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1989 Math. USSR Sb. 64 323

(http://iopscience.iop.org/0025-5734/64/2/A03)

Матем. Сборник
Том 136(178)(1988), Вып. 3

Math. USSR Sbornik
Vol. 64(1989), No. 2

BOUNDARY UNIQUENESS THEOREMS FOR ALMOST ANALYTIC FUNCTIONS, AND ASYMMETRIC ALGEBRAS OF SEQUENCES

UDC 517.5

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ABSTRACT. This article concerns algebras of C^1-functions in the disk $|z| < 1$ such that $|\bar{\partial} f(z)| < w(1-|z|)$, where $w \uparrow$ and $\int_0 \log\log w^{-1}(x)\,dx = +\infty$. For these functions a factorization theorem (on representation of each such function as the product of an analytic function and an antianalytic function, to within a function tending to zero as the boundary is approached) and a number of boundary uniqueness theorems are proved. One of these theorems is equivalent to a result generalizing the classical Levinson-Cartwright and Beurling theorems and consisting in the following. If $f(z) = \sum_{n<0} a_n z^n$, $|z| > 1$, $|a_n| < e^{-p_n}$, $\sum_{n>0} p_n/n^2 = \infty$, F is analytic in the disk $|z| < 1$, and $|F(z)| = o(w^{-1}(c(1-|z|))$ as $|z| - 1$ for all $c < \infty$, where $w(x) = \exp(-\sup_n(p_n - nx))$, then $f = 0$ and $F = 0$ if F has nontangential boundary values equal to the values of f on some subset of the circle $|z| = 1$ of positive Lebesgue measure. Here certain regularity conditions are imposed on p and w. Uniqueness and factorization theorems for almost analytic functions are applied to the description of translation-invariant subspaces in the asymmetric algebras of sequences

$$\mathfrak{A} = \{\{a_n\};\ \forall c\ \exists c_1\colon |a_n| < c_1 e^{-cp_n},\ n < 0,\ \exists c,\ \exists c_1\colon |a_n| < c_1 e^{cp_n},\ n \geq 0\}.$$

Bibliography: 15 titles.

New uniqueness theorems are proved here for functions analytic off the circle $\mathbf{T} = \{z \in \mathbf{C}\colon |z| = 1\}$, sufficiently smooth on one side of $\mathbf{T}$ (and up to $\mathbf{T}$), and having controlled growth on the other side.

These theorems generalize well-known assertions of Levinson and Cartwright and of Beurling, and their proofs are independent of the latter (and elementary in a certain sense). We use the technique of almost analytic extension (with rapid decrease of $|\bar{\partial} f|$ as the boundary is approached), and the results themselves can be expressed in the language of the algebras of almost analytic functions that arise.

These algebras are isomorphic to algebras of sequences with asymmetric asymptotic behavior at infinity, and we obtain a description of the translation-invariant subspaces for them.

§1. Introduction

The first nontrivial uniqueness theorem of the type indicated above is apparently due to Levinson and Cartwright (see [1]).

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30E25; Secondary 30H05.

THEOREM. *Let* $f(z) = \sum_{n<0} a_n z^n$, $|z| > 1$, $|a_n| < e^{-p_{|n|}}$, *where the sequence* $\{p_n\}$ *is monotonically increasing and* $\sum_{n\geq 1} p_n/n_2 = \infty$, *and let* F *be an analytic function on* $\mathbf{D} \overset{\text{def}}{=} \{z \in \mathbf{C}: |z| < 1\}$ *such that* $|F(z)| < v(1-|z|)$, $|z| < 1$, *where* $v \uparrow$ *and* $\int_0 \log\log v(x)\,dx < \infty$. *If* F *can be analytically extended to* f *across some arc of the circle* $\mathbf{T}$, *then* $f = 0$ *and* $F = 0$.

Beurling [2] proved an analogous theorem for functions that can be extended across an arbitrary set of positive measure. The symbol H^2 will denote the Hardy class on the disk $\mathbf{D}$:

$$H^2 = \left\{ f \in A(\mathbf{D}) \colon \sup_{0\leq r<1} \int_{\mathbf{T}} |f(r\zeta)|^2 \frac{|d\zeta|}{2\pi} \overset{\text{def}}{=} \|f\|_2^2 < \infty \right\},$$

$A(\mathbf{D})$ being the space of all functions analytic in $\mathbf{D}$. Each function F in H^2 has nontangential boundary values almost everywhere on $\mathbf{T}$; they will be denoted by the same letter F.

THEOREM. *Let* $f(z) = \sum_{n<0} a_n z^n$, $|z| < 1$, $|a_n| < e^{-p_{|n|}}$, *where* $\{p_n\}$ *is a monotonically increasing sequence with* $\sum_{n\geq 1} p_n/n_2 = \infty$, *and let* $F \in H^2$. *If* $F = f$ *on a subset of* $\mathbf{T}$ *having positive Lebesgue measure, then* $f = 0$ *and* $F = 0$.

As will be shown below, the growth restrictions on F in these theorems can be essentially weakened.

We shall connect the admissible growth of F with the smoothness of f. For this we introduce the following spaces of sequences:

$$\mathfrak{A}_+ \overset{\text{def}}{=} \{\{a_n\}_{n\in\mathbf{Z}} \colon a_n = 0,\ n < 0,\ \exists c, c_1 \colon |a_n| \leq c_1 e^{cp_n}\},$$
$$\mathfrak{A}_- \overset{\text{def}}{=} \{\{a_n\}_{n\in\mathbf{Z}} \colon a_n = 0,\ n \geq 0,\ \forall c\,\exists c_1 \colon |a_n| \leq c_1 e^{-cp_{|n|}}\},$$
$$\mathfrak{A} \overset{\text{def}}{=} \mathfrak{A}_+ + \mathfrak{A}_-,$$

where p is a quasianalytic weight, i.e.,

$$\{p_n\}_{n\geq 1} \text{ is a concave sequence of positive numbers, } p_n = o(n),$$
$$\lim_{n\to\infty} \frac{p_n}{\log n} = \infty, \qquad \sum_{n\geq 1} \frac{p_n}{n^2} = \infty. \tag{1}$$

To shorten the formulations it is convenient to introduce the following (fairly strong) regularity condition on the growth of p:

$$\exists A,\ 0 < A < \infty \colon \frac{p_n}{n} (\log n)^A \uparrow. \tag{2}$$

THEOREM 1. *Let*

$$f(z) = \sum_{n<0} a_n z^n, \quad |z| \geq 1, \qquad F(z) = \sum_{n\geq 0} a_n z^n, \quad |z| < 1,$$

where $\{a_n\} \in \mathfrak{A}$, *and suppose that the weight* p *satisfies condition* (2). *If* F *has nontangential boundary values equal to the values of* f *on some subset of* $\mathbf{T}$ *of positive Lebesgue measure, then* $f = 0$ *and* $F = 0$.

The proof of this theorem below is elementary in the sense that it does not use the Fourier transform, which sometimes obscures a loss of constructive meaning in proofs of uniqueness theorems.

The basis of the proof is the possibility, established by Dyn'kin [3], of almost analytic extension of smooth functions. We write this result in the form we need.

THEOREM. *If the weight p satisfies condition* (1), *then for any sequence* $\{a_n\}$ *in* $\mathfrak{A}_-$ *there is a function* F *of class* $C^1(\mathbf{D})$ *such that*

$$F|\mathbf{T} = \sum_{n<0} a_n z^n|\mathbf{T}, \qquad \forall c \exists c_1 : |\bar{\partial} F(z)| \le c_1 w^{-1}(c(1-|z|)).$$

Such an F is said to be an *almost analytic extension* of the function $\sum_{n<0} a_n z^n$.

