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Approximation in a Class of Banach Algebras of Quasianalytically Smooth Analytic Functions

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We investigate the structure of primary ideals at infinity in spaces of bounded analytic functions in the lower half-plane, having a specified degree of smoothness on the real line (varying from very low to quasianalytic). The problem is related to classical works of Wiener and Beurling. A new method is introduced which permits us to treat both the non-quasianalytic and the quasianalytic cases; previous methods could only handle the non-quasianalytic case. © 1993 Academic Press, Inc.

1. INTRODUCTION

We consider a class of Banach algebras $Q(\mathbf{C}_-, w)$ of analytic functions on the open lower half-plane $\mathbf{C}_-$, closely related to the spaces of Fourier images of the standard weighted L^1 spaces on $\mathbf{R}_+ = [0, \infty[$. The space $Q(\mathbf{C}_-, w)$ consists of restrictions to $\bar{\mathbf{C}}_-$ of functions $f \in C_0(\mathbf{C})$, with $\bar{\partial} f = 0$ on $\mathbf{C}_-$, and with $\bar{\partial} f(z)/w(\Im z)$ belonging to $C_0(\mathbf{C}_+)$. Here, $\mathbf{C}_+$ denotes the open upper half-plane, $C_0(\mathbf{C}_+)$ consists of all continuous functions on the one-point compactification $\mathbf{C}_+ \cup \{\infty\}$ of $\mathbf{C}_+$ that vanish at ∞, and $C_0(\mathbb{C})$ is defined analogously. The weight w is assumed to be continuous on $\mathbf{R}_+ = [0, +\infty[$,

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with value $w(0)=0$ at the left end point, and moreover, it should be positive and increasing on the open interval $]0, \infty[$; the weight w prescribes the degree of smoothness on the real line $\mathbf{R}$ of the functions in $Q(\mathbf{C}_-, w)$. For these spaces $Q(\mathbf{C}_-, w)$ we prove the following assertion. In the formulation, the space $H^\infty(\mathbf{C}_-)$ occurs; it consists of all bounded analytic functions on $\mathbf{C}_-$.

THEOREM. *Let $\mathscr{K}$ be a collection of elements in $Q(\mathbf{C}_-, w)$, and $\mathscr{T}_+(\mathscr{K})$ be the set of all (finite) linear combinations of functions of the type*

$$M_x f(z) = e^{-ixz} \cdot f(z), \qquad z \in \bar{\mathbf{C}}_-,$$

with $f \in \mathscr{K}$ and $x \in \mathbf{R}_+$. Then $\mathscr{T}_+(\mathscr{K})$, being a subset of $Q(\mathbf{C}_-, w)$, is dense in $Q(\mathbf{C}_-, w)$ if and only if

(a) *the functions in $\mathscr{K}$ have no common zeros in $\bar{\mathbf{C}}_-$, and*

(b) *there is no $\varepsilon > 0$, such that*

$$\mathscr{K} \subset e^{-i\varepsilon z} \cdot H^\infty(\mathbf{C}_-).$$

This result is an analogue of Bertil Nyman's 1950 theorem [23] (see also [5, pp. 196–206]) on the completeness of right translates of functions in $L^1(\mathbf{R}_+)$, which itself is an analogue of Norbert Wiener's classical approximation theorem [26]. Nyman's theorem states that the linear span of all right translates of a collection of functions in $L^1(\mathbf{R}_+)$ (to make right translation well defined on $L^1(\mathbf{R}_+)$, one naturally extends the functions to the whole real line by declaring them to vanish on $]-\infty, 0[$) is norm dense in $L^1(\mathbf{R}_+)$ if and only if two conditions are fulfilled: (a) the Fourier transforms of the functions in the given collection should have no common zero on the natural domain of definition, which in this case happens to be the closed lower half-plane $\bar{\mathbf{C}}_-$, and (b) the functions in the collection should not all vanish almost everywhere on an interval of the type $[0, \varepsilon]$, with $\varepsilon > 0$. Wiener's theorem, which states that the linear span of all (right and left) translates of a collection of functions in $L^1(\mathbf{R})$ is norm dense in $L^1(\mathbf{R})$ if and only if the Fourier transforms of the functions in the given collection have no common zeros on the real line $\mathbf{R}$, was generalized by Arne Beurling [1] in 1938 to weighted L^1 spaces on $\mathbf{R}$, for weights of non-quasianalytic type. To be more precise, we explain in some detail what spaces Beurling was working with. The relevant class of weight functions is the collection of all continuous functions ω on the real line having the properties $\omega(t) \geqslant 1$ and $\omega(s+t) \leqslant \omega(s)\,\omega(t)$ (submultiplicativity), where the parameters s and t range over the whole real line; a weight function ω is said to be quasianalytic if

