A new error estimate for a fully finite element discretization scheme for parabolic equations using Crank–Nicolson method

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Aim

We consider a conforming finite element method in which the discretization in time is performed using the Crank-Nicolson method for the non stationary heat equation (as a model for parabolic equations). We provide an error estimate in $W^{1,\infty}(L^2)$-norm. This error estimate seems not to be present in the existing literature.

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Overview : References based on...

- Bradji, A.: An analysis of a second-order time accurate scheme for a finite volume method for parabolic equations on general nonconforming multidimensional spatial meshes. Appl. Math. Comput. **219**/11 (2013), 6354–6371.
- Burman, E.: Crank–Nicolson finite element methods using symmetric stabilization with application to optimal control problems subject to transient advection–diffusion equations. Commun. Math. Sci. **9**/1 (2011), 319–329.

Overview : References based on...(Suite)

- Quarteroni, A. and Valli, A.: Numerical Approximation of Partial Differential Equations. Springer Series in Computational Mathematics 23. Berlin: Springer. (2008)
- Chatzipantelidis, P., Lazarov, R.D., and Thomée, V.: Some error estimates for the lumped mass finite element method for a parabolic problem. Math. Comput. **81**/277 (2012), 1–20.
- Raviart, P. A. and Thomas, J. M.: Introduction à l'Analyse Numérique des Equations aux Dérivées Partielles. Masson, Paris. (1983).

Overview : References based on...(Suite)

- Thomée, V.: Galerkin Finite Element Methods for Parabolic Problems. Springer-Verlag, Second Edition, Berlin (2006).

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Plan of the presentation

1. Problem to be discretized
2. Heat equation and a fully implicit finite element discretization using Crank-Nicolson method as discretization in time.
3. Some known error estimates e
4. Statement of an error estimate (main result)
5. Some applications
6. Conclusion and some perspectives

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Non stationary Heat equation

$$u_t(x,t) - \Delta u(x,t) = f(x,t),\ \ (x,t) \in \Omega \times (0,T), \tag{1}$$

where, Ω is an open bounded polyhedral subset in $\mathbb{R}^d$, with $d \in \mathbb{N}^\star$, $T > 0$, and f is a given function.
An initial condition is given by:

$$u(x,0) = u^0(x),\ \ x \in \Omega. \tag{2}$$

A Dirichlet boundary condition is defined by

$$u(x,t) = 0,\ \ (x,t) \in \partial\Omega \times (0,T), \tag{3}$$

where, we denote by $\partial\Omega = \overline{\Omega} \setminus \Omega$ the boundary of Ω.

About Heat equation?

1. (some physics): Heat equation $u_t - \Delta u$ is typically used in different applications, such as *fluid mechanics*, *heat and mass transfer*,...
2. (existence and uniqueness): existence and uniqueness of a weak solution of heat equation, with (2) (*initial condition*) and (3) (*Dirichlet boundary condition*) can be formulated using Bochner spaces; see for instance Evans book of partial differential equation

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Finite element discretization

1. Let $\{\mathcal{T}_h; h > 0\}$ be a family of shape regular and quasi-uniform triangulations of the domain Ω. The elements of $\mathcal{T}_h$ will be denoted by K. For each triangulation $\mathcal{T}_h$, the subscript h refers to the level of refinement of the triangulation, which is defined by $h = \max_{K \in \mathcal{T}_h} h_K$, where h_K denotes the diameter of the element K.
   Let $\mathcal{V}_0^h$ be the finite element space of continuous, piecewise polynomial functions of degree $k \geq 1$, and vanish on the boundary $\partial\Omega$
2. Uniform mesh on $(0, T)$ with constant step $\tau = T/(N+1)$.

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Discretization of initial conditions

The discretization of the initial condition (2), $u(x,0) = u^0(x)$ is performed using the orthogonal projection : Find $u_h^0 \in \mathcal{V}_0^h$ such that

$$\mathbf{a}(u_h^0, v) = -\left(\Delta u^0,\ v\right)_{\mathbb{L}^2(\Omega)} = \mathbf{a}(u^0, v),\ \forall v \in \mathcal{V}_0^h. \tag{4}$$

where $\mathbf{a}(\cdot,\cdot)$ denotes the bilinear form defined for all $(u, v) \in H^1(\Omega) \times H^1(\Omega)$ by

$$\mathbf{a}(u, v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) dx.$$

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Discretization of the heat equation