Here the majorant w is determined from the weight p by means of the Legendre transformation:

$$w(x) \stackrel{\text{def}}{=} \exp\left(-\sup_n (p_n - nx)\right). \tag{3}$$

Further,

$$\left.\begin{aligned} w \uparrow,\ \log w^{-1}(x) \text{ is convex,} \\ \int_0 \log\log w^{-1}(x) = \infty. \end{aligned}\right\} \tag{4}$$

(See [4] about the last assertion.)

Condition (2) on the weight p implies a certain regularity of the majorant w.

LEMMA 1. *If the majorant* w *is determined by* (3) *from a weight* p *satisfying condition* (2), *then*

$$\varliminf_{x\to 0} \frac{\log \frac{\log w^{-1}(2x)}{\log w^{-1}(x)}}{\log x} > 0. \tag{5}$$

Analogous assertions are encountered in [5] and [6].

To each sequence $\{a_n\}$ in $\mathfrak{A}$ we assign a function $d^*\{a_n\}$ continuous in $\mathbf{D}$: the sum of the function $\sum_{n\ge 0} a_n z^n$ and an almost analytic extension of $\sum_{n<0} a_n z^n$ that exists by the Dyn'kin theorem. Of course, the mapping d^* is not uniquely determined due to the arbitrariness in the choice of the almost analytic extension.

We see what functions lie in the range of d^*. Let

$$\begin{aligned} Q \stackrel{\text{def}}{=} \{f \in C^1(\mathbf{D}),\ \forall c \exists c_1 : |\bar{\partial} f(z)| \le c_1 w(c(1-|z|)); \\ \exists c,\ \exists c_1 : |f(z)| \le c_1 w^{-1}(c(1-|z|))\}. \end{aligned} \tag{6}$$

LEMMA 2. a) $d^*(\mathfrak{A}) \subset Q$.

b) *For any function* f *in* Q *the following limits exist and are finite*:

$$d(f)_n \stackrel{\text{def}}{=} \lim_{r\to 1-0} \frac{1}{2\pi i} \int_{r\mathbf{T}} f(z) z^{-n-1}\, dz, \qquad n \in \mathbf{Z}, \tag{7}$$

and $(f) \stackrel{\text{def}}{=} \{d(f)_n\}_{n\in\mathbf{Z}} \in \mathfrak{A}$.

c) $d(d^*\{a_n\}) = \{a_n\}$ *for any sequence* $\{a_n\}$ *in* $\mathfrak{A}$.

We now find $\operatorname{Ker} d$. Let $\overline{\mathbf{D}}$ be the closure of $\mathbf{D}$, and let

$$J \stackrel{\text{def}}{=} \{f \in C^1(\mathbf{D}),\ \forall c \exists c_1 : |f(z)| + |\bar{\partial} f(z)| \le c_1 w(c(1-|z|))\}.$$

LEMMA 3. a) $d(J) = 0$.

b) *If* $f \in Q$ *and* $d(f) = 0$, *then* $f \in C^1(\overline{\mathbf{D}})$ *and* $f|\mathbf{T} = 0$.

If the majorant w *satisfies condition* (5), *then*

c) Q *is an algebra, and* J *an ideal of this algebra.*

d) *If* $f \in C^1(\overline{\mathbf{D}}) \cap Q$ *and* $f|\mathbf{T} = 0$, *then* $f \in J$.

Theorem 1 now follows from the next assertion.

THEOREM 2. *Let $f \in Q$, and suppose that f has nontangential boundary values equal to zero on a set of positive Lebesgue measure. If w satisfies condition* (5), *then $f \in J$.*

We note that this result is the "almost analytic" analogue of the well-known Luzin-Privalov uniqueness theorem [7].

The property

$$f, g \in A(\mathbf{D}),\ f \cdot g = 0 \Rightarrow f = 0 \text{ or } g = 0$$

of analytic functions in the disk (this is also a uniqueness theorem in essence) is generalized to the case of almost analytic functions in Theorem 3.

THEOREM 3. *If the majorant w satisfies condition* (5), *then f or g is in J whenever $f, g \in Q$ and $f \cdot g \in J$.*

Actually, this theorem asserts that the quotient algebra Q/J is an algebra without divisors of zero.

Theorems 2 and 3 can easily be deduced from the following assertion "on extension of an estimate". The assertion (Lemma 4 below) shows that the smallness of the values of an almost analytic function on some set leads (as in the case of analytic functions, where theorems of the two-constants theorem type are used) to an estimate that is hardly worse, but on a considerably broader set. Let

$$E_{w,c}(f) \stackrel{\text{def}}{=} \{z \in \mathbf{D} \colon |f(z)| < w(c(1-|z|))\}, \qquad E_w(f) \stackrel{\text{def}}{=} E_{w,1}(f).$$

LEMMA 4 (on extension of an estimate). *If the majorant w satisfies condition* (5), *then for each x sufficiently close to* 1 *there exists a number $r(x)$, $\lim_{x\to 1} r(x) = 1$, with the property that if $f \in C^1(\mathbf{D})$, $|f(z)| < w^{-1}(2(1-|z|))$, $|\bar\partial f(z)| < w((1-|z|)/3)$, and the set $x\mathbf{T} \cap E_w(f)$ contains an arc of length $\geq 1-x$, then $r(x)\mathbf{T} \subset E_w(f)$.*

We mention that similar assertions are also presented in [6].

To proceed further it is necessary to essentially refine the lemma on extension of an estimate; the lemma actually asserts that if $f \in Q\backslash J$ and $c < c(f)$, then the set $E_{w,c}(f)$ does not intersect the circles $r\mathbf{T}$ sufficiently close to $\mathbf{T}$ in arcs of length $1-r$.

LEMMA 5. *If the majorant w satisfies condition* (5), *and if $f \in Q\backslash J$, then there exist a number $c > 0$ and a sequence $\{x_k\}_{k\geq u}$ such that*

$$1 - 2^{-k+1} < x_k < 1 - 2^{-k}, \qquad k \geq u,$$
$$\bigcup [x_k, x_k + (\log w^{-1}(c(1-x_k)))^{-1}]\mathbf{T} \cap E_{w,c}(f) = \varnothing.$$

It follows from the proof (see §3, below) that the set of radii of the circles intersecting $E_{w,c}(f)$ is very sparse.

REMARK. Under the conditions of Lemma 5 there exists a number $c > 0$ such that if $R \stackrel{\text{def}}{=} \{r \colon 0 \leq r < 1;\ r\mathbf{T} \cap E_{w,c}(f) = \varnothing\}$, then

$$\lim_{r\to 1}(1-r)\log m(R \cap (r, 1)) = -\infty \tag{8}$$

(here m is linear Lebesgue measure).

Another uniqueness theorem can easily be deduced from Lemma 5—the theorem on the integrability of the logarithm of the modulus of an almost analytic function. We note that the first theorem of this kind was proved by Vol'berg [8]; see also [6], where there are further references.

Each element of the algebra Q admits an additive decomposition

$$f(z) = a_f(z) + \widehat{\bar\partial f}(z), \tag{9}$$

where $a_j \in A(\mathbf{D}) \cap Q \stackrel{\text{def}}{=} Q_+$, and

$$\widehat{\bar\partial f}(z) \stackrel{\text{def}}{=} \frac{1}{\pi} \iint_{|\zeta|<1} \frac{\bar\partial f(\zeta)}{z-\zeta}\, dm_2(\zeta) \in C^1(\hat{\mathbf{C}}) \cap A(\hat{\mathbf{C}} \backslash \overline{\mathbf{D}}).$$

Let $Q_- \stackrel{\text{def}}{=} \{f \in Q \colon a_f = 0\}$.

