LICENCE COURSE ON APPLICATIONS OF MATHEMATICS IN OTHER AREAS

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Abstract. We aim in this note to provide some highlights on the applications of Mathematics in several areas.

Mathematics has become a vastly diverse subject over history, and there is a corresponding need to categorize the different areas of mathematics. A number of different classification schemes have arisen, and though they share some similarities, there are differences due in part to the different purposes they serve. In addition, as mathematics evolves, these classification schemes must evolve as well to account for newly created areas or newly discovered links between different areas. Classification is made more difficult by some subjects, often the most active, which straddle the boundary between different areas.

A traditional division of mathematics is into pure mathematics, mathematics studied for its intrinsic interest, and applied mathematics, mathematics which can be directly applied to real world problems.[1] This division is not always clear and many subjects have been developed as pure mathematics to find unexpected applications later on. Broad divisions, such as discrete mathematics and computational mathematics, have emerged more recently.

1. An overall on Mathematics

Before, we will be able to give some applications of Mathematics, it is useful to provide some idea on how it is Mathematics now.

(1) Set theory
A set can be thought of as a collection of distinct things united by some common feature. Set theory is subdivided into three main areas. Naive set theory is the original set theory developed by mathematicians at the end of the 19th century. Axiomatic set theory is a rigorous axiomatic theory developed in response to the discovery of serious flaws (such as Russell's paradox) in naive set theory. It treats sets as "whatever satisfies the axioms", and the notion of collections of things serves only as motivation for the axioms. Internal set theory is an axiomatic extension of set theory that supports a logically consistent identification of illimited (enormously large) and infinitesimal (unimaginably small) elements within the real numbers. See also List of set theory topics.

(2) Proof theory and constructive mathematics
Proof theory grew out of David Hilbert's ambitious program to formalize all the proofs in mathematics. The most famous result in the field is encapsulated in Gödel's incompleteness theorems. A closely related and now quite popular concept is the idea of Turing machines. Constructivism is the outgrowth of Brouwer's unorthodox view of the nature of logic itself; constructively speaking, mathematicians cannot assert "Either a circle is round, or it is not" until they have actually exhibited a circle and measured its roundness.

(3) Algebra
The study of structure starting with numbers, first the familiar natural numbers and integers and their arithmetical operations, which are recorded in elementary algebra. The deeper properties of whole numbers are studied in number theory. The investigation of methods to solve equations leads to the field of abstract algebra, which, among other things, studies rings and fields, structures that generalize the properties possessed by everyday numbers. Long standing questions about compass and straightedge construction were finally settled by Galois theory. The physically important concept of vectors, generalized to vector spaces, is studied in linear algebra.

(4) Order theory Any set of real numbers can be written out in ascending order. Order Theory extends this idea to sets in general. It includes notions like lattices and ordered algebraic structures. See also the order theory glossary and the list of order topics.

(5) General algebraic systems Given a set, different ways of combining or relating members of that set can be defined. If these obey certain rules, then a particular algebraic structure is formed. Universal algebra is the more formal study of these structures and systems.

(6) Number theory Number theory is traditionally concerned with the properties of integers. More recently, it has come to be concerned with wider classes of problems that have arisen naturally from the study of integers. It can be divided into elementary number theory (where the integers are studied without the aid of techniques from other mathematical fields); analytic number theory (where calculus and complex analysis are used as tools); algebraic

number theory (which studies the algebraic numbers - the roots of polynomials with integer coefficients); geometric number theory; combinatorial number theory; transcendental number theory; and computational number theory. See also the list of number theory topics.

(7) Field theory and polynomials Field theory studies the properties of fields. A field is a mathematical entity for which addition, subtraction, multiplication and division are well-defined. A polynomial is an expression in which constants and variables are combined using only addition, subtraction, and multiplication.

(8) Commutative rings and algebras In ring theory, a branch of abstract algebra, a commutative ring is a ring in which the multiplication operation obeys the commutative law. This means that if a and b are any elements of the ring, then $a \times b = b \times a$. Commutative algebra is the field of study of commutative rings and their ideals, modules and algebras. It is foundational both for algebraic geometry and for algebraic number theory. The most prominent examples of commutative rings are rings of polynomials.

