University of Annaba–"Ecole Preparatoire aux Sciences et Techniques"
Second year undergraduation **2010–2011**

Functions of Several Variables

Taylor expansion, derivation of two composed function, and extrema

Exercise 1. Assume that $f : \mathbb{R}^3 \to \mathbb{R}$ is a given function of class $\mathcal{C}^1$, and g the function defined by

$$g(x, y, z) = f(x - y, y - z, z - x). \tag{1}$$

Show that the following identity holds:

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} = 0. \tag{2}$$

Exercise 2. Let us consider

$$z = x^3 + y^2 + \sqrt{1 + x^2 + y^2}. \tag{3}$$

Compute $\dfrac{dz}{dt}$ when $x(t) = \cos(t)$ and $y(t) = \sin(\frac{t}{3})$.

Exercise 3. Let f be a given function of class $\mathcal{C}^1$ defined on $\mathbb{R}^2$ and consider the function g defined by

$$g(x, y) = f(x^2 - y^2, x + y - y^3). \tag{4}$$

Compute the first partial derivatives of g in terms of the those of f.

Exercise 4. Let f be a given $\mathcal{C}^2$ function defined on $\mathbb{R}^2$. Prove that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}, \tag{5}$$

where (r, θ) the known polar cordinates, that for $r \geq 0$ and $\theta \in [0, 2\pi[$

$$x = r\cos(\theta), \tag{6}$$

$$x = r\sin(\theta). \tag{7}$$

Exercise 5.

1. Recall that the differential of a function f at point (x, y) is a linear form denoted by $df_{(x,y)}$ and defined by

$$df_{(x,y)}(h, k) = \frac{\partial f}{\partial x}(x, y)h + \frac{\partial f}{\partial x}(x, y)k. \tag{8}$$

   Use the defintion (8), to compute the differential of the functions $f(x, y) = x$ and $f(x, y) = y$.

2. Use the known rules of the known differentiation of the sum, product, composed of two functions to compute the differential of the following function without computations of their partial derivatives:

$$f(x, y) = \ln(xy). \tag{9}$$

$$f(x, y, z) = xyz\sin(x^2 + y). \tag{10}$$

$$f(x, y, z) = \sin(x^2 + y)\exp(x + y - z). \tag{11}$$

Exercise 6.

1. Compute the Taylor expansion of order two at the point $\mathcal{P}$ for the following functions:

$$f(x, y) = \sin(x^2 + y)\exp(x + y - z),\ \mathcal{P} = (0, 0). \tag{12}$$

2.

$$f(x, y) = x^2 + y - y^3,\ \mathcal{P} = (1, 0). \tag{13}$$

3. Deduce then in each case the tangent plan at $\mathcal{P}$.

Exercise 7. A student asked to compute the tangent plan at the point $\mathcal{P} = (2, 3, 7)$ for the surface $z = x^4 - y^2$. The answer's was

$$z = 4x^2(x - 2) - 2y(y - 3). \tag{14}$$

1. Whithout checking the computation, show that answer's student is false
2. What are the mistakes done my the student to compute the stated tangent plan.
3. Give the right answer

Exercise 8. Consider the following function f

$$f(x, y) = \exp x \cos y. \tag{15}$$

1. Compute the Taylor expansion of order 0, 1, 2, 3 of f when $(x_0, y_0) = (0, \frac{\pi}{3})$
2. Provide an approximation for $f(-\frac{1}{10}, \frac{\pi}{3} + \frac{1}{50})$ using the computations provided in the previous item.

Exercise 9. Find the points of $z = 4x^2 + y^2$ in which the tangent plan is parallel to $x + y + z = 6$.

Exercise 10. Compute the Hessian matrices for the following functions

1.

$$f(x, y, z) = \cos(xyz). \tag{16}$$

2.

$$f(x, y) = \sin^2(\frac{y}{x}). \tag{17}$$

Exercise 11. Compute the critical points of the following functions:

1.

$$f(x, y) = 2x^2y + 2x^2 + y. \tag{18}$$

2.

$$f(x, y) = xy^2(1 + x + 3y). \tag{19}$$

Exercise 12. Give the type of the critical point $\mathcal{P}$ for the following functions:

1.

$$f(x, y) = x^2 - xy + y^2,\ \mathcal{P} = (0, 0). \tag{20}$$

2.

$$f(x, y) = x^2 + 2xy + y^2 + 6, \ \mathcal{P} = (0, 0). \tag{21}$$

3.

$$f(x, y) = x^3 + 2xy^2 - y^4 + x^2 + 3xy + y^2 + 10, \ \mathcal{P} = (0, 0). \tag{22}$$

Exercise 13. Compute the critical points of the following functions and decide if they are local minimum, local maximum, global minimum, or global maximum:

1.

$$f(x, y) = \sin\, x + y^2 - 2y + 1. \tag{23}$$

2.

$$f(x, y) = \exp\,(x^2 + y^2 - 2x + 2y). \tag{24}$$

3.

$$f(x, y) = \cos(x + y) + \sin\, y. \tag{25}$$

4.

$$f(x, y) = (x + y)\exp\,(-x^2 - y^2). \tag{26}$$

5.

$$f(x, y) = x^2 + xy + y^2 + 2x + 3y. \tag{27}$$

6.

$$f(x, y) = x\exp\,(y) + y\exp\,(x). \tag{28}$$