Using Lemma 5, we can solve the problem of the existence of a multiplicative decomposition $f = f_+ f_-$, $f_\pm \in Q_\pm$. (A whole series of results of this kind are well known, beginning with the Fejér-Riesz theorem on factorization of polynomials, up to the analogous decompositions adapted to various algebras of functions on the circle that arise in the theory of Toeplitz and Wiener-Hopf operators (see [9]).)

THEOREM 4 (on factorization). *If the majorant w satisfies condition* (5), *then for any function f in Q there exist a $g \in Q_+$ and an $h \in Q_-$ such that $f - gh \in J$, and there exist a $k \in \mathbf{N}$ and an $h_1 \in Q_-$ such that $z^k h h_1 - 1 \in J$.*

It can be assumed that any function in Q is, to within an element of J, a "product" of two functions analytic in $\mathbf{D}$ and $\hat{\mathbf{C}} \backslash \overline{\mathbf{D}}$, respectively.

From the factorization theorem we can immediately derive the uniqueness theorems presented above: Theorems 2 and 3 and the result on integrability of the logarithm of the modulus of an almost analytic function. Indeed, if, for example, a function f in Q has nontangential limits equal to zero on a set of positive measure, then the same is true for gh, for $z^k h_1 gh$ (see Theorem 4), for $(z^k h h_1 - 1)g + g$, and hence for g. It remains only to use the Luzin-Privalov theorem. We note that although there is no reference to the Luzin-Privalov theorem in the proof of Theorem 2 in §3, this proof actually contains the classical proof of the Luzin-Privalov theorem. Moreover, Theorem 4 can be used to solve the spectral analysis-synthesis problem in the algebra $\mathfrak{A}$.

In §2 the results on almost analytic functions are applied to the problem of describing the translation-invariant subspaces of convolution algebras.

In §3 we give proofs for the assertions formulated above, and §4 contains a brief discussion of the sharpness of these results.

§2. Convolution equations and invariant subspaces

Problems involving almost analytic functions arise naturally in the description of invariant subspaces of spaces of sequences with a certain asymptotic growth (decrease) at infinity.

If E is a closed proper subspace of some topological space of sequences, then E will be said to be *2-invariant* if $\tau E = E$, and *1-(left)-invariant* if $\tau E \subsetneqq E$, where $\tau\{a_n\} \stackrel{\text{def}}{=} \{a_{n+1}\}$.

As a rule, a description of the 1-invariant subspaces requires considerably greater efforts than a description of the 2-invariant subspaces. In this connection we can mention the well-known Beurling-Helson theorem, the article [10], which treats the space $C^\infty(\mathbf{T})$ isomorphic to the sequence space

$$\left\{ \{a_n\}_{n\in\mathbf{Z}} \colon \lim_{|n|\to\infty} \frac{\log|a_n|}{\log n} = -\infty \right\},$$

and the article [11], which treats spaces of functions analytic in annuli.

We endow the set

$$\mathfrak{A} = \{\{a_n\}_{n\in\mathbf{Z}},\ \forall c\exists c_1\colon |a_n| \le c_1 e^{-cp_{|n|}},\ n<0,\ \exists c,\ \exists c_1\colon |a_n| \le c_1 e^{cp_n},\ n\ge 0\},$$

with the topology of the direct sum of the projective and the inductive limits. Since the weight p satisfies condition (1), the equality $\mathfrak{A}^* = \mathfrak{A}$ is valid, where the duality is defined as

$$(\{an\},\{b_n\}) = \sum_{n\in\mathbf{Z}} a_n b_{-n}.$$

Then, by the Hahn-Banach theorem, the problem of the existence of nontrivial 2-invariant subspaces of $\mathfrak{A}$ turns out to be equivalent to the problem of solvability of the convolution equation

$$f*g=0, \qquad f,g\in\mathfrak{A}\backslash\{0\}. \tag{10}$$

The convolution $\{a_n\}*\{b_n\}$ is defined in the standard way:

$$\{a_n\}*\{b_n\} = \left\{\sum_{n\in\mathbf{Z}} a_n b_{k-n}\right\}_{k\in\mathbf{Z}}.$$

Indeed, if E is 2-invariant, then the equality $(f,\tau^k g) = 0$ holds for any $f\in E^\perp$, $g\in E$, and $k\in\mathbf{Z}$, i.e., $f*g=0$. Conversely, if $f*g=0$, then $f\in E \overset{\text{def}}{=} \{\varphi\colon \varphi*g=0\}$, and E is a 2-invariant subspace of $\mathfrak{A}$.

We show that if p satisfies conditions (1) and (2), then (10) does not have solutions.

THEOREM 5. *If the weight p satisfies condition* (2), *then the space $\mathfrak{A}$ does not contain 2-invariant subspaces.*

Note that under the conditions of the theorem $\mathfrak{A}$ is a convolution algebra, and the theorem itself asserts that there are no divisors of zero in $\mathfrak{A}$. Moreover, Theorem 5 can be interpreted as an assertion about the possibility of spectral analysis-synthesis in $\mathfrak{A}$; more precisely, as an extreme case of such an assertion (there are no exponential-polynomial sequences in $\mathfrak{A}$, but, as Theorem 5 asserts, also no nontrivial translation-invariant subspaces). See [12] about the spectral analysis-synthesis problem.

Theorem 5 can be derived directly from Theorem 3 with the use of the following assertion.

LEMMA 6. *If the weight p satisfies the condition* (2), *then the space $\mathfrak{A}$ is isomorphic to the quotient space Q/J.*

(The natural topology of the sum of the projective and inductive limits is introduced in Q; the ideal J is closed by Lemma 3.)

Thus, to prove the absence of 2-invariant subspaces of $\mathfrak{A}$ it suffices to use the lemma on extension of an estimate.

The factorization theorem is needed to describe 1-invariant subspaces.

It will be assumed that the majorant w satisfies condition (5).

Using Lemma 5, we find for each function f in $Q\backslash J$ a number $c>0$ such that $1\in\operatorname{clos} R$, where $R=\{r\colon r\mathbf{T}\cap E_{w,c}(f)=\varnothing\}$, and

$$\operatorname{wind} f \overset{\text{def}}{=} \lim_{\substack{r\in R\\ r\to 1}} \operatorname{wind} f|r\mathbf{T} > -\infty,$$

where $\operatorname{wind}\varphi$ on the right-hand side is the rotation (index) of a continuous function φ with respect to the point 0. It is clear that $\operatorname{wind} p = \deg p$, for any polynomial p, $\operatorname{wind} fg = \operatorname{wind} f + \operatorname{wind} g$ for any f and g in Q, and $\operatorname{wind} h < 0$ for any h in Q_-.

A canonical factorization of an element $f \in Q$ is defined as $z^k f - f_+ f_- \in J$, where $k \in \mathbf{N}$, $f_+ \in Q_+$, and $f_- \in (1 + Q_-) \cap (1 + Q_-)^{-1}$.

Such a factorization exists by Theorem 4, and $k + \operatorname{wind} f = n(f_+)$, where $n(\varphi)$ is the number of zeros of a function φ in $\mathbf{D}$.