(9) Analysis
Within the world of mathematics, analysis is the branch that focuses on change: rates of change, accumulated change, and multiple things changing relative to (or independently of) one another.
Modern analysis is a vast and rapidly expanding branch of mathematics that touches almost every other subdivision of the discipline, finding direct and indirect applications in topics as diverse as number theory, cryptography, and abstract algebra. It is also the language of science itself and is used across chemistry, biology, and physics, from astrophysics to X-ray crystallography.

(10) Combinatorics
Combinatorics is the study of finite or discrete collections of objects that satisfy specified criteria. In particular, it is concerned with "counting" the objects in those collections (enumerative combinatorics) and with deciding whether certain "optimal" objects exist (extremal combinatorics). It includes graph theory, used to describe inter-connected objects (a graph in this sense is a network, or collection of connected points). See also the list of combinatorics topics, list of graph theory topics and glossary of graph theory. A combinatorial flavour is present in many parts of problem-solving.

(11) Geometry and topology
Geometry deals with spatial relationships, using fundamental qualities or axioms. Such axioms can be used in conjunction with mathematical definitions for points, straight lines, curves, surfaces, and solids to draw logical conclusions. See also List of geometry topics

(12) Convex geometry and discrete geometry Includes the study of objects such as polytopes and polyhedra. See also List of convexity topics

(13) Discrete or combinatorial geometry The study of geometrical objects and properties that are discrete or combinatorial, either by their nature or by their representation. It includes the study of shapes such as the Platonic solids and the notion of tessellation.

(14) Differential geometry The study of geometry using calculus, and is very closely related to differential topology. Covers such areas as Riemannian geometry, curvature and differential geometry of curves. See also the glossary of differential geometry and topology.

(15) Algebraic geometry Given a polynomial of two real variables, then the points on a plane where that function is zero will form a curve. An algebraic curve extends this notion to polynomials over a field in a given number of variables. Algebraic geometry may be viewed as the study of these curves. See also the list of algebraic geometry topics and list of algebraic surfaces.

(16) Topology Deals with the properties of a figure that do not change when the figure is continuously deformed. The main areas are point set topology (or general topology), algebraic topology, and the topology of manifolds, defined below.

(17) General topology Also called point set topology. Properties of topological spaces. Includes such notions as open and closed sets, compact spaces, continuous functions, convergence, separation axioms, metric spaces, dimension theory. See also the glossary of general topology and the list of general topology topics.

(18) Algebraic topology Properties of algebraic objects associated with a topological space and how these algebraic objects capture properties of such spaces. Contains areas like homology theory, cohomology theory, homotopy theory, and homological algebra, some of them examples of functors. Homotopy deals with homotopy groups (including the fundamental group) as well as simplicial complexes and CW complexes (also called cell complexes). See also the list of algebraic topology topics.

(19) Manifolds A manifold can be thought of as an n-dimensional generalization of a surface in the usual 3-dimensional Euclidean space. The study of manifolds includes differential topology, which looks at the properties of differentiable functions defined over a manifold. See also complex manifolds.

(20) Probability and statistics
See also glossary of probability and statistics
Probability theory: The mathematical theory of random phenomena. Probability theory studies random variables and events, which are mathematical abstractions of non-deterministic events or measured quantities. See also

Category:probability theory, and the list of probability topics. Stochastic processes: An extension of probability theory that studies collections of random variables, such as time series or spatial processes. See also List of stochastic processes topics, and Category:Stochastic processes.
Statistics: The science of making effective use of numerical data from experiments or from populations of individuals. Statistics includes not only the collection, analysis and interpretation of such data, but also the planning of the collection of data, in terms of the design of surveys and experiments. See also the list of statistical topics and Category:Statistics.

(21) Computational sciences

(22) Numerical analysis Many problems in mathematics cannot in general be solved exactly. Numerical analysis is the study of iterative methods and algorithms for approximately solving problems to a specified error bound. Includes numerical differentiation, numerical integration and numerical methods; c.f. scientific computing. See also List of numerical analysis topics

(23) Computer algebra This area is also called symbolic computation or algebraic computation. It deals with exact computation, for example with integers of arbitrary size, polynomials or elements of finite fields. It includes also the computation with non numeric mathematical objects like polynomial ideals or series.

