 0, \ \partial\Omega \end{cases}$$

$$-\int_{\partial K} \nabla u \cdot \mathbf{n}_K dx = \int_K g dx$$

$\mathbf{n}_K$: unit normal vector to ∂K outward to K.

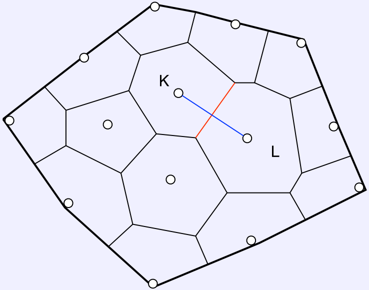
$$\partial K = \bigcup_{\sigma \in \mathcal{E}_K} \sigma \qquad \rightsquigarrow$$

$$-\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla u \cdot \mathbf{n}_{K,\sigma} dx = \int_K g dx$$

Finite Volume scheme for (Ex.)

Orthogonal Property :

$\nabla u \cdot \mathbf{n}_{K,\sigma}$ can be approximated by $\frac{u(x_L)-u(x_K)}{d_{K,L}}$ (result from an easy application of Taylor expansion).



The Algebraic system to be solved for (Ex.)

Recall that

$$-\sum_{\sigma\in\mathcal{E}_K}\int_\sigma \nabla u\cdot \mathbf{n}_{K,\sigma}dx=\int_K gdx$$

and $\nabla u\cdot \mathbf{n}_{K,\sigma}$ can be approximated by $\frac{u(x_L)-u(x_K)}{d_{K,L}}$. **The Algebraic System for Laplace's Equation (Ex.) Generated by the use of Finite Volume Method :**

$$-\sum_{\sigma\in\mathcal{E}_K}\frac{\mathrm{m}(\sigma)}{d_{K,L}}(u_L-u_K)=\int_K gdx$$

(of course, we need some **Boundary Conditions on** u_K).

The Algebraic system to be solved for (Ex.)

Recall that

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$$-\sum_{\sigma\in\mathcal{E}_K}\frac{\mathrm{m}(\sigma)}{d_{K,L}}(u_L-u_K)=\int_K g dx$$

(of course, we need some **Boundary Conditions on** u_K).

The Convergence of the Finite Volume Scheme for (Ex.)

To analyse the Convergence of the finite volume solution, we need the following useful Definition :

Definition

Let Ω be a Polygonal Domain in $\mathbb{R}^d$ ($d = 2$ or 3), and $\mathcal{T}$ a finite volume discretization.
Define $\mathcal{X}(\mathcal{T}) \subset L^2(\Omega)$ as the set of functions from Ω to $\mathbb{R}$ which are constant over each control volume K.

Covergence result for Laplace Equation when $g \in L^2(\Omega)$

Recall that

$$-\sum_{\sigma \in \mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{d_{K,L}}(u_L - u_K) = \int_K g dx$$

and $g \in L^2$

Theorem (Convergence, FVM approximation for smooth data in Laplace equation)

$$\|u_{\mathcal{T}} - u\|_{L^2(\Omega)} \to 0, \ \ as \ \mathrm{size}(\mathcal{T}) \to 0,$$

When the data are irregular

Remark

When $g \notin L^2$, for instance if only $g \in L^1$, the above convergence can be generalized but this requires other tools, cf. **Droniou, Gallouët and Herbin, SIAM 2003**

Discretization by the linear FEM

$\mathcal{M}$ finite element mesh of Ω, triangles or tetrahedra.

$$V_0^{\mathcal{M}} = \{u \in C_c(\Omega); u|_T \in \mathcal{P}_1 \text{ for all } T \in \mathcal{M} \text{ and } u = 0 \text{ on } \partial\Omega\},$$

$$\begin{cases} \text{Find } (u_{\mathcal{M}}, \phi_{\mathcal{M}}) \in V_0^{\mathcal{M}} \times V_0^{\mathcal{M}} \text{ such that :} \\ \displaystyle\int_\Omega \kappa(\cdot, u_{\mathcal{M}}) \nabla \phi_{\mathcal{M}} \cdot \nabla \psi \, dx = \int_\Omega f(\cdot, u_{\mathcal{M}}) \psi \, dx, \forall \psi \in V_0^{\mathcal{M}}, \\ \displaystyle\int_\Omega \lambda(\cdot, u_{\mathcal{M}}) \nabla u_{\mathcal{M}} \cdot \nabla v \, dx = \int_\Omega \kappa(\cdot, u_{\mathcal{M}}) |\nabla \phi_{\mathcal{M}}|^2 v \, dx, \\ \hfill \forall v \in V_0^{\mathcal{M}}. \end{cases}$$

