**University of Annaba–"M.I. Mathematiques et Informatique LMD"**
**First year undergraduation** **2011–2012**

## Analyse

## Complex numbers

Not finished yet; last update Tuesday 25th Oct. 2011

**Exercice 1.** (Third degree equation) Resolve the following third degree equation by remarking that $z_0 = -i$ (recall that $i^2 = -1$):

$$z^3 - (16 - i)z^2 + (89 - 16i)z + 89i = 0. \tag{1}$$

**Exercice 2.** (Some computations) Show that if $z = -1 - i$, then $z^2 + 2z + 2 = 0$

**Exercice 3.** (Complex and real forms)

1. Let $z = x + iy$ and consider its complex conjugate $\bar{z} = x - iy$. Show that
$$x = \frac{z + \bar{z}}{2}, \tag{2}$$
and
$$y = \frac{z - \bar{z}}{2i}. \tag{3}$$
2. Find the complex form, (2)–(3), using of the following equation $x + y = 1$.

**Exercice 4.** (Some computations) Simplify the following complex numbers

1. $$z = (1 + 2i)(3 - 5i); \;\; w = (4 - 3i)^2; \;\; u = i^3; \;\; v = i^4; \;\; k = i^{31} \tag{4}$$
2. $$z = \frac{1}{3 - 5i}; \;\; w = \frac{1 + 2i}{3 - 5i}; \;\; u = \frac{1}{(3 - 5i)^2}; \;\; v = \frac{1 + 2i}{1 - \sqrt{3}i}. \tag{5}$$

**Exercice 5.** (Some properties) Let $z \in \mathbb{C} \setminus \{1\}$. Prove by two methods that $\frac{1 + z}{1 - z} \in i\mathbb{R}$ if and only if $|z| = 1$.

**Exercice 6.** (Some properties) Prove that

1. For all $z, w \in \mathbb{C}$:
$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2). \tag{6}$$
2. For all $z, w \in \mathbb{C}$:
$$|z + w|^2 \leq (1 + |z|^2)(1 + |w|^2). \tag{7}$$
3. In which case there is equality in (7)?

**Exercice 7.** (Some properties) Let $z, w \in \mathbb{C}$ such that $|z| < 1$ and $|w| < 1$. Prove that

$$\left| \frac{z - w}{1 - \bar{z}w} \right| < 1. \tag{8}$$

**Exercice 8.** (Polar form of complex number) Put in the polar form the following complex numbers:

$$z = i;\;\; w = -1 + i;\;\; u = -5 + i\sqrt{75};\;\; v = \left(\frac{\sqrt{3} - i}{2}\right)^6;\;\; k = \left(\frac{2}{-\sqrt{3} + i}\right). \tag{9}$$

**Exercice 9.** (Square roots of a complex number) Find the square roots of the following complex numbers:

$$z = \frac{1}{2}\left(1 - i\sqrt{3}\right);\;\; w = \frac{1}{\sqrt{2}}\left(1 + i\right). \tag{10}$$

**Exercice 10.** (Cubic roots of a complex number) Find the cubic roots of the complex number $z = i$ and the fourth roots of the following complex number:

$$w = \frac{-1 - i\sqrt{2}}{2}. \tag{11}$$

**Exercice 11.** (Moivre rule) Use the Moivre rule to prove the following expansion:

1. $$\cos 2\theta = \cos^2\theta - \sin^2\theta. \tag{12}$$
2. $$\sin 2\theta = 2\sin\theta\cos\theta. \tag{13}$$
3. $$\cos 3\theta = 4\cos^3\theta - 3\cos\theta. \tag{14}$$
4. $$\sin 3\theta = -4\sin^3\theta + 3\sin\theta. \tag{15}$$

**Exercice 12.** (Euler rule) Use the Euler rule to linearize the following expansion:

1. $$\cos^4\theta. \tag{16}$$
2. $$\sin^3\theta\cos\theta. \tag{17}$$
3. $$\sin^4\theta.. \tag{18}$$
4. $$\cos^3\theta\sin^2\theta. \tag{19}$$