The discretization of the heat equation (1), $u_t - \Delta u = f$ is performed as : for any $n \in [\![0, M]\!]$, find $u_h^n \in \mathcal{V}_0^h$ such that, for all $v \in \mathcal{V}_0^h$

$$\left(\partial^1 u_h^{n+1}, v\right)_{\mathbb{L}^2(\Omega)} + \mathbf{a}\left(u_h^{n+\frac{1}{2}},\ v\right) = \left(\frac{1}{\tau}\int_{t_n}^{t_{n+1}} f(t)dt,\ v\right)_{\mathbb{L}^2(\Omega)}, \tag{5}$$

where

$$\partial^1 v^{n+1} = \frac{v^{n+1} - v^n}{k} \text{ and } v^{n-\frac{1}{2}} = \frac{v^n + v^{n-1}}{2}. \tag{6}$$

Some known error estimates: $\mathbb{L}^\infty(\mathbb{L}^2)$–error estimate

Throughout this talk, the notation C stands for a positive constant which is independent of the parameters of the discretization.
$\mathbb{L}^\infty(\mathbb{L}^2)$–error estimate. Under some regularity assumption on the data and on the exact solution, the following $\mathbb{L}^\infty(\mathbb{L}^2)$–error estimate holds, see the books of Quarteroni and Valli (2008) and Raviart and Thomas (1983), for all $n \in [\![0, M+1]\!]$:

$$\| u_h^n - u(t_n)\|_{\mathbb{L}^2(\Omega)} \leq C(h^{k+1} + \tau^2). \tag{7}$$

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Some known error estimates: $\mathbb{L}^2(H^1)$–error estimate

$\mathbb{L}^2(H^1)$–error estimate. Under some regularity assumption on the data and on the exact solution, the following $\mathbb{L}^2(H^1)$–error estimate holds, see the article of Burman (2011), for all $n \in [\![0, M+1]\!]$:

$$\left(\sum_{n=0}^{M} \tau \| e_h^{n+\frac{1}{2}} \|_{H^1(\Omega)}^2 \right)^{\frac{1}{2}} \leq C(h^k + \tau^2). \tag{8}$$

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New error estimate: $W^{1,\infty}(\mathbb{L}^2)$–error estimate

$W^{1,\infty}(\mathbb{L}^2)$–error estimate. Under some regularity assumption on the data and on the exact solution, the following $W^{1,\infty}(\mathbb{L}^2)$–error estimate holds, for all $n \in [\![0, M]\!]$:

$$\| \partial^1 \left(u_h^{n+1} - u(t_{n+1}) \right) \|_{\mathbb{L}^2(\Omega)}, \leq C(h^{k+1} + \tau^2), \tag{9}$$

where ∂^1 is the discrete time derivative

$$\partial^1 v^{n+1} = \frac{v^{n+1} - v^n}{k}. \tag{10}$$

An application of the new error estimate: approximation of the first time derivative

$\partial^1 u_h^{n+1}$ approximates the time derivative of u at $t_{\frac{n+1}{2}}$, i.e. $u_t(t_{\frac{n+1}{2}})$, by order $h^{k+1} + \tau^2$ in $\mathbb{L}^\infty(\mathbb{L}^2)$– norm, where $t_{\frac{n+1}{2}} = (t_{n+1} + t_n)/2$.

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An a priori estimate

Lemma

Assume that there exits $(\eta_h^n)_{n=0}^{M+1} \in \left(\mathcal{V}_0^h\right)^{M+2}$ *such that* $\eta_h^0 = 0$ *and for all* $n \in [\![0, M]\!]$

$$\left(\partial^1 \eta_h^{n+1}, v\right)_{\mathbb{L}^2(\Omega)} + \mathbf{a}\left(\eta_h^{n+\frac{1}{2}},\ v\right) = \left(\gamma^n,\ v\right)_{\mathbb{L}^2(\Omega)},\ \forall\, v \in \mathcal{V}_0^h. \tag{11}$$

Then the following estimate holds:

$$\|\,\partial^1 \eta_h^n\|_{\mathbb{L}^2(\Omega)} \leq\ C(\gamma + \bar{\gamma}),\ \ \forall n \in [\![1, M+1]\!], \tag{12}$$

where

$$\gamma = \max_{n=0}^{M} \|\,\gamma^n\|_{\mathbb{L}^2(\Omega)} \text{ and } \bar{\gamma} = \max_{n=1}^{M} \|\,\partial^1 \gamma^n\|_{\mathbb{L}^2(\Omega)}. \tag{13}$$

Idea on the proof of Lemma 1

First step: Estimate on $\partial^1 \eta_h^{j+1}$, $j \in [\![1, M]\!]$. Acting the discrete operator ∂^1 on (11) to get, for all $n \in [\![1, M]\!]$

$$\left(\partial^2 \eta_h^{n+1}, v \right)_{\mathbb{L}^2(\Omega)} + \mathbf{a}(\partial^1 \eta_h^{n+\frac{1}{2}}, v) = \left(\partial^1 \gamma^n, v \right)_{\mathbb{L}^2(\Omega)} . \tag{14}$$

