Curriculum vitae and a summary of research

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Remark 1 Pages 9–22 are not updated.

1 Research Interests

- Finite Volume and Finite Element Methods
- Improving Convergence Order of these two previous Methods
- Numerical Schemes Approximating Coupled Problems with Irregular Data
- Oblique Derivative Problems and their Approximations
- Defect Correction
- Domain Decomposition
- Volume Approximation for Thermohaline Convective Problems
- Improving Converegence Order of Finite Volume Approximate Solutions of Hyperbolic Equations
- Discontinuous Galerkin finite element method
- A Posteriori estimate in Finite Volume method
- Uses of COMSOL Multiphysics (Femblab)
- Numerical approximation for singular perturbed equations
- Higher order in finite volume methods for time–independent Navier Stokes equations

2 Education

- December 9th 2009: Habilitation (HDR) at the departement of Mathematics–University of Annaba, Algeria.
  Experts (Referees) of my Habilitation:
  - Prof. Robert Eymard, University of Marne la Vallée, Paris, France
  - Prof. Fayssal Benkheldoun, University of Paris Sud, Paris, France
  - Prof. Ammar Boukhmis, University of Annaba, Algeria
- November 14th 2005: Ph.D Thesis in Applied Mathematics, University of Marseille, France. Thesis Advisor: Prof. T. Gallouët.
  Title of Thesis: Improved Convergence Order in Finite Volume and Finite Elements Methods.
  Referees of my Thesis[1] :
  - Prof. Yvon Maday, University of Paris 6, France
  - Prof. Mohand Moussaoui, University of Lyon, France
- Nov. 2002–Nov. 2005 : University of Marseille, France, Ph.D in Applied Mathematics
- July 1996 : Master in Applied Mathematics, Annaba, Algeria
- Sep. 1994–July 1996: University of Annaba, Algeria, Master in Applied Mathematics
- 1988–1993 : B.A. in Mathematics, Annaba, Algeria.

3 Scientific Activities

- Reviewer for Mathematical Reviews of American Mathematical Society (AMS) since March 31, 2008
- Reviewer for Zentralblatt MATH since April 23, 2008
- Chairman for a session in the International Symposium on Finite Volume for Complex Applications VI, Prague: http://fvca6.fs.cvut.cz/
- Reviewer for Journal of the Franklin Institute
- Reviewer for Journal of Mathematical Research (JMR)
- Honorary Peer Reviewer for Global Journal Science Frontier Research (GJSFR)

[1] In France, the Thesis must be refereed by two Referees

4 Positions

- Present Position: Assistant Professor (Maître de Conference) (rank "A") at the University of Annaba, Algeria.
- Since 1999: "Enseignant Chercheur" at the University of Annaba, Algeria.
- June 1st 07–December 31 st 2007 : Postdoc in Nečas Center of Mathematical Modeling, Prague, Czech Rep.
- Mars 1 st 06–Mai 31 st 07 : Postdoc in WIAS: Weierstrass Institute for Applied Analysis and Stochastics, Berlin, Germany
- Sept. 04–Aug. 05 : Teaching Assistant with T. Gallouët, University of Marseille, France
- Sep. 99–Nov.02 : Lecturer in Mathematics, University of Annaba, Algeria
- April 99–July 99 : Lecturer in Mathematics, University of Tebessa, Algeria
- Sep. 97-Mars 99 : Military Service; Mars 15th 98–Mars 15th 99: Teacher of Mathematics in "Académie Militaire Interarmes" (AMIA) de Cherchell, Algérie
- Oct. 93-Jan.97 : Lecturer in Mathematics, University of Annaba, Algeria

5 Courses taught and supervision

- Supervising of "Sebti Habiba" in Master 2, Department of Mathematics–University of Annaba in Algeria, in a subject on "Numerical Methods for Fractional derivative Equations". "Memoire" defended in June 19th 2011.
- Sept. 04–Aug. 05 : Analysis, Undergraduate
- Sept.99–Nov. 00 : Analysis, Statistics, Numerical Analysis, Algebra, Undergraduate.
- April 99–July 99 : Finite Difference methods for Partial Differential Equations, Graduate.
- Oct. 93–Junuary 97 : Analysis and Algebra, Undergraduate.
- Supervision with R. Herbin of two students of Master 2, University of Marseille I–France, in a project entitled: Coupled System with Irregular Data.

6 Publications

6.1 Submitted for publication

- An analysis of a second order time accurate scheme for a finite volume method for parabolic equations on general nonconforming multidimensional spatial meshes.

- Convergence analysis of some high–order time accurate schemes for a finite volume method for second order hyperbolic equations on general nonconforming multidimensional spatial meshes.

