CONVOLUTION EQUATIONS IN SPACES OF SEQUENCES
WITH AN EXPONENTIAL GROWTH CONSTRAINT

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One describes the sets of the solutions of the convolution equations $S*x=0$ (on the set $\mathbb{Z}$ or on $\mathbb{Z}_+=\{n\in\mathbb{Z}:n\geqslant 0\}$) in the spaces of sequences of the type $X=X_{(\beta,\alpha)}$, where $X_{(\beta,\alpha)}=\bigcup_{\gamma<\alpha}\bigcup_{\delta<1/\beta}\{x:|x_n|\leqslant C\gamma^{|n|},n<0,|x_n|\leqslant C\delta^n,n\geqslant 0\}$, $0\leqslant\alpha<\beta\leqslant+\infty$. One proves that any 1-invariant subspace $E, E\subset X$, coincides with $\operatorname{Ker} S$ for some S and, after the Laplace transform $x\to\hat{x}$, $\widehat{E^\perp}$ can be represented in the form $f\cdot A(K_{(\alpha,\beta)})$, where $K_{(\alpha,\beta)}=\{z:\alpha<|z|<\beta\}$. The space E can be written in the form $E=\operatorname{span}\{\{n^k\lambda^n\}_{n\in\mathbb{Z}}:\lambda\in\sigma\}+\{x\in X:x_k=0,k<m\}$, $\sigma\subset\mathbb{C}$, if and only if the representing function f is a pure Weierstrass product (in the ring $K_{(\alpha,\beta)}$, whose zeros do not accumulate to the circumference $|\lambda|=\alpha$.

One of the fundamental problems connected with the homogeneous convolution equations $S*f=0$ and with systems of such equations in $\mathbb{R}^n$ and $\mathbb{Z}^n$ (and, in particular, with differential, difference, etc. equations) consists in the justification of the known formalism defining all the solutions: one has to consider the Laplace transform $\hat{S}$ and from its roots (together with the multiplicities)

$$\hat{S}^{(k)}(\lambda)=0,\quad 0\leqslant k<k(\lambda), \tag{1}$$

to find all the "fundamental" exp-polynomial solutions $p(x)e^{\langle\lambda,x\rangle}$, while all the remaining solutions must be limits of linear combinations of these fundamental solutions. Preliminary information about this subject can be found in [1].

In this note we consider only the case of the group $\mathbb{Z}$ of integers and equations (systems of equations) on the group $\mathbb{Z}$

$$(S*x)_n=\sum_{k\in\mathbb{Z}}S_k x_{n-k}=0,\quad n\in\mathbb{Z}, \tag{2}$$

and on the semigroup $\mathbb{Z}_+$

$$(S*x)_n=0,\quad n\in\mathbb{Z}_+=\{n\in\mathbb{Z}:n\geqslant 0\}. \tag{3}$$

The second of these cases is substantially more complicated than the first one. Practically, it has not been considered in the literature, although its investigation presents a definite interest both from an intrinsic point of view (the description of left-translation-invariant subspaces) and from the point of view of the possible applications (for example, the asymptotic properties of the solutions of convolution equations

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are usually of interest only for positive times $n \to +\infty$) .

The problem of the completeness of the exp -polynomial solutions of Eq. (2) (or of a system of equations), i.e. the completeness of the linear combinations of solutions of the form

$$\{n^k \lambda^n\}_{n \in \mathbb{Z}}, \quad 0 \leqslant k < k(\lambda), \tag{4}$$

in the set of all solutions depends in an essential manner on the space X in which one seeks these solutions. If one does not impose any constraints on the solutions and one considers thus the space $X = \mathcal{F}(\mathbb{Z})$ of all functions (sequences) on $\mathbb{Z}$, then the "distribution" S must have necessarily a compact support (i.e. must be a finite sequence), while the description of the set of the solutions of Eq. (2) (as well as of Eq. (3)) becomes a purely algebraic (and entirely elementary) problem, to which we return at the end of Sec. 2, when the necessary notations will be introduced.

The situation is changed in an essential manner if the space X , where one seeks the solutions, imposes on them some constraints in the neighborhoods of the points at infinity $\pm\infty$. Then the answers to the problems (2) and (3) differ in a more radical manner: if for the equations on $\mathbb{Z}$ everything is determined by the zeros of the characteristic equation, then the solutions of Eq. (3) depend already (through some "Hankel operator") on the essential singularities of the function $\hat{S}$ at the neighborhoods of 0 and ∞ .