Discretization by the cell–centred FVM, balance equation

$$\begin{cases} -\int_{\partial K} \kappa(\cdot, u)\nabla\phi \cdot \mathbf{n}_K dx &= \int_K f(\cdot, u)dx \\ -\int_{\partial K} \lambda(\cdot, u)\nabla u \cdot \mathbf{n}_K dx &= \int_K \kappa(\cdot, u)|\nabla\phi|^2 dx, \end{cases}$$

$\mathbf{n}_K$: unit normal vector to ∂K outward to K.

$$\partial K = \bigcup_{\sigma\in\mathcal{E}_K} \sigma \qquad \rightsquigarrow$$

$$\begin{cases} -\sum_{\sigma\in\mathcal{E}_K}\int_{\sigma} \kappa(\cdot, u)\nabla\phi \cdot \mathbf{n}_{K,\sigma} dx &= \int_K f(\cdot, u)dx \\ -\sum_{\sigma\in\mathcal{E}_K}\int_{\sigma} \lambda(\cdot, u)\nabla u \cdot \mathbf{n}_{K,\sigma} dx &= \int_K \kappa(\cdot, u)|\nabla\phi|^2 dx. \end{cases}$$

Discretization by the cell–centred FVM, flux form

$$\begin{cases} \displaystyle\sum_{\sigma\in\mathcal{E}_K} F^{\kappa}_{K,\sigma} = \mathrm{m}(K) f_K(u_K), \ \forall K \in \mathcal{T}, \\ \displaystyle\sum_{\sigma\in\mathcal{E}_K} F^{\lambda}_{K,\sigma} = \mathrm{m}(K) \mathcal{J}_K(u_{\mathcal{T}}, \phi_{\mathcal{T}}), \ \forall K \in \mathcal{T}, \end{cases}$$

$$F^{\kappa,\lambda}_{K,\sigma} \sim \int_\sigma \begin{pmatrix} \kappa \\ \lambda \end{pmatrix} (\cdot, u) \nabla\phi \cdot \mathbf{n}_{K,\sigma} dx$$

$$\mathcal{J}_K(u_{\mathcal{T}}, \phi_{\mathcal{T}}) \sim \frac{1}{m(K)} \int_K \kappa(\cdot, u) |\nabla\phi|^2 dx$$

Discrete unknowns $\phi_{\mathcal{T}} = (\phi_K)_{K\in\mathcal{T}}$ and $u_{\mathcal{T}}(u_K)_{K\in\mathcal{T}}$.

Discretization by the cell–centred FVM, discrete fluxes

The discrete fuxes can be cmputed without difficult :

$$F^{\kappa}_{K,\sigma} = \begin{cases} \mathrm{m}(\sigma)\tau^{\kappa}_{\sigma}(u_{\mathcal{T}})(\phi_K - \phi_L), \text{ if } \sigma = K|L \in \mathcal{E}_{\text{int}}, \\ \mathrm{m}(\sigma)\tau^{\kappa}_{\sigma}(u_{\mathcal{T}})\phi_K \text{ if } \sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}, \end{cases}$$

$$\tau^{\kappa}_{\sigma}(u_{\mathcal{T}}) = \begin{cases} \dfrac{\kappa_K(u_K)\kappa_L(u_K)}{d_{K,\sigma}\kappa_L(u_L) + d_{L,\sigma}\kappa_K(u_K)} \text{ if } \sigma = K|L, \\ \dfrac{\kappa_K(u_K)}{d_{K,\sigma}}, \text{ if } \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K, \end{cases}$$

Idea of Weak Discrete Gradient of Eymard and Gallouët ; SIAM 2003

Let us introduce Weak Discrete Gradient of Eymard and Gallouët ; SIAM 2003 :
Let $u \in \mathcal{X}(\mathcal{T})$; define $\mathbf{G}_{\mathcal{T}}$ on each $D_{K,\sigma}$

$$\mathbf{G}_{\mathcal{T}}(u)(x) = \frac{d}{d_{K|L}}(u_L - u_K)\mathbf{n}_{K,\sigma},\; forx \in D_{K,\sigma}, \forall K \in \mathcal{T}, \forall \sigma in \mathcal{E}_K,$$

where $\mathbf{n}_{KL}$ is the unit normal vector to σ outward to K.