LEMMA 7. *If the majorant w satisfies condition* (5), *then $f \in Q_-$ if and only if for any canonical factorization $z^k f - f_+ f_- \in J$ the function f_+ is a polynomial of degree at most $k - 1$.*

THEOREM 6. *A closed proper subspace E of the space Q/J is* 1*-invariant, $z^{-1}E \subset E$, $z^{-1}E \neq E$, if and only if there exist a $k \in \mathbf{Z}$ and an $f \in Q_+$ such that $f(z) \neq 0$, $z \in \mathbf{D}$, and $E = z^k f Q_-$.*

Using Lemma 6, we can get the following results from Theorems 4 and 6, respectively.

THEOREM 7. *If the weight p satisfies condition* (2), *then for each sequence $a \in \mathfrak{A}$ there exist sequences $b \in \mathfrak{A}_+$ and $c, d \in \mathfrak{A}_-$ and a number $k \in \mathbf{Z}_+$ such that $a = b * c$ and $c * d = \delta_{-k}$. Here $\delta_m = \{\delta_{mn}\}_{n \in \mathbf{Z}}$, δ_{mn} being the Kronecker symbol.*

THEOREM 8. *Suppose that the weight p satisfies the condition* (2). *A subspace E of $\mathfrak{A}$ is* 1*-invariant if and only if there exist a $k \in \mathbf{Z}$ and an $a = \{a_n\} \in \mathfrak{A}_+$ such that*

$$\sum_{n \geq 0} a_n z^n \neq 0, \quad z \in \mathbf{D}, \qquad E = \tau^k a * \mathfrak{A}_-.$$

Thus, all the 1-invariant subspaces of $\mathfrak{A}$ are generated (algebraically) by a single element (to within a shift)—an invertible element of $\mathfrak{A}_+$.

Finally, we present the following necessary and sufficient condition for periodicity in the mean (under left shifts) in $\mathfrak{A}$.

COROLLARY. *An element a in $\mathfrak{A}$ generates* (*topologically*) *the whole space $\mathfrak{A}$ under left shifts if and only if* $\operatorname{wind} d^*(a) = +\infty$.

§3. Proofs of the assertions

PROOF OF LEMMA 1. Let $\varphi(x) \stackrel{\text{def}}{=} \log w^{-1}(x) = \max_n (p_n - nx)$. Denote by p'_n the difference $p_{n+1} - p_n$. Everywhere outside a countable set of points x we have that $\varphi'(x) = -m$ at $x = p'_m$ if the maximum of $p_n - nx$ is attained at $n = m$.

The fact that $p_n(\log^A n)/n$ is increasing implies that

$$p'_{n+1} \frac{\log^A(n+1)}{n+1} > p_n \left(\frac{\log^A n}{n} - \frac{\log^A(n+1)}{n+1} \right),$$

$$\frac{np'_n}{p_n} > n \frac{\frac{\log^A n}{n} - \frac{\log^A(n+1)}{n+1}}{\frac{\log^A(n+1)}{n+1}};$$

therefore, $np'_n/p_n > 1 - 2A/\log n$ for sufficiently large n. Hence,

$$\frac{x\varphi'(x)}{\varphi(x)} = \frac{-mp'_m}{p_m - mp'_m} < -\frac{\log m}{2A}.$$

We estimate $x = p'_m$:

$$p'_m > \frac{p_m}{2m} > \text{const} \frac{1}{\log^A m}.$$

Then
$$\frac{x\varphi'(x)}{\varphi(x)} < -\operatorname{const}(p'_m)^{-1/A} = -\operatorname{const} x^{-1/A}.$$
Since φ is convex, $\varphi(x) > \varphi(2x) - x\varphi'(2x)$. This gives us that
$$\varliminf_{x\to 0} \frac{\log \frac{\varphi(2x)}{\varphi(x)}}{\log x} \geq \frac{1}{A}.$$

PROOF OF LEMMA 2. a) Since $\{a_n\}$ lies in $\mathfrak{A}$, an estimate of the growth of the analytic function $\sum_{n\geq 0} a_n z^n$ can be obtained from [13], Russian p. 158, English pp. 160–161. Next, by the Dyn'kin theorem, we can extend $\sum_{n\leq -1} a_n z^n$ to a function f in the class $C^1(\overline{\mathbf{D}})$ with the desired estimate on $|\overline{\partial} f|$.

b) Let $f = a_f + \widehat{\overline{\partial} f}$ be the decomposition (9). For $n \geq 0$
$$d(f)_n = \lim_{r\to 1-0} \int_{r\mathbf{T}} (a_f(z) + \widehat{\overline{\partial} f} f(z)) z^{-n-1}\, dz = \lim_{r\to 1-0} \int_{r\mathbf{T}} a_f(z) z^{-n-1}\, dz.$$
The expression under the limit sign does not depend on r. Again we get the desired estimate from [13], loc. cit. For $n < 0$
$$\begin{aligned} d(f)_n &= \lim_{r\to 1-0} \int_{r\mathbf{T}} (a_f(z) + \overline{\partial} f(z)) z^{-n-1}\, dz \\ &= \lim_{r\to 1-0} \int_{r\mathbf{T}} \widehat{\overline{\partial} f}(z) z^{-n-1}\, dz = \int_{\mathbf{T}} \widehat{\overline{\partial} f} f(z) z^{-n-1}\, dz. \end{aligned} \tag{11}$$
Let
$$f_r(z) \stackrel{\text{def}}{=} \frac{1}{\pi} \iint_{|\zeta|<r} \frac{\overline{\partial} f(\zeta)}{z-\zeta}\, dm_2(\zeta).$$
Then
$$|(\widehat{\overline{\partial} f} - f_r)(z)| < \iint_{r<|\zeta|<1} |\overline{\partial} f(\zeta)|\, dm_2(\zeta), \qquad z \in \hat{\mathbf{C}}.$$
Further,
$$\int_{\mathbf{T}} f_r(z) z^{-n-1}\, dz = \int_{r\mathbf{T}} f_r(z) z^{-n-1}\, dz.$$
Therefore, for any $c > 0$ there is a c_1 such that for any $r < 1$
$$|d(f)_n| \leq \frac{\text{const}}{r^n} + c_1 w(c(1-r)), \qquad n < 0,$$
$$|d(f)_n| \leq \inf_r \left(\frac{\text{const}}{r^n} + c_1 w(c(1-r))\right), \qquad n < 0.$$
This yields the desired estimate.

c) This obviously follows from the definitions of d and d^* and Dyn'kin's theorem.

PROOF OF LEMMA 3. a) Since $\lim_{|z|\to 1} |f(z)| = 0$, $d(f)_n$ is equal to zero for each n.

b) As already mentioned, $d(f)_n = \hat{a}_f(n)$ for $n \geq 0$, and $d(f)_n = \int_{\mathbf{T}} \widehat{\overline{\partial} f}(z) z^{-n-1}\, dz$ for $n < 0$. Therefore, if $d(f) = 0$, then $a_f = 0$, $f = \widehat{\overline{\partial} f}$, and $\widehat{\overline{\partial} f}|\mathbf{T} = 0$.

c) It suffices to verify that $c_1 w^2(cx) > w(x)$ for some $c, c_1 > 0$. But this follows from (5).

d) If f_r is defined as in Lemma 2, then
$$|(f - f_r)(z)| \leq \iint_{r<|\zeta|<1} |\overline{\partial} f(\zeta)|\, dm_2(\zeta), \qquad z \in \hat{\mathbf{C}}.$$

Therefore, $\forall c \exists c_1$: $|f_r(z)| \le c_1 w(c(1-r))$ for $|z| > 1$.