Taking $v = \partial^1 \eta_h^{n+1} + \partial^1 \eta_h^n$ in (14) to get

$$\begin{aligned} \| \partial^1 \eta_h^{n+1} \|^2_{\mathbb{L}^2(\Omega)} &- \| \partial^1 \eta_h^n \|^2_{\mathbb{L}^2(\Omega)} + \frac{\tau}{2} \left| \partial^1 \left(\eta_h^{n+1} + \eta_h^n \right) \right|^2_{1,\Omega} \\ &= \tau \left(\partial^1 \gamma^n, \partial^1 \left(\eta_h^{n+1} + \eta_h^n \right) \right)_{\mathbb{L}^2(\Omega)} . \end{aligned}$$

Summing the previous equality over $n \in [\![1, j]\!]$, where $j \in [\![1, M]\!]$ and using some technical steps with a discrete Poincaré inequality yields

$$\| \partial^1 \eta_h^{j+1} \|^2_{\mathbb{L}^2(\Omega)} \leq \| \partial^1 \eta_h^1 \|^2_{\mathbb{L}^2(\Omega)} + 4T \, (C_p)^2 \, (\bar{\gamma})^2 . \tag{15}$$

Idea on the proof of Lemma 1 (Suite)

Second step: Estimate on $\partial^1 \eta_h^j$. Taking $n = 0$ in (11) to get (note that $\eta_h^0 = 0$)

$$\left(\partial^1 \eta_h^1, v\right)_{\mathbb{L}^2(\Omega)} + \frac{1}{2}\mathbf{a}(\eta_h^1, v) = \left(\gamma^0, v\right)_{\mathbb{L}^2(\Omega)}.$$

Taking $v = \partial^1 \eta_h^1$ in the previous equality and using some technical steps leads to

$$\|\partial^1 \eta_h^1\|_{\mathbb{L}^2(\Omega)} \leq \gamma. \tag{16}$$

This with (15) implies the desired estimate of Lemma 1. ∎

Idea on the proof of the new $W^{1,\infty}(\mathbb{L}^2)$–error estimate

The proof of the new $W^{1,\infty}(\mathbb{L}^2)$–error estimate is based essentially on the comparison with the following finite element scheme:
For each $n \in [\![0, M+1]\!]$, we compute $\bar{u}_h^n \in \mathcal{V}_0^h$ such that

$$\mathbf{a}(\bar{u}_h^n, v) = -\left(\Delta\, u(t_n),\ v\right)_{\mathbb{L}^2(\Omega)} = \mathbf{a}(u(t_n), v),\ \forall\, v \in \mathcal{V}_0^h. \tag{17}$$

The following convergence result can be shown using the classical error estimates in finite element methods

$$\left|\bar{u}_h^n - u(t_n)\right|_{1,\Omega} \leq\ Ch^k, \tag{18}$$

and, for all $j \in \{1, 2\}$

$$\|\partial^j \bar{u}_h^n - \partial^j u(t_n)\|_{\mathbb{L}^2(\Omega)} \leq\ Ch^{k+1}. \tag{19}$$

Idea on the proof of the new $W^{1,\infty}(\mathbb{L}^2)$–error estimate (Suite)

We write the error between the exact solution u and its finite element approximate solution as

$$u(t_n) - u_h^n = \left(u(t_n) - \bar{u}_h^n \right) + \left(\bar{u}_h^n - u_h^n \right) . \tag{20}$$

The error $\left(u(t_n) - \bar{u}_h^n \right)$ is already estimated in (17)–(18). It remains now to estimate $\eta_h^n =: \bar{u}_h^n - u_h^n$. Using the schemes satisfied by $\bar{u}_h^n$ and u_h^n, we get

$$\left(\partial^1 \eta_h^{n+1}, v \right)_{\mathbb{L}^2(\Omega)} + \mathbf{a}\,(\eta_h^{n+\frac{1}{2}},\ v) = \left(\mathbb{K}^{n,1} - \mathbb{K}^{n,2},\ v \right)_{\mathbb{L}^2(\Omega)}, \tag{21}$$

where

$$\mathbb{K}^{n,1} = -\partial^1 \left(u(t_{n+1}) - \bar{u}_h^{n+1} \right) \ \text{and}\ \mathbb{K}^{n,2} = -\frac{1}{\tau}\int_{t_n}^{t_{n+1}} \Delta\, u(t)dt + \frac{\Delta\, u(t_{n+1}) + \Delta\, u(t_n)}{2}.$$

Idea on the proof of the new $W^{1,\infty}(\mathbb{L}^2)$–error estimate (Suite)