- Some second order time accurate for a finite volume method for the wave equation using a spatial multidimensional generic mesh.

6.2 Articles in Journals

- A theoretical analysis of a new finite volume scheme for second order hyperbolic equations on general nonconforming multidimensional spatial meshes.
  Accepted, in October 2011, for publication in Numerical Methods for Partial Differential Equations.

- Some abstract error estimates of a finite volume scheme for a nonstationary heat equation on general nonconforming multidimensional spatial meshes, 31 pages.
  With J. Fuhrmann. Accepted, June 2011, for publication in the journal "Applications of Mathematics" of Prague.

- Error estimates of the discretization of linear parabolic equations on general nonconforming spatial grids.
  With J. Fuhrmann. Comptes rendus - Mathématique 348/19-20 , 1119–1122, 2010.

- Some simples error estimates for finite volume approximation of parabolic equations.
  Comptes Rendus de l'Académie de Sciences, Paris, 346/9-10 pp. 571-574, 2008.

- Discretization of the coupled heat and electrical diffusion problems by the finite element and the finite volume methods.
  With R. Herbin. IMA Journal of Numerical Analysis, **28** (3), 469–495, 2008.

- Optimal defect corrections on composite nonmatching finite element meshes.
  With A.-S. Chibi. IMA Journal of Numerical Analysis, **27** (4), 765– 780, 2007

- Error Estimate for Finite Volume Approximate Solutions of Some Oblique Derivative Boundary Problems.
  With T. Gallouët. International Journal on Finite Volumes. **3** (2), 35 pages (electronic), 2006

- Impropved Convergence Order for Finite Volume Solutions. Part I: 1D Problems.
  With B. Atfeh. Arab Journal of Mathematical Sciences. **11** (1), 1–30, 2005.

- Impropved Convergence Order for Finite Volume Solutions. Part II: 2D Problems.
  With B. Atfeh. Arab Journal of Mathematical Sciences. **11** (2), 1–53, 2005.

6.3 Articles in Proceedings

- Some abstract error estimates of a finite volume scheme for the wave equation on general nonconforming multidimensional spatial meshes.
  Accepted in Finite Volumes for Complex Applications VI, Proceedings of the 6th International Symposium on Finite Volume for Complex Applications/ edited by J. Fořt, J. Fürst, J. Halama, R. Herbin, and F. Hubert, Springer, 2011.

- Some Error Estimates for the Discretization of Parabolic Equations on General Multidimensional Nonconforming Spatial Meshes.
  With J. Fuhrmann. Accepted for publication in LNCS (Lecture Notes in Computer Science) "Numerical Methods and Applications"Volume 6046, eds. I. Domov, S. Dimova, and N. Kolkovska, 2010.

- Towards an approach to improve convergence order in finite volume and finite element methods.
  Proceedings of ICNAAM "International Conference in Numerical Analysis and Applied Mathematics", Edited by T. E. Simos, G. Psihoyios, and Ch. Tsitouras, 1162–1165, 2009

- Some error estimates in finite volume methods for parabolic equations.
  With J. Fuhrmann. Finite Volumes for Complex Applications V, Proceedings of the 5th International Symposium on Finite Volume for Complex Applications/ edited by R. Eymard and J.-M. Hérard, Wiley, 233–240, 2008.

- On the discretization of Ohmic losses.
  With R. Herbin. Proceedings of Tamtam, 2007,Tipaza, Algeria, 217–222. AMNEDP-USTHB, 2007.

- On the discretization of the coupled heat and electrical diffusion problems.
  With R. Herbin. Numerical Methods and Applications. 6 th International Conference, NMA 2006, Borovets, Bulgaria, Aug. 20–24, 2006. Lecture Notes in Computer Science 4310 Springer 2007, pp. 1–15.

- Finite volume approximation for an oblique derivative boundary problem.
  with T. Gallouët. Finite Volumes for Complex Applications IV, Proceeding of the 4th International Symposium on Finite Volume for Complex Applications/edited by F. Benkhaldoun, D. Ouazar, and S. Raghay, Hermes-Penton, pp. 143–152, 2005.

- Improved convergence order of finite solutions and application in finite elements methods. Proceedings of ICNAAM: International Conference in Numerical Analysis and Applied Mathematics, Simos, G. Psihoyios and C. Tsitouras (eds), Wiley -VCH, pp. 94-98, 2005.

7 Contributions using COMSOL Multiphysics

- Convergence rates for models with coupled 1D/2D subdomains.
  With E. Holzbecher and M.-S. Litz. COMSOL Conference of Paris, 2010.

- On the convergence order of the COMSOL solutions in Sobolev norms.
  With Holzbecher. CD Proceedings of the COMSOL Conference of Budapest, November 2008.

