On a necessary condition for the uniform convergence of the series of functions

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Aim of this note

The aim of this note is to prove a necessary condition for the uniform convergence of the series of functions. This necessary condition is similar to that known for the convergence of numerical series.

1 Introduction and some preliminaries

Let $\sum_{n\geq 0} U_n$ be a series with the general term U_n. It is known that if the series $\sum_{n\geq 0} U_n$ converges then

$$\lim_{n\to\infty} U_n = 0. \tag{1}$$

Let us now consider a series of function $\sum_{n\geq 0} f_n(x)$. Assume that the series of function $\sum_{n\geq 0} f_n(x)$ convergences point-wise on D to a limit denoted by $S = \sum_{n=0}^{\infty} f_n(x)$. We say that $\sum_{n\geq 0} f_n(x)$ convergences uniformly to $\sum_{n=0}^{\infty} f_n(x)$ on the subset $D' \subset D$ if

$$\lim_{n\to\infty} \sup_{x\in D'} \left| \sum_{k=0}^{n} f_k(x) - \sum_{k=0}^{\infty} f_k(x) \right| = 0. \tag{2}$$

The condition (1) can be written in the following form:

$$\lim_{n\to\infty} \sup_{x\in D'} \left| \sum_{k=n+1}^{\infty} f_k(x) \right| = 0. \tag{3}$$

We justify in this note that a necessary condition to get the uniform convergence (3) is that

$$\lim_{n\to\infty} \|f_n\|_{D'} = 0, \tag{4}$$

where $\|\varphi\|_{D'}$ denotes, as usual, the following norm:

$$\|\varphi\|_{D'} = \sup_{x\in D'} |\varphi(x)|. \tag{5}$$

1. **First Proof**. This proof is based on the use of Cauchy criterion. The uniform convergence (3) can be written as: For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $p > q > N$

   $$\left\| \sum_{k=q}^{\infty} f_k - \sum_{k=p}^{\infty} f_k \right\|_{D'} \leq \epsilon. \tag{6}$$

   Which is

   $$\left\| \sum_{k=q}^{p-1} f_k \right\|_{D'} \leq \epsilon. \tag{7}$$

   Let us take in particular $q = p - 1$. This gives for all $q > N$

   $$\|f_q\|_{D'} \leq \epsilon. \tag{8}$$

   Which is the $\lim_{n\to\infty} \|f_n\|_{D'} = 0$.

2. **Second Proof**. For simplicity of notation, let us set

$$S_n(x) = \sum_{k=0}^{n} f_k(x) \tag{9}$$

and

$$S(x) = \sum_{k=0}^{\infty} f_k(x). \tag{10}$$

Uniform convergence (2) (or also (3)) can be written as

$$\lim_{n\to\infty} \|S_n - S\|_{D'} = 0. \tag{11}$$

On another hand, using the triangle inequality, we have

$$\begin{aligned} \|f_n\|_{D'} &= \|S_{n+1} - S_n\|_{D'} \\ &= \|S_{n+1} - S\|_{D'} + \|S - S_n\|_{D'}. \end{aligned} \tag{12}$$

Which implies that, thanks to (11)

$$\lim_{n\to\infty} \|f_n\|_{D'} = 0. \tag{13}$$

References

[1] K. Allab, Elément d'Analyse: Fonction d'une Variable Réelle. OPU, 1990

[2] W. F. Trench, Introduction to Real Analysis. ISBN 0-13-045786-8, Free Hyperlinked Edition 2.03, November 2012.