Recall that

$$\left(\partial^1 \eta_h^{n+1}, v\right)_{\mathbb{L}^2(\Omega)} + \mathbf{a}\,(\eta_h^{n+\frac{1}{2}},\ v) = \left(\mathbb{K}^{n,1} - \mathbb{K}^{n,2},\ v\right)_{\mathbb{L}^2(\Omega)}. \qquad (22)$$

Using Lemma 1 to obtain

$$\|\,\partial^1 \eta_h^n\|_{\mathbb{L}^2(\Omega)} \leq\ C(\gamma + \bar{\gamma}),\ \ \forall n \in [\![\,1, M+1]\!], \qquad (23)$$

where

$$\gamma = \max_{n=0}^{M} \|\,\mathbb{K}^{n,1} - \mathbb{K}^{n,2}\|_{\mathbb{L}^2(\Omega)}\ \text{ and }\ \bar{\gamma} = \max_{n=1}^{M} \|\,\partial^1 \left(\mathbb{K}^{n,1} - \mathbb{K}^{n,2}\right)\|_{\mathbb{L}^2(\Omega)}. \qquad (24)$$

Error estimates (17)–(18) imply that, for $j \in \{0, 1\}$

$$\|\partial^j \mathbb{K}^{n,1}\|_{\mathbb{L}^2(\Omega)} \leq\ Ch^{k+1}. \qquad (25)$$

Idea on the proof of the new $W^{1,\infty}(\mathbb{L}^2)$–error estimate (Suite)

Recall that

$$\mathbb{K}^{n,2} = -\frac{1}{\tau}\int_{t_n}^{t_{n+1}} \Delta\, u(t)dt + \frac{\Delta\, u(t_{n+1}) + \Delta\, u(t_n)}{2}.$$

We use

$$\mathbb{K}^{n,2} = -\frac{1}{\tau}\int_0^{\tau} \left(\frac{\left(t - \frac{\tau}{2}\right)^2}{2} - \frac{\tau^2}{8} \right) \Delta\, u_{tt}(t + t_n)dt. \tag{26}$$

We can easily check that $\frac{\left(t - \frac{\tau}{2}\right)^2}{2} - \frac{\tau^2}{8}$ is non–positive for $t \in [0, \tau]$ and by some elementary calculations, we get

$$\int_0^{\tau} \left(\frac{\left(t - \frac{\tau}{2}\right)^2}{2} - \frac{\tau^2}{8} \right) dt = -\frac{\tau^3}{12}.$$

This with allows us to get, for $j \in \{0, 1\}$

$$\|\partial^j \mathbb{K}^{n,2}\|_{\mathbb{L}^2(\Omega)} \le C\tau^2. \tag{27}$$

Idea on the proof of the new $W^{1,\infty}(\mathbb{L}^2)$–error estimate (Suite)

Recall that

$$\gamma = \max_{n=0}^{M} \| \mathbb{K}^{n,1} - \mathbb{K}^{n,2} \|_{\mathbb{L}^2(\Omega)} \quad \text{and} \quad \bar{\gamma} = \max_{n=1}^{M} \| \partial^1 \left(\mathbb{K}^{n,1} - \mathbb{K}^{n,2} \right) \|_{\mathbb{L}^2(\Omega)}, \tag{28}$$

and the following estimates have been obtained

$$\|\partial^j \mathbb{K}^{n,1}\|_{\mathbb{L}^2(\Omega)} \leq Ch^{k+1} \quad \text{and} \quad \|\partial^j \mathbb{K}^{n,2}\|_{\mathbb{L}^2(\Omega)} \leq C\tau^2. \tag{29}$$

Consequently, with (28)

$$\|\gamma\|_{\mathbb{L}^2(\Omega)} \leq Ch^{k+1} \quad \text{and} \quad \|\bar{\gamma}\|_{\mathbb{L}^2(\Omega)} \leq C\tau^2. \tag{30}$$

This with (23) implies that

$$\| \partial^1 \eta_h^n \|_{\mathbb{L}^2(\Omega)} \leq C(h^{k+1} + \tau^2), \quad \forall n \in [\![1, M+1]\!]. \tag{31}$$

This completes the proof of new $W^{1,\infty}(L^2)$–error estimate. ■

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Conclusion

We considered the heat equation (as model), with initial and homogeneous boundary conditions in any space dimension. A new error error estimate in $W^{1,\infty}(L^2)$-norm is derived for an implicit Crank-Nicolson finite element scheme.

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Perspectives

- Is it possible to prove a convergence in $W^{1,\infty}(L^2)$-norm under a weak-regularity ?
- Is it possible to extend the obtained error estimate to other complex equations (or systems), e.g. time dependent incompressible Navier-Stokes equations, in which heat equation is involved?

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