Theorem

Let u smooth enought then $\mathbf{G}_{\mathcal{T}}(u) \to \nabla u$ weakly in $(L^2)^2$,i.e. :

$$\int_\Omega \mathbf{G}_{\mathcal{T}}(u) \cdot \psi dx \to \int_\Omega \nabla u \cdot \psi dx,\; \forall \psi \in (L^2)^2.$$

Discrete Ohmic Losses using Discrete Gradient of Eymard and Gallouët ; SIAM 2003

For the sake of simplicity, we assume that $\kappa = 1$; we use the idea of Discrete Gradient to find a **Discrete Ohmic Losses** :

$$\mathcal{J}_K(u_\mathcal{T}, \phi_\mathcal{T}) \sim \frac{1}{\mathrm{m}(K)} \int_K |\nabla \phi|^2 \, dx$$

$$\mathcal{J}_K(u_\mathcal{T}, \phi_\mathcal{T}) = \frac{1}{\mathrm{m}(K)} \sum_{\sigma \in \mathcal{E}_K} \mathrm{m}(\mathcal{D}_{K,\sigma}) \mathcal{J}_\sigma(\phi_\mathcal{T}), \; K \in \mathcal{T} \text{ and } \sigma \in \mathcal{E}_K,$$

$$\mathcal{D}_{K,\sigma} = \{t x_K + (1-t)x, \; (x,t) \in \sigma \times (0,1)\},$$

$$\mathcal{J}_\sigma(\phi_\mathcal{T}) = \frac{d}{d_{K|L}} (\phi_L - \phi_K)^2$$

Existence of discrete solutions

Theorem

1. *There exists at least a solution* $(u_{\mathcal{M}}, \phi_{\mathcal{M}}) \in (V_0^{\mathcal{M}})^2$ *to the FE scheme.*
2. *There exists at least a solution* $(u_{\mathcal{T}}, \phi_{\mathcal{T}}) \in (H((\mathcal{T}))^2$ *the. FV scheme.*

Proof : Brouwer's theorem.

Remark

Existence for the continuous problem :

1. *by Schauder's theorem (Gallouët–Herbin. 94)*
2. *by the convergence of the FEM or FVM schemes*

Convergence result, FVM scheme

Theorem (Convergence, FVM approximation)

$(\mathcal{T}_n)_{n\in\mathbb{N}}$; $h_n = \sup\{diam(K), K \in \mathcal{T}_n\} \to 0$, *as* $n \to \infty$ $\exists\zeta > 0$;

$d_\sigma \leq \zeta d_{K,\sigma}, \forall\sigma \in \mathcal{E}_n, \forall K \in \mathcal{T}_n.$

Then, up to a subsequence :

$$\begin{cases} \|\phi^n - \phi\|_{L^2(\Omega)} \to 0, \text{ as } n \to +\infty, \\ \|u^n - u\|_{L^q(\Omega)} \to 0, \text{ as } n \to +\infty, \text{ for all } q < \frac{d}{d-2}. \end{cases}$$

with $(\phi, u) \in H_0^1(\Omega) \times \cap_{p<\frac{d}{d-1}} W_0^{1,p}$ *solution to (WF).*

$$\int_\Omega \mathcal{J}^n(u^n, \phi^n)\psi dx \to \int_\Omega \kappa(\cdot, u)|\nabla\phi|^2\psi dx \text{ as } n \to +\infty, \forall\psi \in \mathcal{C}(\overline{\Omega})$$

Assumption on the FEM scheme

$$T_{i,j}^{\lambda}(u_{\mathcal{M}}) = -\int_{\Omega} \lambda(\cdot, u_{\mathcal{M}}) \nabla \xi_i \cdot \nabla \xi_j \geq 0, \forall i \neq j. \qquad (\mathrm{Hyp}(T_{i,j}))$$

Usual condition for the maximum principle

Needed here for the convergence proof

Meshes which satisfy this condition :

1. 2D : triangles with Delaunay condition
2. 3D : tetrahedra with acute angles
3. What if λ is a matrix instead of a scalar ?