- On the convergence order of the COMSOL solutions.
  With Holzbecher. CD Proceedings of the COMSOL Conference of Grenoble, October 2007.

7.1 Technical Reports and Preprints

- Numerical schemes to a non linear elliptic system with irregular data. Preprint.
  With R. Herbin.

- Some improvements of convergence order of finite volume solutions. Preprint of LATP n 04-13 .
  with B. Atfeh.

- Improved Convergence Order of Finite Element Solutions on Non-Uniform Meshes. Part I: 1D Problems.

- Improved Convergence Order of Finite Element Solutions on Non-Uniform Meshes. Part II: Bilinear Finite Elements on Rectangular Region.

- Improved Convergence Order of Finite Volume Solutions for Unstructured Meshes on Rectangle.

- Improved Convergence Order of Finite Volume Solutions for some Nonlinear and Parabolic Equations.

- Defect correction and discrete Schwarz method for second order boundary value problems. Preprint of the Department of Mathematics, Annaba, Algeria, 2001.
  With A.-S. Chibi.

- Une contribution à l'amélioration de convergence de la méthode des éléments finis sur une région circulaire. Preprint of the Department of Mathematics, Annaba, Algeria n26, 1998.
  With A.-S. Chibi.

8 Talks and Workshops

- Towards an approach to improve convergence order in finite volume and finite element methods. In "International Conference in Numerical Analysis and Applied Mathematics", Greece, September, 2009.

- On the convergence order of the COMSOL solutions.
  Presented by E. Holzbecher in the COMSOL Conference of Grenoble, October 2007.

- On the discretization of ohmic losses.
  Oral Presentation by R. Herbin in " TAMTAM07, 3 éme Colloque sur les Tendances dans les Applications Mathématiques en Tunisie, Algerie, Maroc". 14–15 Avril 2007.

- Numerical Schemes for Ohmic Losses.
  With Raphaele Herbin. Oral Presentation in the Workshop of "Modelling and Simulation of PEM Fuel Cells", September 18-20, 2006. WIAS, Berlin, Germany.

- Improved Convergence Order of Finite Solutions and Application in Finite Elements Methods.
  Oral Presentation in International Conference on Numerical Analysis and Applied Mathematics, ICNAAM , September 16-20, 2005, Rhodes, Greece.

- Finite Volume Approximation for an Oblique Derivative Boundary Probem.
  With Thierry Gallouët. Oral Presentation In the Fourth International Symposium in Finite Volumes for Complex Applications-Problems and Perspectives, July 4-8, 2005, Morocco.

- Towards highly accurate approximations through defect correction and discrete Schwarz method.
  With A.-S. Chibi. Oral Presentation by A. -S. Chibi in the Second International Conference on Mathematical Sciences ICM December 2004, United Arab Emirates University.

- Amélioration de l'Ordre de Convergence pour l'Approximation de Problèmes elliptiques.
  With Bilal Atfeh. Oral Presentation in "Congré National d'Analyse Numérique", June 2004.

- Some Improvements of Convergence Order of Finite Volume solution.
  Work-Shop of Volumes finis/Galerkin Discontinu , organized by Pr. R. Herbin in CMI, Marseille.

- Defect Correction Technique through Domain Decomposition Method.
  "Séminaire d'Analyse Appliquée e au CMI, univ. de Provence, Marseille,
  http://www.latp.univ-mrs.fr/equipes/analyse_appliquee/eqANAP-seminaire.html

- On the Dependence of the Convergence of the Corrections on Subdomains on the Degree of the Interpolation Operators.
  With Ahmed Salah Chibi. Oral Presentation in the Third Mathematical Colloque in Analysis and its Applications, 2002, Algeria.

- "Sur la Convergence de l'Alternative de Schwarz Discrete Accelerée".
  With Ahmed Salah Chibi. Oral Presentation in the Meeting of Algerian Mathematiciens RMA, 2000, Algeria.

- "Une Contribution à l'Amélioration de Convergence de la Méthode des Eléments Finis sur une Région Circulaire".
  With Ahmed Salah Chibi. Oral Presentation in the "Congré National de Mathématiques", Annaba, Algeria, 1999.

9 Invitation and Lectures

- March 10th 2009 at 11 AM: Invited for an interview for scientific research position in the scientific French organization CNRS. Unfortunately I could not reach "Institut Henri Poincaré" (IHP) in Paris, where my interview is scheduled, because I had no ticket flight from Algeria to Paris.

- Colloquium Tuesday, January 15th, 2008, in the Department of Mathematics and Statistics of Memorial University, St Johns, Newfoundland–Canada.

- Lecture in Necas Center of Mathematical Modeling, June 2007.