Below we give the exact description of the spaces which will be considered; now we note that the spaces $\operatorname{Ker} S$ of the solutions of Eqs. (2), (3) are, respectively, translation-invariant

$$\tau_k \operatorname{Ker} S \subset \operatorname{Ker} S, \quad k \in \mathbb{Z},$$
$$\tau_k \{x_n\}_{n \in \mathbb{Z}} = \{x_{n-k}\}_{n \in \mathbb{Z}}$$

and left-translation-invariant $\tau_k \operatorname{Ker} S \subset \operatorname{Ker} S$, $k \leqslant 0$. A natural extension of the problem of the determination of all solutions of convolution equations is the problem of the description of all (closed) translation-invariant subspaces. Indeed, in the cases considered below it turns out that each such subspace† E coincides with the set of solutions $E = \operatorname{Ker} S$ of a certain equation $S * f = 0$ (on $\mathbb{Z}$ or on $\mathbb{Z}_+$). Moreover, it is clear that the investigations of left- and right-invariant spaces constitute equivalent problems.

We shall consider closed right-invariant subspaces $E \subset X$,

$$\tau_k E \subset E, \quad k \in \mathbb{Z}_+,$$

which will be called 1-invariant if they are not left-invariant. The subspaces $E \subset X$ for which $\tau_k E \subset E$, $k \in \mathbb{Z}$, are said to be 2-invariant.

1. Spaces. We shall consider spaces distinguished by conditions of exponential growth (or decrease) at infinity, and, in general, the conditions are different near $+\infty$ and $-\infty$. For this we set

†The exact formulations can be found in Theorems 1 and 2.

$$M_{\gamma,\delta} = \left\{ x : \exists\, c,\ |x_n| \leqslant c\,\gamma^{|n|},\ n<0;\ |x_n| \leqslant c\,\delta^{n},\ n \geqslant 0 \right\}.$$

We shall investigate the convolution equations in the spaces

1. $X_{[\alpha,\beta]} = \bigcap_{\gamma>\alpha} \bigcap_{\delta>1/\beta} M_{\gamma,\delta},\ 0<\beta\leqslant+\infty,\ 0\leqslant\alpha<+\infty,$
2. $X_{[\alpha,\beta)} = \bigcap_{\gamma>\alpha} \bigcup_{\delta<1/\beta} M_{\gamma,\delta},\ 0\leqslant\beta<+\infty,\ 0\leqslant\alpha<+\infty,$
3. $X_{(\alpha,\beta]} = \bigcup_{\gamma<\alpha} \bigcap_{\delta>1/\beta} M_{\gamma,\delta},\ 0<\beta\leqslant+\infty,\ 0<\alpha\leqslant+\infty,$
4. $X_{(\alpha,\beta)} = \bigcup_{\gamma<\alpha} \bigcup_{\delta<1/\beta} M_{\gamma,\delta},\ 0\leqslant\beta<+\infty,\ 0<\alpha\leqslant+\infty.$

Each of these spaces is provided with the topology of the direct sum of projective and/or inductive limits (separately on each of the semiaxes $\mathbb{Z}_-$, $\mathbb{Z}_+$). Relative to the duality

$$(a, b) = \sum_{n\in\mathbb{Z}} a_n b_{-n}$$

the conjugate space is given by the equality

$$X^*_{\langle\alpha,\beta\rangle} = X_{>\beta,\alpha<},$$

where the parentheses in the right- and left-hand sides for α and β have opposite meanings.

In the scale of the spaces $X_{\langle\alpha,\beta\rangle}$, the classes $X = X_{(\alpha,\alpha]}$ and $X = X_{[\alpha,\alpha)}$ occupy exceptional places. Namely, one can verify that for these spaces X we have:

a) X does not contain nontrivial 2-invariant subspaces.†

b) All the closed 1-invariant subspaces of X are exhausted by the spaces $\{x \in X : x_n = 0 \text{ for } n < k\},\ k\in\mathbb{Z}$.