(24) Physical sciences
Mechanics Addresses what happens when a real physical object is subjected to forces. This divides naturally into the study of rigid solids, deformable solids, and fluids, detailed below.

(25) Particle mechanics In mathematics, a particle is a point-like, perfectly rigid, solid object. Particle mechanics deals with the results of subjecting particles to forces. It includes celestial mechanics–the study of the motion of celestial objects.

(26) Mechanics of deformable solids Most real-world objects are not point-like nor perfectly rigid. More importantly, objects change shape when subjected to forces. This subject has a very strong overlap with continuum mechanics, which is concerned with continuous matter. It deals with such notions as stress, strain and elasticity. See also continuum mechanics.

(27) Fluid mechanics Fluids in this sense includes not just liquids, but flowing gases, and even solids under certain situations. (For example, dry sand can behave like a fluid). It includes such notions as viscosity, turbulent flow and laminar flow (its opposite). See also fluid dynamics.

(28) Other mathematical sciences
Operations research (OR), also known as operational research, provides optimal or near-optimal solutions to complex problems. OR uses mathematical modeling, statistical analysis, and mathematical optimization. Mathematical programming (or optimization) minimizes (or maximizes) a real-valued function over a domain that is often specified by constraints on the variables. Mathematical programming studies these problems and develops iterative methods and algorithms for their solution.

Applications of Mathematics are various and is impossible to quote all them. Mathematics becomes present everywhere. We will focus first on the applications of Mathematics in Biology and will consider the so called Bio–Mathematics.

2. Une definition pour les Mathematiques

Les mathematiques sont un ensemble de connaissances abstraites resultant de raisonnements logiques appliques a divers objets tels que les nombres, les figures, les structures et les transformations. Les mathematiques sont aussi le domaine de recherche developpant ces connaissances, ainsi que la discipline qui les enseigne. Les mathematiques se distinguent des autres sciences par un rapport particulier au reel. Elles sont de nature entiÃ¨rement intellectuelle, Ã©tant fondÃ©es sur des axiomes declares vrais (c'est-a-dire que les axiomes ne sont pas soumis Ã l'experience, mÃªme s'ils en sont souvent inspires) ou sur des postulats provisoirement admis. Un enonce mathematique - denomme generalement theoreme, proposition, lemme, fait, scholie ou corollaire –est considere comme valide lorsque le discours formel qui etablit sa verite respecte une certaine structure rationnelle appelee demonstration, ou raisonnement logico-deductif.

Bien que les resultats mathematiques soient des verites purement formelles, ils trouvent cependant des applications dans les autres sciences et dans differents domaines de la technique. C'est ainsi qu'Eugene Wigner parle de "la deraisonnable efficacite des mathematiques dans les sciences de la nature.

3. Elaboration graphiques et numerique des donnee experimentales

Supposons que nous avons des resultats experimentaux. Pour analyser ces resultat, on utilise par exemple les methodes de Statistiques pour collecter le maximum des information de l'experience en question.

3.1. **Le traitement graphique des donnees.** Ceci ce qui concerne la presentation des resultats. Celle ci doit etre la plus claire et la plus parlante possible.Elle se fait principalement en deux formes:

- Presentation sous forme des tableaux qui sont la liste des resultats des observations, eventuellement presentes et regroupes d'une facon bien adaptee.

- Presentation sous forme de graphes qui peuvent etre de formes tres diverses suivant le nombre des variables considerees.

3.2. Le traitement numerique des donnees. C'est ceci concerne l'elaboration des resultats. A partir des resultats bruts d'une serie d'observations, il faudra essayer d'obtenir des resultats globaux:

- Soit la verification d'une loi, c'est a dire dependence fonctionnelle entre plusieures variables.
- Soit, dans le cas ou les resulats presentent une grande variabilite (cas frequent dans les sciences de la vie), un resume des donnees par des constantes numeriques, de facon a pouvoir faire des comparaisons entre resultats relatifs a une meme variable ou pouvoir mettre en evidence des relations entre les variables differentes.

