Some highlights on the coefficients of the Fourier series

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Aim of this note

Assume that Ψ is a given smooth function in such way that it can developed using Fourier series. Assume that the function Ψ is π-periodic. Therefore the function Ψ is also 2π-periodic. Consequently, the coefficients of the Fourier series can be computed using two manners:

1. Using the fact that Ψ is also 2π-periodic
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \Psi(x)\cos(nx)dx \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \Psi(x)\sin(nx)dx. \tag{1}$$
2. Using the fact that Ψ is π-periodic
$$a_n = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Psi(x)\cos(2nx)dx \quad b_n = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Psi(x)\sin(2nx)dx. \tag{2}$$

We will show that the computation of the coefficients of the Fourier series using the formulas (1) and (2) leads to the same results.

Proof of the stated results

Indeed, let us consider the formulas given by (1). The coefficient a_n given in (1) can be written as
$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{-\frac{\pi}{2}} \Psi(x)\cos(nx)dx + \int_{-\frac{\pi}{2}}^{0} \Psi(x)\cos(nx)dx + \int_{0}^{\frac{\pi}{2}} \Psi(x)\cos(nx)dx + \int_{\frac{\pi}{2}}^{\pi} \Psi(x)\cos(nx)dx\right) \tag{3}$$
Let the new variable $t = x + \pi$ in the first integral of (3), we get, since Ψ is π-periodic
$$\int_{-\pi}^{-\frac{\pi}{2}} \Psi(x)\cos(nx)dx = \int_{0}^{\frac{\pi}{2}} \Psi(t)\cos(nt - \pi n)dt = (-1)^n \int_{0}^{\frac{\pi}{2}} \Psi(t)\cos(nt)dt. \tag{4}$$
Let the new variable $t = x - \pi$ in the fourth integral of (3), we get, since Ψ is π-periodic
$$\int_{\frac{\pi}{2}}^{\pi} \Psi(x)\cos(nx)dx = \int_{-\frac{\pi}{2}}^{0} \Psi(t)\cos(nt + \pi n)dt = (-1)^n \int_{-\frac{\pi}{2}}^{0} \Psi(t)\cos(nt)dt. \tag{5}$$
Inserting (4) and (5) in (3) yields
$$a_n = \frac{1 + (-1)^n}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Psi(t)\cos(nt)dt. \tag{6}$$
Which gives that
$$a_{2n+1} = 0 \qquad \text{and} \qquad a_{2n} = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Psi(t)\cos(2nt)dt. \tag{7}$$
In the same manner, we justify that
$$b_{2n+1} = 0 \qquad \text{and} \qquad b_{2n} = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Psi(t)\sin(2nt)dt. \tag{8}$$
This means that the Fourier series $S(f)(x)$ using the fact that Ψ is also 2π-periodic is
$$\begin{aligned} S(f)(x) &= \frac{a_0}{2} + \sum_{n\geq 1}(a_n\cos(nx) + b_n\sin(nx)) \\ &= \frac{a_0}{2} + \sum_{n\geq 1}(a_{2n}\cos(2nx) + b_{2n}\sin(2nx)), \end{aligned} \tag{9}$$

where a_n and b_n are given by (7) and (8).

We remark that the expansion (9) is exactly the one of the Fourier series when we use the fact that Ψ is π-periodic, i.e. the coefficients are computed using (2).