Everywhere below we shall assume that

$$\alpha \geqslant \beta \text{ (in case 1)};\quad \alpha > \beta \text{ (in the remaining cases).} \tag{5}$$

This allows us to give to the symbol $\langle\beta,\alpha\rangle$ the meaning of the interval with the same name ‡ of the real axis (which will be done in the sequel). Then

$$\lambda\in\mathbb{C},\ \{\lambda^n\}_{n\in\mathbb{Z}} \in X_{\langle\alpha,\beta\rangle} \Longleftrightarrow |\lambda|^{-1} \in \langle\beta,\alpha\rangle$$

In addition, if α,β run over the pairs indicated in (5), then the spaces $X_{\langle\alpha,\beta\rangle}$, $X^*_{\langle\alpha,\beta\rangle}$ cover in their totality the entire scale of spaces X , except the above mentioned exceptional subfamily. The 2-invariant subspaces of the spaces X and X^* are in a one-to-one correspondence. Similarly, to the right-invariant subspaces of X there correspond the right-invariant subspaces of X^* . This correspondence is realized by passing to the polar $E \mapsto E^{\perp} = \{ f : \forall q \in E\ \ f, q = 0 \}$. Finally, under the mapping $\{x_n\}_{n\in\mathbb{Z}} \to \{x_{-n}\}_{n\in\mathbb{Z}}$, to the left-invariant subspaces of $X_{\langle\alpha,\beta\rangle}$ there correspond the

†We note that at the same time we obtain an elementary example of a space X , which is the direct sum of a Frechet space and of a space of type LN^* , and of an operator τ_1 in X which does not have common invariant subspaces with its inverse $\tau_1^{-1} = \tau_{-1}$. The problem of the construction of such operators in Frechet spaces is investigated in [2], where one has considered spaces that are similar to X .
‡In which the parentheses for α,β have the same meaning as in $X_{\langle\alpha,\beta\rangle}$.

right-invariant subspaces of $X_{\langle 1/\beta, 1/\alpha\rangle}$ (where the parentheses for α and β have the same meaning as in $X_{\langle\alpha,\beta\rangle}$).

Thus, for the description of all 1- and 2-invariant subspaces in the scale $X_{\langle\alpha,\beta\rangle}$, it is sufficient to consider only the pairs (5) and the right-invariant subspaces in them.

Since, under the assumptions (5), $X^*_{\langle\alpha,\beta\rangle}$ is an algebra with respect to convolution, the 2-invariant subspaces of $X_{\langle\alpha,\beta\rangle}$ coincide with the subspaces invariant relative to the convolutions with elements of $X^*_{\langle\alpha,\beta\rangle}$, while the 1-invariant subspaces are those that are invariant with respect to the convolutions with the elements of $X^*_{\langle\alpha,\beta\rangle}$, concentrated on $\mathbb{Z}_+$.

2. Laplace Transform. We define the Laplace transform of the sequence $\{a_n\}$ as the formal power series

$$\{a_n\}^\wedge = \sum_{n\in\mathbb{Z}} a_n z^n$$

Then $\widehat{X^*}_{\langle\alpha,\beta\rangle} = A(K_{\langle\beta,\alpha\rangle})$ is the algebra of all functions, analytic in the annulus $K = K_{\langle\beta,\alpha\rangle}$,

$$K_{\langle\beta,\alpha\rangle} = \{ z : |z| \in \langle\beta,\alpha\rangle \},$$

endowed with the natural topology.

We set $\hat{E} = \{\hat{f} : f \in E\}$. If $f \in A(K)$, then let $z \longmapsto k_f(z)$ be an integer-valued function, defined on K , equal to the multiplicity of zero of the function f at the point z (the divisor of f), while the divisor of the subset $F \subset A(K)$ is defined by the equality $k_F = \min_{f\in F} k_f$. The set F is said to be divisorial if $F = \{ f \in A(K) : k_f \geqslant k_F \}$. The following lemma is well known in the spectral analysis of translation-invariant subspaces.

LEMMA. 1. E is a 2-invariant subspace of $X^*_{\langle\alpha,\beta\rangle}$ if and only if $\hat{E}$ is an ideal in $\widehat{X^*}_{\langle\alpha,\beta\rangle}$.

2. E is a 1-invariant subspace of $X^*_{\langle\alpha,\beta\rangle}$ if and only if $\hat{E}$ is a submodule (but not an ideal) in $\widehat{X^*}_{\langle\alpha,\beta\rangle}$ over the ring $A(K_{[0,\alpha\rangle})$.