Avant de donner les detailles techniques, on va donner une idee sur quelques modeles mathematiques

4. Modeleles mathematiques en Biologie: exemple evolution de poulations

Il s'agit d'etudier l'evolution dans le temps d'une population formee d'individus susceptibles de reproduire, de mourir, ou eventuellement, d'agir les uns sur l'autre. Exemples

- Microbes, bacteries
- Cellules dans une organe en cours de developpement ou de regression.
- Desintegration des atomes radioactifs
- Propagation des epidemies
- Demographie (etude des populations humaines)
- Ecologie (etude des populations animales)

4.1. Lois exponentielles: mise en equation. On considere une population formee de N individus et evoluant en fonction de temps t: N est donc fontion du temps t. On suppose que pendant un petit intervalle de temps Δt, la variation de ΔN des nombre d'individus de la population est propontionnelle a N et t. On ecrit alors

$$\frac{\Delta N}{\Delta t} = kN. \tag{1}$$

- Si $k > 0$, la population croit
- Si $k < 0$, la population decroit

Si l'on suppose que Δt peut prendre des valeurs suffisament petites, (1) nous donne

$$N' = kN. \tag{2}$$

4.2. Resolution mathematique. On a donc a resoudre une equation dont l'inconnue est une fonction et existe sa derivee. Telle equation s'appelle une equation differentielle. Il s'agit de chercher des fonctions (si possible toutes les fonctions) qui verifient l'equation (2). Les techniques d'analyse nous donne

$$N(t) = N_0 \exp(kt), \tag{3}$$

ou N_0 est la valeur initiale de la population.

4.3. Representation graphique. Deux types de representation graphique

(1) representation naturel (ordinaire) $(t, N(t))$

  (a) $k > 0$, croissance tres rapide

  (b) $k < 0$, decroissance tres rapide

(2) representation semi–logaritmique $(t, \ln N(t))$

4.4. **Quelques exemples concrets.** Les lois exponentielles se rencontrent assez souvent dans la nature.

- Radiocative: le nombre d'atomes radioactives suit une loi exponentielle decroissante. La constante $\lambda = -k$ s'appelle *constante de desintegration radioactive*
- Biologie: le debut de developpement d'un organisme ou d'un organe est en general bien decrit par une loi exponentielle. Les cellules se devisent en deux au bout d'un temps T: supposons que a l'instant $t = 0$ nous avans N_0 cellules. Donc dans T, $3T$, $4T$,...., nous avans $2N_0$, $4N_0$, $8N_0$, $16N_0$,...Ceci correspond a la fonction

$$N(t) = N_0 2^{\frac{t}{T}} = N_0 \exp\left(\frac{t}{T}\ln 2\right) = N_0 \exp(kt), \tag{4}$$

avec

$$k = \frac{\ln 2}{T}. \tag{5}$$

- Demographie: si on admet le taux de naissance τ_n et le taux de mortalite τ_m. On admet que la population suit une loi exponentielle (*loi de Malthus*) de coefficient $k = \tau_n - \tau_m$.

4.5. **Validation du model.** On connait experimantalement le nombre $N(t)$ d'individus de la population a divers instants t. Il est deficile, en utilisant la representation ordinaire de savoir si les points experimentaux sont proches d'une courbe exponentielle. Par contre, la representation semi–logaritmique, les points doivent etre presque alignes.

5. Simulation mathematique

En general, on passe par les etapes suivants:

(1) Observation, tests experimentaux,....
(2) Modelisation mathematique: on essaye de mettre les observations, les tests experimentaux sous formes mathematiques, exemples, equation, equations differentielles, ...
(3) Simulation mathematiques: en general, ces models mathematiques peuvent pas etre resoudre d'une facon exacte. On fait alors une simulation, exemple simulation numerique
(4) On verfie avec la realite si nos calculs sont conformes avec les reultats reels.

Pour obtenir description de l'evolution de $N(t)$ en fontion du temps, on est passe par l'intermediaire d'une fonction $N(t)$ qui se trouvait etre solution d'une equation differentielle (ou plus generalement, une fontion fonctionnelle). Nous avons la chance que N peut etre ecrire explicitement. Souvent, on connait pas la solution, donc on cherche souvent a approximer le modele que nous avons.
Pour approximer par exemple le modele

$$N' = kN \tag{6}$$

existent plusieures methodes. Parmi ces methodes (qui nous permettent a approximer les equations differentielles, ou eventuellement, equation au derivees partielles), on cite

(1) Methode de differences finies (la plus ancienne)
(2) Methode des elements finis
(3) Methodes spectrales
(4) Methode de volumes finis.