Convergence result, FEM scheme

Theorem (Convergence, FEM approximation)

$(\mathcal{M}_n)_{n\in\mathbb{N}}$ *satisfying* $(\mathrm{Hyp}(T_{i,j}))$.

$h_n = \sup\{diam(K), K \in \mathcal{M}_n\} \to 0, \quad as \quad n \to \infty,$

Then, up to a subsequence,

$$\begin{cases} \|\phi^n - \phi\|_{L^2(\Omega)} \to 0, \ \ as \ n \to +\infty, \\ \|u^n - u\|_{L^q(\Omega)} \to 0, \ \ as \ n \to +\infty, \ \ for\ all\ q < \frac{d}{d-2}. \end{cases}$$

with $(\phi, u) \in H_0^1(\Omega) \times \cap_{p<\frac{d}{d-1}} W_0^{1,p}$ *solution to (WF).*

$$\int_\Omega \kappa(\cdot, u^n)\nabla\phi^n \cdot \nabla\phi^n dx \to \int_\Omega \kappa(\cdot, u)|\nabla\phi|^2 \ dx \ \ as \ n \to +\infty.$$

Keeping in our minds

1. To approximate the **Ohmic Losses**, we used a **Discrete Ohmic Losses** based on the use of **Discrete Gradient of Eymard and Gallouët, SIAM 2003.**
2. The Convergence of boths schemes is obtained.
3. The Convergence of FES requires a triangulation with acute angles, but this is hard to create in 3D.
4. But in FV, mesh in 3D which requires othogonality Property is Possible using Voronöi mesh, for instance. A work is doing by Hang Si (WIAS) (Generation of the Voronöi mesh in 3D).

Conclusions, perspectives

1. It is also possible, provided some **Assumption on the Regularity of the exact solution** (ϕ, u) , to a get a **Convergence Order**, i.e. "**Quantify the Convergence**". The idea is to use the article of S. Clain 1996.
   Some partial results are obtained in this direction for the **Finite Element Scheme**.
2. Using the previous item, it is also possible to **Improve Convergence Order**.

Contribution concerning the approximation of problem with irregular data

- Discretization of the coupled heat and electrical diffusion problems by the finite element and the finite volume methods.
  With R. Herbin. IMA Journal of Numerical Analysis, **28** (3), 469–495, 2008.

- On the discretization of the coupled heat and electrical diffusion problems.
  With R. Herbin. Numerical Methods and Applications. 6 th International Conference, NMA 2006, Borovets, Bulgaria, Aug. 20–24, 2006. Lecture Notes in Computer Science 4310 Springer 2007, pp. 1–15.

Some simulation : COMSOL Multiphysis

Definition (What is COMSOL Multipysics)

The COMSOL COMSOL Multiphysis (or FEMLab, as many people called it) is a commercial software allows us to resolve (represent) the solutions of differential and partial differential equations of different types.
Among the wonderful of this Software is that it contains a Script which allow us do some programmes and to import (or to export) programms written in Matlab.

Contribution concerning the uses of COMSOL Multiphysis (FEMLab)

- On the convergence order of the COMSOL solutions in Sobolev norms.
  With Holzbecher. CD Proceedings of the COMSOL Conference of Budapest, November 2008.

- On the convergence order of the COMSOL solutions.
  With Holzbecher. CD Proceedings of the COMSOL Conference of Grenoble, October 2007.

Some works under preparation

- Higher order finite volume approximations on lower order schmes using general grids.
- An H^1 capacity result and piecewise higher order finite element approximations.

Some perspectives : question of the regularity of the exact solution

The regularity of the exact solution
is required to get higher order numerical approximation. It seems then important (if it is not so difficult) to think how to get higher order numerical approximation with “basic” assumptions on the exact solution.