3. The subspace $E \subset X_{\langle\alpha,\beta\rangle}$ is the closure of the linear hull $\mathcal{L}(\{n^k\lambda^n\})$ of the set of exp -polynomial sequences $\{n^k\lambda^n\}$, $n \in \mathbb{Z}$, contained in E , if and only if $\widehat{E^\perp}$ is divisorial. If $\operatorname{clos}\mathcal{L} \neq X_{\langle\alpha,\beta\rangle}$, then $\operatorname{clos}\mathcal{L} = \mathcal{L}$.

Clearly, this lemma can be applied also to the space $X = \mathcal{F}(\mathbb{Z})$ of all sequences; here $\widehat{X^*}$ is the algebra $\mathcal{P}$ of all polynomials in z and $1/z$, while the role of $A(K_{[0,\alpha)})$ is played by the ring $\mathcal{P}_A$ of the polynomials in z . It is also clear that the polynomials $p, q \in \mathcal{P}$ generate the submodule $\mathcal{L}(z^n p, z^k q : n, k \geqslant 0)$, equal to $Q\mathcal{P}_A$, where $Q = \mathrm{HO}\mathcal{D}(p, q)$ (the greatest common divisor of p and q over the ring $\mathcal{P}_A$). From here it follows that for every submodule $\hat{E} \subset \mathcal{P}$ either a) $\hat{E} \subset z^N \mathcal{P}_A$ for some $N \in \mathbb{Z}$ and then $\hat{E} = Q\mathcal{P}_A$, $k_Q = k_{\hat{E}}$, or b) $\hat{E} \not\subset z^N \mathcal{P}_A$ for all $N \in \mathbb{Z}$ and then $\hat{E} = Q\mathcal{P}$, $k_Q = k_{\hat{E}}$.

In particular, the translation-invariant subspaces E in $\mathcal{F}(\mathbb{Z})$ have the form $\mathcal{X}(\{n^{\kappa}\lambda^{n}\}_{n\in\mathbb{Z}}: 0\leq\kappa<k_{\hat{E}}(\lambda)) \overset{\text{def}}{=} \mathcal{X}(k_{\hat{E}})$ if they are 2-invariant, and the form

$$\mathcal{X}(\kappa_{\hat{E}}) \dotplus \{x\in\mathcal{F}(\mathbb{Z}): x_{\kappa}=0,\ \kappa<N\}$$

for some $N\in\mathbb{Z}$ if they are 1-invariant.

3. 2-Invariant Subspaces in $X_{\langle\alpha,\beta\rangle}$. The description of the ideals in $\widehat{X}^{*}_{\langle\alpha,\beta\rangle}$ is well known; see, for example [3]. They are divisorial and are principal ideals: $\hat{E} = \mathfrak{f}A(K)$, for some $\mathfrak{f}\in A(K)$. Therefore, the 2-invariant subspaces E for $X^{*}_{\langle\alpha,\beta\rangle}$ have the form $E=S*X^{*}_{\langle\alpha,\beta\rangle}$, while for $X_{\langle\alpha,\beta\rangle}$ they are represented in the form of (the closure of) a linear hull of exp-polynomial sequences, contained in E.

4. Description of 1-Invariant Subspaces in $X^{*}_{\langle\alpha,\beta\rangle}$. Let $\mathcal{E}\subset A(K)$. By $W_{\mathcal{E}}$ we shall denote the Weierstrass product with respect to the common zeros of $\mathcal{E}$, i.e. the function $W_{\mathcal{E}}=W^{\circ}W^{\infty}$, where $W^{\circ}=W^{\circ}(1/z)$, $W^{\infty}=W^{\infty}(z)$ are the Weierstrass products with respect to the zeros converging to the circumferences $|z|=\beta$ and $|z|=\alpha$, respectively; regarding this, see [4]. The function $W_{\mathcal{E}}$ also lies in $A(K)$. We set

$$X_{[*,\beta\rangle} \overset{\text{def}}{=} \{x: x|_{\mathbb{Z}_-}=0,\ x|_{\mathbb{Z}_+}\in X_{\langle\alpha,\beta\rangle}\},$$
$$X_{\langle\alpha,*]} \overset{\text{def}}{=} \{x: x|_{\mathbb{Z}_-}\in X_{\langle\alpha,\beta\rangle},\ x|_{\mathbb{Z}_+}=0\}.$$

THEOREM 1. Let $\hat{E}$ be a closed subspace of the space $A(K_{\langle\beta,\alpha\rangle})$. If $z\hat{E}\subset\hat{E}$, $z\hat{E}\neq\hat{E}$, then there exist $p\in\mathbb{Z}$ and $q\in A(K_{\langle\beta,\infty]})$ such that

$$\begin{aligned}\hat{E} &= \mathfrak{f}A(K_{[0,\alpha\rangle}),\\ \mathfrak{f} &= z^{p+1}e^{q}W_{\hat{E}}\end{aligned} \tag{6}$$

Conversely, any subspace of the mentioned form is 1-invariant in $A(K_{\langle\beta,\alpha\rangle})$.