- Invited to do a Seminar, by Departamento de Matematica, Instituto Superior Tecnico, Lisboa, Portugal (June 2006).

- Invited by the Weierstrass Institute for Applied Analysis and Stochastics of Berlin WIAS , February 2006.
  Title of the talk: Finite Volume Methods for Elliptic Problems.

10 Funded projects

- Chef of PNR project "Etude Mathématique et numériques de quelques modeles de dynamique de Gaz, exemple systeme des equation d'Euler de la dynamique de Gaz".

- Chef of the CNEPRU project "L' Analyse mathématique et numérique de la récupération assistée des hydrocarbures" (Mathematical and numerical analysis of enhanced oil recovery).

11 Summary of my Research

11.1 Introduction

My principal subject is to "improve" convergence order (in some sense given below) of the finite element and finite volume solutions, and some related to topics (which are expected to help us to "improve" convergence order of numerical solutions).
I'm also interested with Numerical Approximation of Coupled Problems with Irregular Data.
In my short Postdoctoral position, between First April 2006–End February 2007, in (WIAS) Weierstrass Institute of Applied Analysis ans Stochastics, Berlin (Germany), I had a beautiful apportunity to:

- to learn some basic aspects in the Geology of salt water,
- to do some simulation, using codes of C++, in order to justify some gelogical hypotheses, and also to compare the results obtained by these codes and other results in the literature.
- to use Femlab in order to justify some results obtained for Double Diffusive Convection Problems in Porous media.

11.2 Improved convergence order of Finite Element/Finite Volume solutions

Finite volume method have been widely used for the numerical simulation of various types (elliptic, parabolic or hyperbolic equations) of conservation laws. Some of the important features of finite volume method are similar to those of finite element method: it may be used on arbitrary geometries, using structured or unstructured meshes, and leads to robust schemes. An additional feature is the local conservativity of the numerical fluxes.
The convergence order of the finite volume solution depends on the degree of the finite volume scheme used in the discretization of the equation to be solved.
To be more precise, we consider, for instance,

$$\mathcal{L}u = f, \tag{1}$$

be an elliptic equation posed on a sufficiently smooth domain Ω of $\mathbb{R}^d$, $d = 1, 2$ or 3 .
We consider an admissible mesh $\mathcal{T}$ in the sense of the Definition 9.1 of [26]. Thus the finite volume approximation of the equation (1) leads to a system to be solved:

$$\mathcal{L}^{\mathcal{T}} u^{\mathcal{T}} = f^{\mathcal{T}}, \tag{2}$$

where $\mathcal{L}^{\mathcal{T}}$ is a "good" matrix (in terms of computational cost) and $u^{\mathcal{T}}$ is the finite volume approximate solution.
In general, the convergence order of the finite volume solution $u^{\mathcal{T}}$ is $O\,(\text{size}(\mathcal{T}))$.
To get highly convergence order, we have to use, in general, high order schemes (see, for instance, [28]).
This leads to complex systems to be solved.
Nevertheless, we can use the "basic" finite volume solution $u^{\mathcal{T}}$ to get new finite volume approximations of higher convergence orders, successively. These new approximation can be computed using the same matrix $\mathcal{L}^{\mathcal{T}}$ and changing only the second members of the systems we resolve. Thus the computational costs of these approximations are "comparable" to that

of the basic finite volume solution $u^{\mathcal{T}}$.

Some of my reserach interests are concerned with the techniques which allow us to obtain these approximations on arbitrary admissible meshes in the sense of the Definition 9.1 of [26]. Note that, in case of some restricted Meshes (for instance, uniform Meshes in finite difference/element methods) we can obtain some improvements, (using some known techniques in the context of finite element and finite difference methods). Therefore, the results we obtained are new from the following points of view:

- Such "improvements" are done in finite volume method, which is not classical in the literature.
- The techniques we use to obtain such "improvements" are new. These techniques take into consideration the convergence order of the basic finite volume solution $u^{\mathcal{T}}$, namely the convergence order of the finite volume solutions is $O\left(\text{size}(\mathcal{T})\right)$ in the both norms of H^1 and L^2.

11.3 Outline of my Achievements to Improve Convergence Order

- In the Paper [1], we are interested with improving convergence order of finite volume solutions arising from arbitrary admissible mesh in 1D Problems.

  We consider linear elliptic equaton with Dirichlet boundary condition

  $$-u_{xx} + \alpha u_x + \beta u = f, \tag{3}$$

  where $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$ (but this can be extended to the case $\alpha \in \mathbb{R}$, it suffices only to change the scheme).

  We consider the finite volume scheme of three points on an admissible mesh $\mathcal{T}$.

