Non-Newtonian fluid models: some existing results and advances

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Aim of the presentation

In the first part, we will present some existing results on Non-Newtonian fluid, e.g. existence, uniqueness and the regularization methods. In particular, we will present a new result on the existence of a weak solution for homogeneous incompressible Bingham fluid.

The second part is to investigate finite volume approximation of a Bingham type problem.

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Plan of the presentation

1. Introduction
   - Non Newtonian fluid
   - Bingham rheology
   - **about the seconde part
2. Bingham-Navier–Stokes equations
   - Setting of the problem
   - Functional spaces
   - New result on the existence of solution
   - Proof strategy
     - Approximate solutions
     - Compactness results
     - Passing to the limit
     - Uniqueness of solutions
3. **part 2
4. **Conclusion and outlook

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- To study more complex fluids, such as molten plastics, synthetic fibers, biological fluids, paints, and greases, etc., it is necessary to consider a generalized Navier Stokes system that models the behavior of fluids whose viscosity depends on the rate of deformation (i.e., non-Newtonian fluids).

This complex behavior is translated into a mathematical complexity which gives rise to complex stress-strain laws, such as the Carreau-Yasuda, Bingham, power law, Cross, Casson, Herschel-Bulkley, etc..

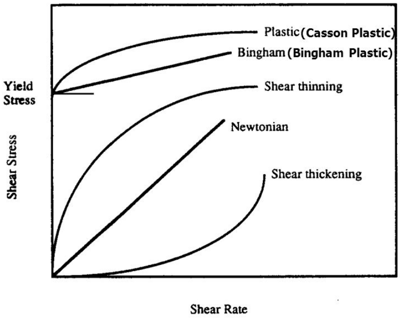

Figure: Examples of Non-Newtonian fluid models.

- Among the various classes of non-Newtonian materials, those exhibiting viscoplastic properties are particularly interesting by their ability to strain only if the stress rate exceeds a minimum value.
- The most commonly used model to account for this particular behavior is the Bingham model [2].

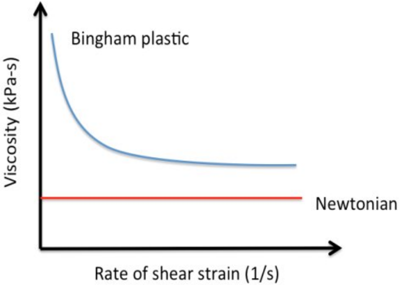

Figure: Bingham viscosity.

Difficulties

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To overcome these difficulties

- Duvaut and Lions [2] exclude the stress tensor by passing to a variational inequality for the velocity field.
- Shelukhin et al. [4, 5] use Bercovier and Engelman model as an approximation of the Bingham fluid.
- **We approximate the Bingham fluid using a bi-viscosity fluid.**

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Let Ω be a smooth domain in $\mathbb{R}^2$ with Lipschitz boundary and Ω_T the open set $\Omega \times (0, T)$, where $T > 0$ is the final time.
We consider an unsteady flow of incompressible Bingham fluid in 2D which is governed by the following Navier-Stokes system

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nabla \cdot (\tau(Du)) + \nabla p = f & \text{in } \Omega_T, \\ \nabla \cdot u = 0 & \text{in } \Omega_T. \end{cases} \tag{1}$$

Here u is the velocity vector, p is the pressure, and τ is the stress tensor where the strain tensor is defined as

$$Du = \frac{1}{2}(\nabla u + \nabla u^t),$$

and $f : \Omega_T \to \mathbb{R}^2$ represents the external forces (such as gravity).

The system (1) is equipped with the following initial condition

$$u(\cdot, 0) = u_0 \qquad \text{in } \Omega, \tag{2}$$

and the homogeneous Dirichlet boundary condition

$$u = 0 \qquad \text{on } \partial\Omega \times (0, T)\cdot \tag{3}$$

The Bingham stress–strain constitutive law is defined as

$$\begin{cases} \tau(Du) = \left(2\mu + \frac{\tau_y}{|Du|}\right) Du & \text{if } \mid \tau \mid > \tau_y, \\ \\ Du = 0 & \text{if } \mid \tau \mid \leq \tau_y\cdot \end{cases} \tag{4}$$

Here μ is the viscosity, τ_y is the yield stress and $\mid A \mid^2 = A : A$ where the inner product is defined as $A : B = \displaystyle\sum_{i,j} A_{ij} B_{ij}$.