For the case $\langle\beta,\alpha\rangle=(0,\infty)$ this theorem has been proved, by a different method, by K. Petrenko in his thesis (Leningrad State University, 1983).

THEOREM 2. E is a closed right-invariant subspace of the space $X^{*}_{\langle\alpha,\beta\rangle}$ if and only if there exists a sequence $S\in X^{*}_{\langle\alpha,\beta\rangle}$ such that $E=S*X^{*}_{[*,\alpha\rangle}$.

5. Description of 1-Invariant Subspaces in $X_{\langle\alpha,\beta\rangle}$. Clearly, a certain formula for the 1-invariant subspaces is contained implicitly in Theorems 1 and 2 but the determination of an appropriate coordinate description of the polar $(S*X^{*}_{[*,\alpha\rangle})^{1}$ in the space $X_{\langle\alpha,\beta\rangle}$ is not as simple as in the case of 2-invariant subspaces (Sec. 3). Below we make use of the following notation. We write each sequence $w\in X_{\langle\alpha,\beta\rangle}$ as a pair of sequences $w=\{x,y\}$, $x=w|_{\mathbb{Z}_-}$, $y=w|_{\mathbb{Z}_+}$. In the subspace E, $\tau_{\kappa}E\subset E$, $\kappa\geq 0$, we isolate a part E_{0}, having the form of a graph: $\{x,y\}\in E_{0}$, $y=0\Rightarrow x=0$ and, moreover, $E_{0}|_{\mathbb{Z}_+}=X_{\langle\alpha,\beta\rangle}|_{\mathbb{Z}_+}$ (indeed, some shift of the space E has these properties). This subspace has the form $E_{0}=\{\{\Gamma y,y\}: y\in X_{\langle\alpha,\beta\rangle}|\mathbb{Z}_+\}$, while the operator is sought from the condition of invariance of E_{0} with respect to right shifts. The entire space E is the sum

$$E = E_0 \dotplus \{R, 0\},$$

where R is some explicitly written subspace, depending on the values of the divisor in the neighborhoods of the circumference $\{|z|=\beta\}$.

In order to realize this plan, we shall represent each sequence $x \in X_{<\alpha,\beta>}$ in the form of a pair of functions

$$x \mapsto \hat{x} = \Big(\sum_{n<0} x_n z^n, \sum_{n\geq 0} x_n z^n\Big),$$

the first of which lies in $A(K_{>\alpha,\infty]})$, and the second in $A(K_{[0,\beta<})$; here (and in the sequel) the parentheses for α,β have the same meaning as the corresponding parentheses in $X_{>\beta,\alpha<}$. The inverse mapping (in fact, the mapping inverse to the Laplace transform) is denoted by $(\cdot,\cdot)^\vee$ so that $(\hat{x})^\vee = x$. In the sequel, for a function f, regular at the point ∞ of the extended complex plane, by the symbol f_∞ we shall denote the germ of the function f at this point, while for formal power series $a = \sum_{n\in\mathbb{Z}} a_n z^n$ we set

$$P_- a = \sum_{n<0} a_n z^n.$$

Finally, we consider a 1-invariant subspace E, $E \subset X_{<\alpha,\beta>}$. By Theorem 2, its polar can be written in the form $E^\perp = \check{f} * X^*_{[*,\alpha>}$, where $f \in A(K_{<\beta,\alpha>})$; moreover (see (6))

$$f = z^{p+1} x y, \text{ where } p \in \mathbb{Z}, \tag{7}$$

$$x \in A(K_{[0,\alpha>}); \ x = W_{X^{-1}(0)}; \ x(\lambda) \neq 0, |\lambda| \in [0,\beta<, \tag{8}$$

$$y \in A(K_{<\beta,\infty]}); \ y(\lambda) \neq 0, \ \lambda \in <\lambda,\infty]. \tag{9}$$

THEOREM 3. Let E be a (closed) 1-invariant subspace of the space $X_{<\alpha,\beta>}$. Then there exist $p \in \mathbb{Z}$ and functions x, y, possessing the properties (8), (9) and such that

$$\tau_p E = \{\{\hat{\Gamma} g, g\}^\vee : g \in A(K_{[0,\beta<})\} \dotplus \mathcal{L}(\{(y^{-1}(z-\lambda)^{-k})_\infty, 0\}^\vee : 0 < k \leq k_x(\lambda)), \tag{10}$$

where $\hat{\Gamma} g = -(y^{-1} P_-(gy))_\infty$. Conversely, any p, x, y of the indicated form define by formula (10) a 1-invariant subspace E.