  This leads to a finite volume solution $u^{\mathcal{T}}$ which can be computed using the system (2) where $\mathcal{L}^{\mathcal{T}}$ is a tridiagonal matrix.

  We prove that finite volume approximations $u_k^{\mathcal{T}}$ of orders $\left(\text{size}(\mathcal{T})\right)^{k+1}$, k is integer, can be obtained, successively. These approximations can be computed using the same matrix $\mathcal{L}^{\mathcal{T}}$ and changing the second member of the the algebraic systems we resolve.

- In the Paper [2], we consider an homogeneous linear elliptic equation in two dimensinal space, without convection term:

  $$\mathcal{L}(u) = -\Delta u + pu = f, \text{ on } \Omega = (0,1)^2. \tag{4}$$

We introduce an admissible mesh $\mathcal{T}$ on Ω and we consider a finite volume scheme of five points (cartesian mesh).

The finite volume approximation of (4) leads to the system (2) where $\mathcal{L}^{\mathcal{T}}$ is a block tridiagonal matrix.

The convergence order of $u^{\mathcal{T}}$ is in general $O(h)$ in both discrete L^2 and H_0^1-norms. This allows, as in first dimensional space, to approximate u of (4) and its first derivatives, in a discrete L^2-norm.

We prove that finite volume approximations $u_k^{\mathcal{T}}$ of orders $(\text{size}(\mathcal{T}))^{k+1}$ can be obtained, successively. These approximations can be computed using the same matrix $\mathcal{L}^{\mathcal{T}}$ and changing the second member of the the algebraic systems we resolve.

- An Abstract of [6] extends our results to some non-linear equations. We consider the following semi-linear equation with homogeneous boundary conditions

$$\begin{cases} -u_{xx}(x) = f(x, u(x)), \ x \in I = (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{5}$$

here f is smooth function satisfies

$$\left(f(x,s) - f(x,t)\right)(s-t) \leq \gamma (s-t)^2, \tag{6}$$

where $0 < \gamma < 1$.

Note that the condition (6) does not imply that f is a contraction.

A motivation for improving the convergence order of the finite volume approximate solution $u^{\mathcal{T}}$ is that the problems (5) arise when applying a method of line procedure for solving parabolic partial differential equations in one space dimensional or partial differential equations in two variables (cf. [21]). In [21], the authors developed a new finite difference formulae of arbitrary order on the so called superconvergent meshes under some mild conditions which differs from the condition (6).

Note that, the condition (6) allow us to obtain convergence Rate $O(h)$ (cf. [26], Remark 8.1).

The finite volume solution $u^{\mathcal{T}}$ can be computed using the non-linear equation

$$\mathcal{L}^{\mathcal{T}} u^{\mathcal{T}} = f^{\mathcal{T}}(u^{\mathcal{T}}), \tag{7}$$

We prove that, for each integer k, a finite volume approximations $u_k^{\mathcal{T}}$, called k^{th} correction, of order $O(h^{k+1})$ can be obtained. In addition to this, each k^{th} correction can be computed using the matrix $\mathcal{L}^{\mathcal{T}}$, i.e. $u_k^{\mathcal{T}}$ is defined by

$$\mathcal{L}^{\mathcal{T}} u_k^{\mathcal{T}} = f^{\mathcal{T}}(u_k^{\mathcal{T}}) + d_k^{\mathcal{T}}, \tag{8}$$

where $d_k^{\mathcal{T}}$ is a convenient defect defined in terms of the $(k-1)^{\rm st}$ correction.

- Our technique can be extended to some parabolic equations and some partial results are given in [6], [15], and [16].

  We can improve the convergence order of finite volume solutions approximating the linear parabolic equations

$$u_t(\mathbf{x},t) - \Delta u(\mathbf{x},t) = f(\mathbf{x},t),\ (\mathbf{x},t) \in \Omega \times (0,T), \tag{9}$$

  An initial condition is given by:

$$u(\mathbf{x},0) = u_0,\ \mathbf{x} \in \Omega, \tag{10}$$

  and an homogeneous boundary condition (for the sake of simplicity) is given by:

$$u(\mathbf{x},t) = 0,\ (\mathbf{x},t) \in \partial\Omega \times (0,T), \tag{11}$$

  where Ω is the either an interval in $\mathbb{R}$ or rectangle in $\mathbb{R}^2$.

  We introduce an admissible mesh in the sense of the Definition 5.1 of [26], in case of Ω is an interval, and an admissible mesh with rectangles in case of Ω is a rectangle.

  Let $\{K,\ K \in \mathcal{T}\}$ denote the control volumes and h denote the mesh size of the space discretization. The time discretization may be performed with a variable time step; in Order to simplify the notations, we choose a constant time step $k \in (0,T)$. Let $N_k = \max\{n \in \mathbb{N},\ nk < T\}$, and we denote $t_n = nk$, for $n \in \{0, ..., N_k+1\}$.