Further,

$$|f_r(z)| \le \iint_{|\zeta|<1} |\bar\partial f(\zeta)|\, dm_2(\zeta), \qquad z \in \hat{\mathbf{C}}.$$

Using the logarithmic convexity of the function

$$\sup_{|z|=t} \log|f_r(z)|, \qquad r < t \le +\infty,$$

we get that

$$\forall c \exists c_1 : \ \log|f_r|\sqrt{r}\mathbf{T}| \le \tfrac{1}{2}(c_1 + \log w(c(1-r)) + \text{const}).$$

Using (5), we get that $\forall c \exists c_1$: $\forall r$ $|f_r|\sqrt{r}\mathbf{T}| \le c_1 w(c(1-r))$. Therefore, $\forall c \exists c_2$: $\forall r$ $|f|r\mathbf{T}| \le c_2 w(c(1-r))$. The desired estimate is proved.

PROOF OF LEMMA 4. It follows from (5) that

$$\frac{\log w^{-1}(1-x^2)}{\log w^{-1}(1-x)} < \frac{1}{20e}$$

for all x sufficiently close to 1, $x > x_*$. We set $x_n = x^{2^n}$ and define for $x > x_*$ the sequence of numbers

$$y_0 = 1 - x, \qquad y_{n+1} = \min\left\{1, y_n + (1-x_n)\log\left(\frac{1}{20}\frac{\log w^{-1}(1-x_n)}{\log w^{-1}(1-x_{n+1})}\right)\right\}, \quad n \ge 0.$$

It is clear that $y_n > 1 - x_n$ for $x_n > x_*$. We prove by induction that

$$\left.\begin{array}{l}\text{for } x_n > x_* \text{ the set } x_n\mathbf{T} \cap E_w(f) \text{ contains}\\ \text{an arc of length at least } y_n.\end{array}\right\}. \tag{12}$$

If this has already been proved, then by using the monotonicity of the majorant w, we get that in fact $y_k = 1$ if $x_k > x_*$, and

$$\begin{aligned}\frac{1}{2}\int_{1-x}^{2^{k(1-x)}} \log\log w^{-1}(x)\,dx > 1 + (1-x_k)\log\log w^{-1}(1-x_{k+1})&\\ + 2^{k+1}(1-x)\log 20.&\end{aligned} \tag{13}$$

Indeed, if $y_k \ne 1$, then $y_n < 1$ for all $n = 0, \dots, k$, and so

$$\begin{aligned} y_k &= y_0 + \sum_{n=0}^{k}(1-x_n)\log\left(\frac{1}{20}\frac{\log w^{-1}(1-x_n)}{\log w^{-1}(1-x_{n+1})}\right) \\ &> \sum_{n=1}^{k}(x_{n-1}-x_n)\log\log w^{-1}(1-x_n) \\ &\quad - (1-x_k)\log\log w^{-1}(1-x_{k+1}) - \log 20 \sum_{n=1}^{k}(1-x_n). \end{aligned}$$

Further,

$$\begin{aligned}\sum_{n=1}^{k}(x_{n-1}-x_n)\log\log w^{-1}(1-x_n) &> \sum_{n=1}^{k} 2^{n-2}(1-x)\log\log w^{-1}(2^n(1-x)) \\ &\ge \frac{1}{2}\sum_{n=1}^{2^k-1}(1-x)\log\log w^{-1}(n(1-x)) \ge \frac{1}{2}\int_{1-x}^{2^k(1-x)} \log\log w^{-1}(x)\,dx.\end{aligned}$$

Therefore, from (13) and the fact that $y_k \neq 1$ it follows that $y_k > 1$, which is impossible. Hence $y_k = 1$.

Thus, by using the condition $\int_0 \log\log w^{-1}(x)\,dx = \infty$ we get that for any fixed $x_* < 1$ and for all x sufficiently close to 1 there is a number k such that $x_k > x_*$ and (13) holds, and therefore $y_k = 1$ and $x_k\mathbf{T} \subset E_w(f)$. This gives us the lemma.

We return to the proof of (12). The induction base is obvious. We prove the induction step.

Suppose that $\Omega \stackrel{\text{def}}{=} \{x_n^3 < |z| < x_n\}$ and $g = a_j + f_{x_n^3} \in A(\Omega)$. Denote by I an arc of the circle $x_n\mathbf{T}$ of length y_n such that $I \subset E_w(f)$. Then

$$|g|I| \leq w(1 - x_n) + w\left(\frac{1 - x_n^3}{3}\right), \qquad |g|\partial\Omega| \leq w^{-1}(2(1 - x_n)) + w\left(\frac{1 - x_n^3}{3}\right).$$

For the points ζ of the circle $x_n^2\mathbf{T}$ such that

$$\omega(\zeta, I, \Omega) > 2\frac{\log w^{-1}(1 - x_n^2)}{\log w^{-1}(1 - x_n)}, \tag{14}$$

the quantity $\log|g(\zeta)|$ can be estimated as

$$\begin{aligned}\log|g(\zeta)| &\leq \log(2w(1 - x_n))2\frac{\log w^{-1}(1 - x_n^2)}{\log w^{-1}(1 - x_n)} + 2\log w^{-1}(2(1 - x_n)) \\ &< \frac{3}{2}\log w^{-1}(1 - x_n^2),\end{aligned}$$

since the function $\log|g|$ is subharmonic in Ω. Therefore, $|f(\zeta)| < w(1 - |\zeta|)$ for all ζ in $x_n^2\mathbf{T}$ satisfying (14). It remains to estimate the measure of the set I_1 of such points ζ. (Obviously, I_1 is an arc.) Using the fact that $y_n > 1 - x_n$, we choose the arc

$$I_2 = \{x_n e^{i\theta},\ a < \theta < a + (1 - x_n)\} \subset I.$$

If $\zeta = x_n^2 e^{it}$, $t > a + (1 - x_n)$, then a simple computation of the harmonic measure shows that

$$\omega(\zeta, I_2, \Omega) \geq \frac{1}{10}\exp\left(-\frac{\pi}{2(1 - x_n)}(t - a - (1 - x_n))\right).$$

This implies that the length I_1 is at least y_{n+1}. The induction step for (12) is proved, and with it the whole lemma.

PROOF OF THEOREM 2. We use the sawtooth construction of Luzin and Privalov (see [7]). Suppose that the nontangential boundary values of f are equal to zero on a set e with $me > 0$. Then there exists a set $e_1 \subset e$ with $me_1 > 0$ such that the domain

$$\Omega = \bigcup_{z \in e_1} \{\zeta \in \mathbf{D}\colon |\zeta - z| < 2(1 - |\zeta|)\},$$

has the properties $\|f|\Omega\|_\infty < \infty$ and $\lim_{z\in\Omega, |z|\to 1} f(z) = 0$. We assume without loss of generality that

$$|f(z)| < \tfrac{1}{2}, \quad z \in \Omega; \qquad |f(z)| < w^{-1}(2(1 - |z|)), \quad z \in \mathbf{D}.$$

By the lemma on extension of an estimate, it suffices to show that there is a sequence $r_n \to 1$ such that the circles $r_n\mathbf{T}$ intersect $E_w(f)$ in arcs of length at least $1 - r_n$. Then the values $|f(z)|$ turn out to be small on entire circles with radii tending to 1. If we take the limit in (7) with respect to this sequence of radii, then we get that $d(f) = 0$ for $f \in J$.