COROLLARY. If E is a 1-invariant subspace in $X_{<\alpha,\beta>}$, then there exists $p \in \mathbb{Z}$ such that

$$\tau_p E|_{\mathbb{Z}_+} = X_{<\alpha,\beta>}|_{\mathbb{Z}_+}.$$

6. Discussion. First we note that the linear hull from formula (10) is necessarily closed (for the same reason for which the linear hull $\mathcal{L}(\{n^k \lambda^n\})$ in part 3 of the lemma is closed). It is also clear that the decomposition (10) is not uniquely determined by the space E but depends also on the factorization (7). In particular, if such a factorization is possible with $x = 1$ (or, in other words, the divisor $\hat{E}$ is trivial in the neighborhood of the circumference $\{|\lambda| = \alpha\}$), then it is possible to have also a representation (10) in which the second term is equal to zero.

It is natural to attempt to write a 1-invariant subspace E in the form $E_1 \dotplus E_2$, where E_2 is 2-invariant while E_1 is 1-invariant and does not contain 2-invariant subspaces. This is possible if and only if $k_{\hat{E}} \neq 0$ in the neighborhood of the circumference $\{|\lambda| = \beta\}$.

Another natural question is the following: for which 1-invariant subspaces E is it possible to have a representation

$$\tau_p E = \left\{ \{x, y\} : x \in \mathcal{L}\left(\{n^K \lambda^n\}_{n \in \mathbb{Z}} : 0 < K < k_{\hat{E}}(\lambda)\right), y \in X_{\langle \alpha, \beta \rangle}|\mathbb{Z}_+ \right\}, \tag{11}$$

similar to the representation for the spaces $E \subset \mathfrak{F}(\mathbb{Z})$? The answer: only for those $E \subset X_{\langle \alpha, \beta \rangle}$ for which $k_{\hat{E}} \neq 0$ in the neighborhood of the circumference $\{|\lambda| = \beta\}$, while the representing function f is a pure Weierstrass product (i.e. $g = 0$ in formula (6)).

As an illustration to the formula (10) we consider the case when the function y is rational. (It is clear that in this case the space E has the form (11) but in this model case we wish to compute the operator $\hat{\Gamma}$ and the second term $\mathcal{L}$ in the formula (10)). Let $y = \prod_{K=1}^{N}\left(1 - \frac{\lambda_K}{z}\right), \lambda_K \neq \lambda_j; x = W_{\{\mu_n\}}$. Then

$$\hat{\Gamma}\left(\sum_{K=0}^{\infty} a_K z^K\right) = \sum_{s=1}^{\infty} z^{-s} (\lambda_1^s, \dots, \lambda_N^s)\left(\{\lambda_K^{-\tau}\}_{\substack{K=1\\ \tau=1}}^{N}\right)^{-1} \begin{pmatrix} a_0 \\ \vdots \\ a_{N-1} \end{pmatrix},$$

$$\mathcal{L} = \mathcal{L}_n\left(\left(\sum_{\substack{\tau+\tau_1+\dots+\tau_N = -K-1\\ \tau, \tau_i \geq 0}} \lambda_1^{\tau_1} \cdot \ldots \cdot \lambda_N^{\tau_N} \cdot \mu_n^{\tau}\right)_{K=-1}^{-\infty}, 0\right).$$

7. Concluding Remark. With appropriate modifications, Theorems 1-3 can be carried over to certain sequence spaces of superexponential growth (decrease), for example, defined like $X_{\langle \alpha, \beta \rangle}$ with the aid of weight sequences of the form

$$W_n = \exp(c|n|^{\alpha}),\ \alpha > 1; \quad W_n = \exp(c \exp \alpha |n|),\ \alpha > 0.$$

In these cases one has to consider analytic functions in $\mathbb{C} \setminus \{0\}$, having a prescribed growth around zero and infinity.

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