On va choisir la methode de differences finies pour approximer l'equation $N' = kN$, on utilise la methode de differences finis pour sa simplicite. On se fixe d'abord un pas h (une valeur positive qui peut etre suffisamment petit). On considere en suite des points s'appellent les points de maillages t_0, $t_1 = t_0 + h$, $t_2 = t_0 + 2h$, $t_3 = t_0 + 3h$,...$t_n = t_0 + nh$. On doit des valeurs approchees a $N(t_1)$, $N(t_2)$, ..., $N(t_n)$.
On ecrit alors notre equation $N'(t) = kN(t)$ dans les points des maillages t_0, $t_1 = t_0 + h$, $t_2 = t_0 + 2h$, $t_3 = t_0 + 3h$,...$t_n = t_0 + nh$, pour obtenir

$$N'(t_n) = kN(t_n), \tag{7}$$

avec

$$N(t_0) = N_0. \tag{8}$$

On suppose que l'etat initial N_0 au point t_0 est connu.
On rapproche $N'(t_n)$ par le rapport $(N(t_{n+1}) - N(t_n))/h$ (car $(N(t_{n+1}) - N(t_n))/h = N'(t_n) + (h/2)N'(\xi_n)$ et donc le reste $(h/2)N'(\xi_n)$ essez petit pour h suffisamment petit). Soit N_n une valeur approximee de $N(t_n)$, on obtient alors

$$(N_{n+1} - N_n)/h = kN_n. \tag{9}$$

Ceci nous donne alors

$$N_{n+1} = hkN_n + N_n = (hk+1)N_n. \tag{10}$$

La suite N_n est donc une suite reccurente (de premier ordre).
On trouve alors

$$N_n = (hk+1)^n N_0. \tag{11}$$

Les methodes numeriques s'interessent non seulement par l'approximation numerique du modele en question mais aussi s'interessent par les methodes pour prouver la convergence, exemple, comment justifier que N_n converge ou non vers $N(t_n)$ lorsque h tend vers zero.
D'apres ce que nous avons remarque, existent deux domaines interessants pour la simulation: statistiques et analyse numeriques.

6. Calcul numerique: une methode de simulation

Les calculs numeriques nous peuvent aider par exemple

(1) calcul des constantes connues en mathematiques ou physiques, exemple π, e, $\sqrt{2}$, les aires, les volumes, moments d'inertie....

(2) evolutions des systemes caracterises par certaines variables: trajectoires des particules, evolution chronologique d'une population; developpement d'un organisme.

(3) calcul des valeurs des fonctions: fonctions usuelles sinus, cosinus,

Dans ces situations, il intervient d'abord une partie d'elaboration theorique, qui est souvent des domaines des sciences appliquees: recherche des lois ou des relations sous forme d'equations ou des formules. Les quantities sont pas generalement determinees exactement mais on cherche a construire des approximations aussi precise qu'on le desire.

7. Approximation numerique

Le probleme etudie dans cette section est le calcul numerique de certaines constantes impliquees en mathematiques (ou physiques).

7.1. **Some background.** Nous avons besoin de quelques outils

(1) Suites: suite une famille de nombres reells (ou complexes) u_n indexes par les nombres entieres, $0, 1, 2, ...$ On s'interesse au comportement de u_n lorsque n augmente.

(2) Serie s_n est une suite de la forme u_0, $u_0 + u_1$, $u_0 + u_1 + u_2$, ..., $u_0 + u_1 + u_2 + ... + u_n$,
Le terme u_n et s_n s'appellent respectivement le terme general et la some partielle d'indice n.

Comme le cours est destine pour des etudiants de Maths, je suppose que certain bagage est deja connu comme par exemple les criteres de convergences des suites et series

8. Differential and partial differential equations

Differential and partial differential equations are one of the important mathematical models in Physics

9. Chapter on the discretization of differential and partial differential equations by finite diffience methods: model by model

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