Thus, let z be a point of density of the set e_1,

$$\lim_{\varepsilon\to 0}\frac{m(e_1\cap\{ze^{i\theta}:\ |\theta|<\varepsilon\})}{m\{ze^{i\theta}:\ |\theta|<\varepsilon\}}=1. \tag{15}$$

We compute $\omega(\zeta, e_1, \Omega\backslash(1-2\varepsilon)\mathbf{D})$ for $\zeta\in\{(1-\varepsilon)e^{i\theta}z:\ |\theta|<\varepsilon\}$. It is clear that

$$\omega(\zeta, e_1, \Omega\backslash(1-2\varepsilon)\mathbf{D}) > \tfrac{1}{2}\omega(\zeta, e_1, \Omega) > \tfrac{1}{2}(\omega(\zeta, e_1, \mathbf{D}) - \omega(\zeta, \mathbf{T}\backslash e_1, \mathbf{D}))$$

(since the values of the harmonic function $\omega(\zeta, e_1, \mathbf{D}) - \omega(\zeta, \mathbf{T}\backslash e_1, \mathbf{D})$ on $\partial\Omega$ are not greater than the values of $\omega(\zeta, e_1, \Omega)$). Therefore,

$$\omega(\zeta, e_1, \Omega\backslash(1-2\varepsilon)\mathbf{D}) > \omega(\zeta, e_1, \mathbf{D}) - \tfrac{1}{2}.$$

Using (15), we get that $\omega(\zeta, e_1, \mathbf{D}) > 3/4$ for sufficiently small ε. Then

$$\omega(\zeta, e_1, \Omega\backslash(1-2\varepsilon)\mathbf{D}) > 1/4.$$

Since

$$f_{1-2\varepsilon}+a_j\in A(\mathbf{D}\backslash(1-2\varepsilon)\overline{\mathbf{D}}),\qquad |(f_{1-2\varepsilon}+a_f)|\Omega\backslash(1-2\varepsilon)\mathbf{D}|<1,$$

and $|(f_{1-2\varepsilon}+a_f)|e_1|<cw(\varepsilon)$ for some $c<\infty$ independent of ε, we get that

$$\log|(f_{1-2\varepsilon}+a_f)(\zeta)|<\tfrac{1}{4}\log cw(\varepsilon)\quad \text{for } \xi\in\{(1-\varepsilon)e^{i\theta}z, |\theta|<\varepsilon\}.$$

Therefore, $|(f_{1-2\varepsilon}+a_f)(\zeta)| < (cw(\varepsilon))^4$ and $|f(\zeta)| < w(1-|\zeta|)$ for ζ in an arc of length $1-|\zeta|$. Applying the lemma on extension of an estimate, we conclude the proof of the theorem.

Proof of Theorem 3. It can be assumed without loss of generality that

$$|f(z)|<w^{-1}(2(1-|z|)),\quad z\in\mathbf{D},\qquad |\overline{\partial}f(z)|<w((1-|z|)/3),\quad z\in\mathbf{D},$$

and there exists a sequence $r_n\to 1$ such that $m(r_n\mathbf{T}\cap E_w(f))>\frac{1}{2}$. It follows from (5) that

$$\frac{\log w^{-1}(1-r_n^2)}{\log w^{-1}(1-r_n)}<\frac{1}{100}$$

for sufficiently large n.

We prove that the intersections $r_n^2\mathbf{T}\cap E_w(f)$ contain arcs of length $1-r_n^2$ for large n. After this, as in Theorem 2, it remains only to use the lemma on extension of an estimate.

We choose an arc $I\subset r_n\mathbf{T}$ such that $mI=1-r_n^2$ and $mI_1\geq\frac{9}{10}(1-r_n)$, where $I_1\stackrel{\text{def}}{=}I\cap E_w(f)$. Then $J_1\stackrel{\text{def}}{=}r_nI\subset r_n^2\mathbf{T}$. We prove that $J_1\subset E_w(f)$. Let

$$\Omega=\{r_n^3<|z|<r_n\},\qquad g=a_f+f_{r_n^3}\in A(\Omega).$$

Then

$$|g|I_1|<w(1-r_n)+w\left(\frac{1-r_n^3}{3}\right),\qquad |g|\partial\Omega|<w^{-1}(2(1-r_n))+w\left(\frac{1-r_n^3}{3}\right).$$

Since $\omega(\zeta, I_1, \Omega)\geq 1/30$ for each ζ in J_1, we get that for $\zeta\in J_1$

$$\log|g(\zeta)|<\log 2w^{-1}(2(1-r_n)+\tfrac{1}{30}\log 2w(1-r_n)<\tfrac{1}{2}\log w(1-r_n^2)).$$

Thus, $|f(\zeta)|<w(1-r_n^2)=w(1-|\zeta|)$ for any $\zeta\in J_1$.

The theorem is proved.

Proof of Lemma 5. It follows from (5) that for some $c>0$ and sufficiently small x

$$x^{-1}\log w^{-1}(x)<\log w^{-1}(cx).$$

It can be assumed without loss of generality that

$$|f(z)| < w^{-1}(2(1-|z|)), \tag{16}$$

$$|\bar{\partial} f(z)| < w((1-|z|)/3). \tag{17}$$

Using the lemma on extension of an estimate, each r sufficiently close to 1 we can distinguish a collection $\{x_k\} \subset r\mathbf{T} \cap E_w(f)$ of points that separate the circle $r\mathbf{T}$ into segments I such that $(1-r)/3 < |I| < \frac{4}{3}(1-r)$. For a given r we introduce $g \stackrel{\text{def}}{=} a_f + f_{r^2}$. Define $g_a = \int_{a\mathbf{T}} \log|g|\,dm$, $0 < a < 1$. We prove that

$$g_a > \log w((1-r)/2) \quad \text{for } r^{7/9} < a < r^{2/3} \text{ and for } r^{4/3} < a < ar^{11/9}. \tag{18}$$

Suppose first that $r^{7/9} < a < r^{2/3}$. Let $\Omega = \{r^2 < |z| < r^{2/3}\}$. It follows from (16) and (17) that

$$|g|\Omega| < w^{-1}(\tfrac{4}{3}(1-r)) + w(\tfrac{1}{3}(1-r^2)).$$

If (18) does not hold, i.e., $g_a \le \log w((1-r)/2)$, then there is an arc I of length $1-r$ on the circle $a\mathbf{T}$ such that

$$\int_I \log|g|\,dm \le (1-r)\log w((1-r)/2).$$

All the more so,

$$\int_I \log^-|g|\,dm \le (1-r)\log w((1-r)/2).$$

(Here $\log^- x$ is 0 if $\log x \ge 0$, and it is $\log x$ if $\log x < 0$.) Let x_k be the point among those marked that is closest to the arc I, $f(x_k) > w(1-r)$. Then

$$\begin{aligned}
\log\left(w(1-r) - w\left(\frac{1-r^2}{3}\right)\right) &< \log|g(x_k)| \le \int_{\partial\Omega} \log|g(\zeta)|\omega(x_k, d\zeta, \Omega) \\
&\le \sup_{z\in\partial\Omega} \log^+|g(z)| + \int_{\partial\Omega} \log^-|g(\zeta)|\omega(x_k, d\zeta, \Omega) \\
&\le \log 2w^{-1}(\tfrac{4}{3}(1-r)) + \int_I \log^-|g(\zeta)|\omega(x_k, d\zeta, \Omega) \\
&\le \log 2w^{-1}(\tfrac{4}{3}(1-r)) + \inf_{\zeta\in I} \frac{\omega(x_k, d\zeta, \Omega)}{dm} \int_I \log^-|g|\,dm \\
&\le \log 2w^{-1}(\tfrac{4}{3}(1-r)) + \tfrac{1}{30}w((q-r)/2) < \log w(\tfrac{2}{3}(1-r)),
\end{aligned}$$

which is impossible for r sufficiently close to 1.

Consequently, condition (18) holds for $r^{7/9} < a < r^{2/3}$. The proof is analogous for $r^{4/3} < a < r^{11/9}$.