  Let $u^{\mathcal{T}} = (u_K^n)_{\{K\in\mathcal{T};n\in\mathbb{N}\}}$ be the finite volume approximate solution (see equation (17.1) in [26]).

  Eymard et al. [26] (cf. Theorem 17.1) proved that the following Error Estimate holds:

$$\left(\sum_{K\in Tt} \mathrm{m}(K)\,(e_K^n)^2\right)^{\frac{1}{2}} \le C(h+k),\ \forall n \in \{1, ..., N_k+1\}, \tag{12}$$

  where C is a constant independent of (h,k) and $e_K^n = u(\mathbf{x}_K, t_n) - u_K^n$.

  To improve the convergence order of $u^{\mathcal{T}}$, we justify at first the following Error Estimate (which allow us to approximate the derviative, in space direction, of u):

$$\left(\sum_{n=1}^{N_k+1} k\|e_n^{\mathcal{T}}\|_{1,\mathcal{T}}^2\right)^{\frac{1}{2}} \le C(h+k), \tag{13}$$

  where $\|\cdot\|_{1,\mathcal{T}}$ is the discrete H_0^1-norm defined in the Definition 9.3 of [26]; and $e_n^{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$ defined by $e_n^{\mathcal{T}}(\mathbf{x}) = u(\mathbf{x}_K, t_n) - u_K^n$ *a.e.* $\mathbf{x} \in K$, for all $K \in \mathcal{T}$.

Remark 2 We can prove also the following Error Estimate

$$\left(\sum_{n=1}^{N_k+1} \sum_{K \in Tt} \mathrm{m}(K) (\frac{e_K^{n+1} - e_K^n}{k})^2 \right)^{\frac{1}{2}} \leq C \frac{h+k}{\sqrt{k}}. \quad (14)$$

Remark 3 Note that the Error Estimates (13) and (14) are not given in [26].

We prove, provided that the solution u is smooth enough, that finite volume approximations $u_m^{\mathcal{T}}$, with m is an integer, of Order $O((h+k)^m)$, can be obtained using the same matrix that was used to compute $u^{\mathcal{T}}$.

- In the Paper [8], we discretize the following Oblique Boundary Value Problem

$$\begin{cases} -\Delta u(\mathbf{x}) = f(\mathbf{x}), \mathbf{x} \in \Omega \\ u_\eta(\mathbf{x}) + \alpha u_t(\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in \partial\Omega, \end{cases} \quad (15)$$

where $v_\eta = \nabla v \cdot \mathbf{n}$ and $v_t = \nabla v \cdot \mathbf{t}$, with $\mathbf{n} = (\mathbf{n}_x, \mathbf{n}_y)^t$ (resp. $\mathbf{t} = (-\mathbf{n}_y, \mathbf{n}_x)^t$) is the normal vector to the boundary $\partial\Omega$ and outward to Ω (resp. is a tangential derivative), and α is a constant on each line of the polygone.

And we assume that

$$\int_\Omega u(\mathbf{x}) d\mathbf{x} = 0. \quad (16)$$

We present a finite volume scheme to approximate (15)-(16) and we prove that the finite volume solutions arising from this scheme converge to the weak Solution of the Problem, when the mesh size tends to 0.

- In the Paper [9], we extend the previous result to some other cases:

  - We consider the case

$$\begin{cases} -\Delta u(\mathbf{x}) = f(\mathbf{x}), \mathbf{x} \in \Omega \\ u_\eta(\mathbf{x}) + (\alpha u)_t(\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in \partial\Omega, \end{cases} \quad (17)$$

  where α is a smooth function satisfing the following conditions:

$$\alpha \in \mathcal{C}^1(\overline{\Omega}) \text{ and } \alpha_t \geq -\delta, \quad (18)$$

  where δ is a given positive constant only depending on Ω. Note the conditions (18) allows us to use the classical Lax-Milgram theorem.

  We present a finite volume scheme to approximate the Problem (17)-(16) and we prove that the finite volume solutions arising from this scheme converge to the weak solution of the Problem (17)-(16), when the mesh size tends to 0.

In addition to this, we prove an Error Estimate of Order $O\left(\sqrt{\mathrm{size}(\mathcal{T})}\right)$, when the solution u of (16)-(17) is belonging to $\mathcal{C}^2(\overline{\Omega})$.

- We study also the case α is constant on each line of the boundary $\partial\Omega$.

  We assume that u satisfies:

  * $u \in \mathcal{C}^2(\overline{\Omega})$.
  * $u(\mathcal{S}_j) = 0$, for any corner of $\partial\Omega$.