Let us choose some spaces. Let X be a Banach space, for each $1 \leq p < \infty$, we defined the following function spaces :

$$H = \left\{ v \in L^2(\Omega) : \; \nabla \cdot v = 0, \; v_{|\partial\Omega} = 0 \right\},$$

$$V = \left\{ v \in H_0^1(\Omega) : \; \nabla \cdot v = 0 \right\} \cdot$$

These two spaces are Hilbert space equipped with the scalar products respectively induced by those of $L^2(\Omega)$ and of $H_0^1(\Omega)$, i.e

$$\| u \|_H^2 = \int_\Omega | v |^2 \, dx \qquad \text{and} \qquad \| u \|_V^2 = \int_\Omega | \nabla v |^2 \, dx \cdot$$

We also use the following spaces:

$$L^p(0,T;X) = \left\{ v : (0,T) \mapsto X : \| v \|^p_{L^p(0,T;X)} = \int_0^T \| v \|^p_X < \infty \right\},$$

$$L^\infty(0,T;X) = \left\{ v : (0,T) \mapsto X : \| v \|_{L^\infty(0,T;X)} = \mathop{sup\ ess}_{t\in(0,T)} \| v \|_X < \infty \right\}.$$

The space

$$E_{2,2}(V) = \left\{ v \in L^2(0,T;V) : \ \partial_t v \in L^2(0,T;V') \right\},$$

is a Banach space with the following norm

$$\| u \|_{E_{2,2}} = \| v \|_{L^2(0,T,V)} + \| \partial_t v \|_{L^2(0,T,V')} .$$

Where V' is the topological dual of V, and we denote by $\langle \cdot, \cdot \rangle$ the duality bracket between V and V'.

We call function $u \in E_{2,2}$ a weak solution of the problem (1)-(4) if

$$\int_0^T \langle \partial_t u, \varphi \rangle dt + \int_{\Omega_T} \tau(Du) : D\varphi \, dxdt + \int_{\Omega_T} (u \cdot \nabla) u \cdot \varphi \, dxdt = \int_0^T \langle f, \varphi \rangle dt, \tag{5}$$

for all $\varphi \in L^2(0, T; V)\cdot$

Theorem 1

Assume that $f \in L^2(0, T; V')$ and $u_0 \in H$, then the Navier Stokes equation for a Bingham fluid (1)-(4) has a weak solution such that

$$u \in L^2(0, T; V) \cap L^\infty(0, T; H), \quad \partial_t u \in L^2(0, T; V'), \quad \tau(Du) \in L^2(\Omega_T).$$

Remarks

- Theorem 1 ensure the existence of a classical weak solution $(u, p) \in E_{2,2} \times \mathcal{D}'(\Omega_T)$ for the system (1)-(4).
- The solution u is more then a classical weak solution. In fact, we have $u \in C^0([0, T]; H)$, moreover u satisfies the following energy equality

$$\frac{1}{2} \parallel u(s_2) \parallel^2_{L^2(\Omega)} + \int_{s_1}^{s_2} \int_\Omega \tau(Du) : Du = \int_{s_1}^{s_2} \langle f, u \rangle dt + \frac{1}{2} \parallel u(s_1) \parallel^2_{L^2(\Omega)}, \tag{6}$$

for all $s_1, s_2 \in [0, T]\cdot$

Proof strategy

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- Step 4 : Prove the uniqueness of solutions.

Step 1: Approximate solutions

In this step, we build an approximate problem by regularizing the Bingham tensor (4) with another operator that approximates the physical behavior of Bingham fluids and has some analytical properties.