We now estimate the number of zeros of g. The union of the disks of radii $\frac{3}{4}(1-r)$ about the points x_k contains the annulus $K = (r^{4/3}, r^{2/3})\mathbf{T}$. In each of these disks the number of zeros of the analytic function g can be estimated with the help of Jensen's formula:

$$\begin{aligned}
\int_0^{3/4(1-r)} \frac{n(t)}{t}\,dt &\le \int_{|\zeta - x_k| = 3/4(1-r)} \log|g(\zeta)|\,dm(\zeta) - \log|g(x_k)| \\
&\le \log[w^{-1}(\tfrac{1}{2}(1-r)) + w(\tfrac{2}{3}(1-r))] - \log[w(1-r) - w(\tfrac{2}{3}(1-r))],
\end{aligned}$$

where $n(t)$ is the number of zeros of g in the disk $|z - x_k| < t$. Further,

$$n(\tfrac{3}{4}(1-r)) \le \tfrac{1}{2}\int_{1/2(1-r)}^{3/4(1-r)} \frac{n(t)}{t}\,dt.$$

Therefore, if N is the number of zeros of g in K, then

$$N \le \frac{\text{const}}{1-r} \log w^{-1}(1-r) < \log w^{-1}(c^{3/2}(1-r)). \tag{19}$$

Let $h(z) = \prod(z - z_n)$, where $\{z_n\}$ is the sequence of zeros of g in the annulus K, taken according to multiplicity; then

$$\sup_{z\in\mathbf{D}} |h(z)| < \left(\sup_{z,n} |z - z_n|\right)^N < w^{-1}(c^2(1-r)). \tag{20}$$

A lower estimate of $h(z)$ can be obtained according to Cartan's theorem (see [14]), namely, there is a set B—a union of at most N disks with sum of radii $(1-r)/12$—such that

$$|h(z)| > ((1-r)/24e)^N > w(c^3(1-r)), \qquad z \in K\backslash B. \tag{21}$$

Let $F = g/h$. Then $\log|F|$ is a harmonic function on the annulus K. We choose numbers a and b in the respective intervals $(r^{7/9}, r^{2/3})$ and $(r^{4/3}, r^{11/9})$ such that $a\mathbf{T} \cap B = b\mathbf{T} \cap B = \varnothing$. Then it follows from (18), (20), and (21) that

$$\int_{a\mathbf{T}} \log|F|\,dm > 2\log w(c^2(1-r)), \qquad \int_{b\mathbf{T}} \log|F|\,dm > 2\log w(c^2(1-r)),$$
$$\int_{a\mathbf{T}} \log^-|F|\,dm > 2\log w(c^3(1-r)), \qquad \int_{b\mathbf{T}} \log^-|F|\,dm > 2\log w(c^3(1-r)).$$

Now let $\Omega_1 \stackrel{\text{def}}{=} \{r^{10/9} < |z| < r^{8/9}\}$. For $z \in \Omega_1$

$$\begin{aligned}\log|F(z)| &= \int_{\partial\Omega_1} \log|F(\zeta)|\omega(z, d\zeta, \Omega) \ge \int_{\partial\Omega} \log^-|F(\zeta)|\omega(z, d\zeta, \Omega)\\ &\ge \sup_{\zeta\in\partial\Omega} \frac{\omega(z, d\zeta, \Omega)}{dm} \int_{\partial\Omega} \log^-|F(\zeta)|\,dm > \frac{50}{1-r} 4\log w(c^3(1-r))\\ &> \log w(c^5(1-r)).\end{aligned}$$

For $z \in \Omega_1\backslash B$

$$|g(z)| > w(c^5(1-r))w(c^3(1-r)) > w(c^6(1-|z|)).$$

Thus, for $z \in \Omega_1\backslash B$ (for r sufficiently close to 1)

$$|f(z)| > \tfrac{1}{2} w(c^6(1-|z|)). \tag{22}$$

It follows from the properties of B mentioned above that the set $\{x \in [r^{10/9}, r^{8/9}]: x\mathbf{T} \cap B = \varnothing\}$ has length at least $(1-r)/4$ and consists of at most N segments. Therefore, it follows from (19) that for some x in the segment $[r^{10/9}, r^{8/9}]$

$$[x, x + (\log w^{-1}(c^3(1-r)))^{-1}]\mathbf{T} \cap B = \varnothing.$$

Letting r go to 1 and using (22), we get the desired result.

Proof of Theorem 4. Applying Lemma 5 to the given function f, we get that for some $c > 0$ there exist a sequence $\{x_k\}_{k\ge 0}$ and a number u such that

$$1 - 2^{-k+1} < x_k \le 1 - 2^{-k}, \quad x_0 = 0, \quad [x_k, y_k]\mathbf{T} \cap E_{w,c}(f) = \varnothing, \qquad k \ge u,$$

where $y_k = x_k + (\log w^{-1}(c(1-x_k)))^{-1}$.

We expand $\widehat{\bar\partial}$ in the sum

$$\widehat{\bar\partial f}(z) = \sum_{k\ge 0} \iint_{x_k \le |\zeta| \le x_{k+1}} \frac{\bar\partial f(\zeta)}{z - \zeta}\,dm_2(\zeta) \stackrel{\text{def}}{=} \sum_{k\ge 0} g_k.$$

Let

$$h_k(z) \overset{\text{def}}{=} \chi_{[x_k,y_k]}(|z|)\frac{6(|z|-x_k)(y_k-|z|)}{(y_k-x_k)^3}g_{k-1}(z), \qquad k>0,$$
$$h(z) \overset{\text{def}}{=} \sum_{k>0} h_k(z)\frac{z}{|z|}.$$

It is clear that

$$|h_k(z)| < (\log w^{-1}(c(1-|z|)))^{-1}\frac{3}{2}\sup_{|\zeta|>x_{k-1}}|\bar\partial f(\zeta)|.$$

Therefore,

$$\hat h \overset{\text{def}}{=} \iint_{|\zeta|<1}\frac{h(\zeta)}{z-\zeta}\,dm_2(\zeta) \in Q_-.$$

Since

$$\int_{x\mathbf{T}}\frac{g_k(\zeta)}{z-\zeta}\,d\zeta = g_k(z)$$

for $|z|>x>x_{k+1}$, it follows that

$$\iint_{|\zeta|<1}\frac{h(\zeta)}{z-\zeta}\,dm_2(\zeta) = \widehat{\bar\partial f}(z)$$

for $z\in\hat{\mathbf{C}}\backslash\mathbf{D}$. Further, $|(\widehat{\bar\partial f}-\hat h)(z)| \le |h_k(z)+f_k(z)|$, for $x_k<|z|<x_{k+1}$. Hence,

$$|(\widehat{\bar\partial f}-\hat h)(z)| \le \tfrac{1}{2}w(c(1-|z|))$$

for z sufficiently close to the circle $\mathbf{T}$.