  We prove in this case that a finite volume solution approximates u by Order $O\left(\sqrt{\mathrm{size}(\mathcal{T})}\right)$, can be obtained.

- Since the meshes in finite volume method are admissible, hence our technique (which allowed to improve the convergence order of the finite volume solutions) can be applied in order to improve convergence order of finite element solutions on nonuniform meshes (which is not classical in the literature).

  Indeed

  - In the Paper [17], we consider an elliptic equation in one dimensional space:

  $$\begin{cases} -(pu_{\mathbf{x}})_{\mathbf{x}} + qu = f, \text{ in } I = (0,1), \\ +\text{homogeneous Dirichlet conditions.} \end{cases} \tag{19}$$

    We introduce a nonuniform mesh $\mathcal{T}$ of mesh size h and we consider linear finite element $u^{\mathcal{T}}$ to approximate u.

    We prove that finite element approximations of order $O(h^{2k+2})$, with k is integer, can be obtained successively. These new approximations can be computed using the same matrix that used to compute $u^{\mathcal{T}}$.

  - In the Paper [18], we consider the following model

  $$\begin{cases} -\Delta u = f, \text{on } \Omega = (0,1)^2 \\ u|_{\partial\Omega} = 0. \end{cases} \tag{20}$$

    We introduce a triangulation $\mathcal{T}_h = \{e\}$ by rectangles. Let $(h_e \times k_e)$ denote the size of the rectangle e and we assume that there exists some $\zeta > 0$ independent of the discretization such that $\min(h_e, k_e) \geq \zeta \max(h_e, k_e)$, for all $e \in \mathcal{T}_h$ (see [23]). The convergence order of the bilinear finite element solution u^h is $O(h^2)$ in discrete H^1 norm.

    We prove that a finite element approximation of order $O(h^4)$ can obtained using the same matrix that used to compute u^h.

- Defect Correction method (A huge study of this technique can be found in [5] and in the references therein.), is one of the well known methods to obtain, successively, using the same matrix and changing only the second member, linear (or bilinear) finite element approximations of higher order. An approach for defect correction technique, in finite element methods, was suggested by [34]. The Moore's approach is based on some "higher order interpolatory" mappings. Roughly speaking, among the conditions required to apply the "classical" defect correction, in the context of finite element method, is the uniformity of the mesh and the regularity of the exact solutions. Some results are achieved in the case of one dimensional case, and on rectangular region in the case of two space dimensions , see [20], [21], [22] and the references therein.

  Our aim in the work [14], is to generalize the previous results of defect correction for more general smooth domain problems. Since, theoretically, smooth regions can be mapped, using conformal mappings, into the unit disque, we present in [14] an idea which allows us to apply the so called "defect correction" on the a disque Ω , and then to obtain, using the first iteration of the defect correction process, a linear finite element approximation u_1^h of order $O(h^4)$ in discrete H_0^1 norm. According to the literature of defect correction, we should have first a "basic" linear finite element approximation u^h of order $O(h^2)$ in the different higher order "discrete Sobolev norms".

  The idea is to decompose the whole domain Ω into overlapping sub–domains; the first one is an annulus Ω_1 and the second is a rectanglar region Ω_2 . Linear finite elements are introduced on each sub– region, with uniform meshes, in the cartesian coordinates. To compute a finite element solution on the whole region, we use a discrete Schwarz alternating. On each sub-region, there is a linear finite element approximation. These two approximations are coupled on the interior boundaries. To obtain the order $O(h^2)$ in discrete Sobolev norms for these two approximations, we introduce higher order interpolatory mappings on the interior boundaries and then we use these mappings to relate these two approximations on these boundaries. We give explict relations between the orders of these higher order interpolatory mappings such that each finite elements approximation on the the two sub–region are of order $O(h^2)$ in the different "discrete Sobolev norms". This entitles, as we explained before, to define on each sub–region, via the first iteration of defect correction process combined with a discrete Schwarz alternating, a new lineae finite element approximation of order $O(h^4)$.