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bi-viscosity model

The regularizing tensor chosen is the bi-viscosity tensor:

$$\tau_m(A) = \begin{cases} 2m\mu A & \text{if } \mid A \mid \leq \gamma_m, \\ \left(2\mu + \frac{\tau_y}{|A|}\right) A & \text{if } \mid A \mid > \gamma_m. \end{cases} \tag{7}$$

Where $A \in \mathbb{M}_2$ and $\gamma_m = \dfrac{\tau_y}{2\mu(m-1)}, \ m \geq 2.$

Why the bi-viscosity model ?

- It approximate the physical behavior of Bingham fluid :
  The idea is to consider the Bingham fluid when $\mid \tau \mid \leq \tau_y$ (which is practically solid) as highly viscous Newtonian fluid, by involving a second artificial viscosity $\mu_m = m\mu$.

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  The idea is to consider the Bingham fluid when $\mid \tau \mid \leq \tau_y$ (which is practically solid) as highly viscous Newtonian fluid, by involving a second artificial viscosity $\mu_m = m\mu$.
- The bi-viscosity model is coercive, growing, monotonic and continuous, which are the conditions of an existence theorem.

Other regularization methods

Other regularisation choices are possible, such as the Papanastasiou and Bercovier and Engelman model :

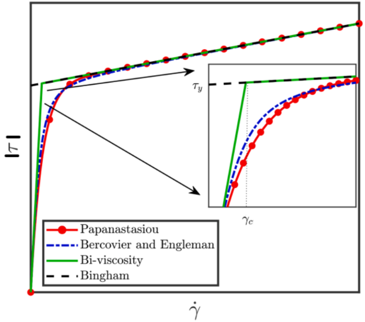

Figure: Different viscosity regularization models and Bingham model.

Lemma 1

Assume that $f \in L^2(0, T; V')$ and $u_0 \in H$, then the approximate problem (1)-(3), (7), has at least a solution $u_m \in E_{2,2}$ in the following sense :

$$\int_0^T \langle \partial_t u_m, \varphi \rangle + \int_{\Omega_T} \tau_m(Du_m) : D\varphi + \int_{\Omega_T} (u_m \cdot \nabla) u_m \cdot \varphi = \int_0^T \langle f, \varphi \rangle, \quad (8)$$

for all $\varphi \in L^2(0, T; V)\cdot$

Proof

This Lemma is an application of Theorem 1 proved by Dreyfuss and Hungerbühler in [3].

Step 2: Compactness results

Lemma 2

The approximate solution u_m, constructed in Step 1 satisfied the following estimations

(i) The sequence u_m is bounded in $L^2(0, T; V) \cap L^\infty(0, T; H)$.

(ii) The sequence $(u_m \cdot \nabla)u_m$ is bounded in $L^2(0, T; V')$.

(iii) The sequence $\tau_m(Du_m)$ is bounded in $L^2(\Omega_T)$.

(iv) The sequence $\partial_t u_m$ is bounded in $L^2(0, T; V')$.

Step 3: Passing to the limit

Lemma 3

The following convergence are proved for subsequences which are denoted by u_m.

(i) $u_m \to u$ weakly in $L^2(0, T; V)$ and *-weakly in $L^\infty(0, T; H)$.

(ii) $\partial_t u_m \to \partial_t u$ weakly in $L^2(0, T; V')$.

(iii) $(u_m \cdot \nabla)u_m \to (u \cdot \nabla)u$ weakly in $L^2(0, T; V')$.

(iv) $\tau_m(Du_m) \to \tau(Du)$ weakly in $L^2(\Omega_T)$.

Step 4: Uniqueness of solutions

To prove that the problem (1)-(4) has a unique solution, We consider u_1 and u_2 to be two weak solutions of (5) and introduce $u = u_1 - u_2$.

Outline of the proof

- We take the function $\varphi = u\mathbf{1}_{(0,t]}$, $t \in (0, T)$, as a test function.
- We use Lions-Magenes theorem and the following inequality

$$(\tau(A) - \tau(B)) : (A - B) \geq 2\mu \mid A - B \mid^2, \qquad \forall A, B \in \mathbb{M}_2. \quad (9)$$

- Finlay, we obtain uniqueness using Gronwall lemma.

Conclusion and outlook

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Thank you for your attention