Let $f_1 = a_f + \hat h$. Then $f - f_1 \in J$, and $|f(z)| > w(c(1-|z|))$ for $x_k \le |z| \le y_k$, $|f_1(z)| > \frac{1}{2}w(c(1-|z|))$ if $k \ge v$ is sufficiently large; hence $|f_1(z)| > \frac{1}{2}w(c(1-|z|))$ for $x_k \le |z| \le y_k$. Moreover, $\bar\partial f_1(z) = 0$ for $z \notin \bigcup_{k>0}[x_k,y_k]\mathbf{T}$. Therefore, we can define

$$f_2(z) \overset{\text{def}}{=} \exp\left(-\int_{\mathbf{D}\backslash x_v\bar{\mathbf{D}}}\frac{\bar\partial f_1}{f_1}(\zeta)\frac{dm_2(\zeta)}{z-\zeta}\right), \qquad f_2\in 1+Q_-,\ 1/f_2\in 1+Q_-.$$

It is easy to see that $f_3 \overset{\text{def}}{=} f_1f_2 \in A(\mathbf{D}\backslash x_v\bar{\mathbf{D}})\cap Q$. We decompose f_3 into a product $f_3 = f_4f_5$, $f_4\in A(\mathbf{D})\cap Q$, $f_5\in A(\hat{\mathbf{C}}\backslash x_v\bar{\mathbf{D}})$. Further, it can be assumed without loss of generality that $f_5(z)\ne 0$ for $r=(1+x_v)/2\le|z|<\infty$. In the disk $\mathbf{D}$ we define a function f_6 of class C^1 coinciding with f_5 on $\mathbf{D}\backslash r\mathbf{D}$. Then $f_6\in Q_-$, $f_3-f_4f_6\in J$. It is clear that for some m (m is the order of the zero of f_5 at infinity) it is possible to define a function f_7 of class C^1 in the disk that coincides with $z^{-m-1}f_5(z)$ on $\mathbf{D}\backslash r\mathbf{D}$. Further, $f_7\in Q_-$ and $f_6f_7z^{m+1}-1\in J$.

In summary:

$$f-f_4(f_6f_2^{-1})\in J, \qquad f_4\in Q_+, \qquad f_6f_2^{-1}\in Q_-,$$
$$f_2f_7\in Q_-, \qquad (f_2f_7)(f_6f_2^{-1})z^{m+1}-1\in J,$$

which is what was required.

Proof of Lemma 6. If in Q we introduce the topology of the sum of the projective and inductive limits, then it follows from the proof of Lemma 2b) that the mapping d acts continuously from Q to $\mathfrak{A}$. By Lemma 3a), $J=\operatorname{Ker} d$ is a closed ideal of Q. The linearity of d is obvious. Therefore, it can be assumed that d acts from

Q/J to $\mathfrak{A}$. The bijectivity of d follows from Lemma 2a), c). Consequently, d is a homeomorphism of Q/J onto $\mathfrak{A}$; it suffices to verify the multiplicativity of d on the set of polynomials, where it is obvious. Thus, d is an isomorphism of topological algebras. The lemma is proved.

PROOF OF LEMMA 7. a) If f_+ is a polynomial of degree at most $k-1$ and $f_- \in 1+Q_-$, then $f_+f_- \in z^kQ_-$; therefore, $z^kf \in z^kQ_-$ and $f \in Q_-$.

b) Conversely, if $f \in Q_-$, then $f_+f_- \in z^kQ_-$ and $f_+ \in z^kQ_- \cap Q_+$; consequently, f_+ is a polynomial of degree at most $k-1$.

PROOF OF THEOREM 6. It follows from the Hahn-Banach theorem that if $f \in E^\perp \subset (Q/J)^* = Q/J$ and $g \in E$, then $fg \in Q_-/J$. Conversely, if $fE \subset Q_-/J$, then $f \in B^\perp$. It follows from Lemma 7 that wind $f < +\infty$ for any representative f of an element in $E \cup E^\perp$. Therefore, they can be factored as follows:

$$z^kf - pf_0f_- \in J, \tag{23}$$

where p is a polynomial formed from the zeros of f_+ in $\mathbf{D}$, and f_0 is a function in Q_+ nonvanishing in the disk and hence invertible in Q_+ (this can easily be proved by using the Carathéodory inequality (see [14]), and $f_- \in (1+Q_-) \cap (1+Q_-)^{-1}$. Now

$$f \in E,\ g \in E^\perp \Rightarrow f_0g_0 = 1, \qquad \operatorname{wind} f + \operatorname{wind} g < 0 \Rightarrow fg \in Q_-.$$

Therefore, if

$$k_1 = \sup_{f\in E} \operatorname{wind} f, \qquad k_2 = \sup_{g\in E^\perp} \operatorname{wind} g,$$

then $k_1 + k_2 = -1, f_0z^kzQ_-/J \subset E$, and $f_0^{-1}z^{k_2}zQ_-/J \subset E^\perp$, where f_0 is the function in the decomposition of a representative of some element $f \in E$ according to (23). But

$$(f_0z^{k_1}zQ_-/J)^\perp = (f_0^{-1}z^{k_2}zQ_-/J), \qquad (f_0^{-1}z^{k_2}zQ_-/J)^\perp = (f_0z^{k_1}Q_-/J),$$

and therefore $E = f_0z^{k_1}Q_-/J$. Arguing in the same way, we get that, also conversely, every subspace E of the form z^kfQ_-/J is 1-invariant.

§4. Discussion of the results

The Levinson-Cartwright results and Theorem 1 forbid the extension of a function analytic in $\hat{\mathbf{C}}\setminus\overline{\mathbf{D}}$ and quasianalytically smooth in $\hat{\mathbf{C}}\setminus\mathbf{D}$ to an analytic function in the disk with certain growth restrictions. V. P. Khavin has constructed an example showing that these restrictions cannot be completely removed.

Here is a sketch of Khavin's construction. We construct a function $f \in A(\hat{\mathbf{C}}\setminus\{1\})$ that is quasianalytically smooth in $\hat{\mathbf{C}}\setminus\mathbf{D}$. Let $\psi(x)$ be a positive function with sufficiently rapid decrease as $|x| \to \infty$, and let $F(z)$ be an entire function such that $|F(x)| < \psi(x)$ and $F \not\equiv 0$ (F exists by Carleman's theorem (see [15]).

We map the upper half-plane $\mathbf{C}_+$ conformally onto the disk $\mathbf{D}$: $\zeta = z/(z+i)$. Let $\varphi(\zeta) = F(z(\zeta))$, $D = \frac{1}{2} + \frac{1}{2}\mathbf{D}$, and $\gamma = \partial\mathbf{D}$. Then the desired function f is defined as the Cauchy integral of φ:

$$f(z) = \int_\gamma \frac{\varphi(\zeta)}{z-\zeta}\,d\zeta, \qquad |z| > 1.$$

By contracting the contour γ, we can see that $f \in A(\hat{\mathbf{C}}\setminus\{1\})$; f becomes arbitrarily smooth in $\hat{\mathbf{C}}\setminus\mathbf{D}$ if ψ decreases rapidly enough. The function f is not identically equal to zero, because $\varphi \notin H^1$ if ψ decreases rapidly enough.

The above theorems use a fairly strong regularity condition. In some of them (Theorems 2, 3, and 5) these conditions are excessive in order to simplify the proofs, while in others (Theorems 4, 7, and 8) they are best possible of those known to the author. At the same time the classical theorems (for example the Beurling theorem) do not contain conditions of this kind. The question arises as to how essential these conditions are here. Vol'berg constructed an example showing that the theorem on integrability of the logarithm of the modulus of an almost analytic function ceases to be valid if the condition $x \log w^{-1}(x) \uparrow \infty$ is violated. Consequently, Lemma 5 and the factorization theorem cease to be valid.

A description of 1-(left)-invariant subspaces of the algebra $\mathfrak{A}$ was obtained in §2. A similar result can be obtained also for 1-(right)-invariant subspaces.

The author thanks N. K. Nikol'skiĭ for his undivided attention to this work, and A. L. Vol'berg, V. P. Khavin, and D. V. Yakubovich for useful discussions.

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Received 4/APR/87

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Translated by H. H. McFADEN