12 Numerical Approximations of a Coupled Problem with an Irregular Data

In a collaboration [10]–[11] with Prof. Herbin, we provided with Finite Volume and Finite Element schemes for a Coupled System of Elliptic Equations, modelling electrical conduction and heat diffision with ohmic losses. The "Ohmic Losses" generate an L^1 right hand side, which requires an adequate procedure, adapted from L^1 theory of Boccardo and Gallouët. We are interested with is the following:

$$
\begin{aligned}
-\nabla(\kappa(\mathbf{x}, u(\mathbf{x})) \cdot \nabla\phi(\mathbf{x})) &= f(\mathbf{x}, u(\mathbf{x})),\ \mathbf{x} \in \Omega && (21)\\
\phi(\mathbf{x}) &= 0,\ \mathbf{x} \in \partial\Omega, && (22)\\
-\nabla(\lambda(\mathbf{x}, u(\mathbf{x})) \cdot \nabla u(\mathbf{x})) &= \kappa(\mathbf{x}, u(\mathbf{x}))|\nabla\phi|^2(\mathbf{x}),\ \mathbf{x} \in \Omega, && (23)\\
u(\mathbf{x}) &= 0,\ \mathbf{x} \in \partial\Omega, && (24)
\end{aligned}
$$

where Ω, of boundary $\partial\Omega$, is a convex polygonal open subset of $\mathbb{R}^d$; the functions f, κ and λ are given bounded functions from $\Omega \times \mathbb{R}$ to $\mathbb{R}$, continuous with respect to $y \in \mathbb{R}$ for a.e. $\mathbf{x} \in \mathbb{R}$, and measurable with respect to $\mathbf{x} \in \Omega$ for any $y \in \mathbb{R}$. These three functions have to be satisfied (which implies that Problem 21, for a given function u, is H^1–elliptic), there exists $\alpha \in \mathbb{R}^+$ such that:

$$\alpha \leq \kappa(\mathbf{x}, y) \text{ and } \alpha \leq \lambda(\mathbf{x}, y),\ \forall y \in \mathbb{R},\ \text{for a.e. } \mathbf{x} \in \Omega. \quad (25)$$

The domain Ω of may be seen as a domain made up of a thermally and electronically conducting material; u (resp. ϕ) denotes the temperature (resp. electrical potential); κ (resp. λ) is electrical conductivity (resp. thermal conductivity).Diffusion of electricity in a resistive medium induces some heating knows as "Ohmic Losses". The "Ohmic Losses" may be written as $\kappa|\nabla\phi|^2$.

It is expected, thanks to the fact that $f \in L^2(\Omega \times \mathbb{R}, \mathbb{R})$ and κ satisfies (25), that $\phi \in H^1(\Omega)$ (of course when ϕ exists), and then the second member $\kappa(\cdot, u(\cdot)|\nabla\phi|^2(\cdot)$ is only in $L^1(\Omega)$. Hence there is no "classical" variational formulation for Problem (24), for a given ϕ.

The existence of a weak solution to the Problem (21)- (24) is given in the article [30].

Note that $|\nabla\phi|^2$ belongs to $L^1(\Omega)$. Hence there is no "classical" variational formulation to this Problem. Therefore, there are no "classical" numerical schemes to this nonlinear coupled Problem.

We present two numerical schemes to approximate (21)- (24).

The first one is a linear finite element scheme. The existence of such approximation is ensured thanks to the fact that all the norms in a finite dimensional spaces are equivalent and the closed balls are compact.

The convergence of a finite element solution can be done using results of [29].

The second one is a finite volume scheme. The idea of a this scheme is inspired from the article [27].

The existence of such approximation is ensured thanks to the fact that all the norms in a finite dimensional spaces are equivalent and the closed balls are compact.

The convergence of a finite volume solution can be done using the weak convergence of [29].

13 Simulations on some thermhaline (double-diffusive) problems

In my Postodoctoral position in WIAS (Weierstrass Institute of Applied Analysis and Stochastics), in Berlin–Germany, I was interested with the finite volume approximations of a thermohaline problem.

13.1 A description of the problem and some achievement

A full description of the Thermohaline Problem I worked on, with Dr. J. Fuhrmann, can be found in the Thesis [33]. For the sake of completeness, we give here some basic concepts for this Problem.

The project is concerned with numerical simulations of Thermohaline convection in the North–East Germany Basin, and the identification of the qualitative state of the flow regim. The tasks are:

- the developement and validation of a simulation code for thermohaline convection in porous media on the base of a more general software framework for finite volume methods for systems of diffusion-convection -reaction equations,
- the application of this code to aformentioned problem.

Our achievement is that we introduced a small thickness between two layers and to see the behaviour of the fluid, namely "Rupelton Layer".

The result of simulations show that the convection force drives salt plumes reaching the surface whose tipes are moving apperantly randomly.

In addition to this, we remarked oscillations in these simulations which confirm some known results in thermohaline (see, for instance, [35] and [36]).

14 Simulation using COMSOL Multiphysics

Under the influence of my colleague of WIAS Dr. E. Holzbecker, I became very interested to use COMSOL Multiphysics in some goals of the numerical simulation. In [12]–[13], we use COMSOL Multiphysics, to justify numerically some known theoretical results concerning the convergence order of finite element solutions.

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