$$\log \omega(t) = o(t), \qquad \text{as} \quad |t| \to +\infty,$$

and

$$\int_{-\infty}^{+\infty} (1+t^2)^{-1} \log \omega(t)\, dt = +\infty,$$

and non-quasianalytic if

$$\int_{-\infty}^{+\infty} (1+t^2)^{-1} \log \omega(t)\, dt < +\infty.$$

The weighted spaces Beurling considered were the spaces $L^1(\mathbf{R}, \omega)$, consisting of all (equivalence classes of) Lebesgue measurable functions f on $\mathbf{R}$ satisfying the integrability condition

$$\int_{-\infty}^{+\infty} |f(t)|\, \omega(t)\, dt < +\infty.$$

For some reason Beurling restricted his considerations to symmetric weight functions, $\omega(-t) = \omega(t)$, but this condition has later been shown to be irrelevant. Using what has been known in closer circles as the "Nyman bottle construction," Nyman later [23] showed that in Beurling's theorem, the non-quasianalyticity condition on the weight is essential; that is, for some quasianalytic weights, the natural analogue of Wiener's theorem is false. In fact, more recently, Vretblad [25] and Domar [8] have shown that we have a Wiener-type theorem if and only if the weight is non-quasianalytic.

In 1964, Vladimir Gurariĭ and Boris Levin [17] (see also [15]) extended Nyman's theorem for the space $L^1(\mathbf{R}_+)$ to weighted spaces $L^1(\mathbf{R}_+, \omega)$, but again it was necessary to assume that the weight ω was of non-quasianalytic type, which in this case is supposed to mean that it has an extension $\tilde{\omega}$ to the whole real line which is a non-quasianalytic weight function. The need for that assumption arose in the proof at the same point as it does for Beurling's generalization of Wiener's theorem, if we follow the method of proof indicated in [18, pp. 142–143], which relies on duality and complex function theory, instead of the standard regular algebra argument. The critical technical point in the two proofs, of the theorems of Beurling and Gurariĭ–Levin, is the need to use the classical Beurling–Levinson–Sjöberg log–log theorem [21, p. 376], which cannot be done for quasianalytic weights, because the initial estimate is too weak. In Beurling's situation, the assertion fails for quasianalytic weights, which is exactly when it is not permitted to apply the Beurling–Levinson–Sjöberg log–log theorem. One might think that in the case of $\mathbf{R}_+$, a dichotomy like that which happens for the real line $\mathbf{R}$ would occur as well; that is, the natural extension of Nyman's theorem might not be valid for quasianalytic weights on $\mathbf{R}_+$. Here, we should declare what we mean by a quasianalytic weight on $\mathbf{R}_+$: it should have an extension $\tilde{\omega}$ to the wole real line that is a quasianalytic weight function on $\mathbf{R}$, and in addition, we require that the divergence of the logarithmic integral occurs on $\mathbf{R}_+$, that is,

$$\int_0^{+\infty} (1+t^2)^{-1} \log \omega(t)\, dt = +\infty.$$

However, the examples which show that Beurling's theorem does not extend to quasianalytic weights do not carry over to the half-line situation, and Gurariĭ [16] now believes that Nyman's $L^1(\mathbf{R}_+)$ theorem should extend to the spaces $L^1(\mathbf{R}_+, \omega)$, no matter whether ω is quasianalytic or not.

In the context of the spaces $Q(\mathbf{C}_-, w)$, the non-quasianalyticity condition amounts to

$$\int_0^{\delta} \log\log(1/w(t))\, dt < +\infty,$$

for some $\delta > 0$, chosen so that $w(\delta) \leqslant 1/e$, so as to make the integrand nonnegative. It clearly plays no role in our main theorem stated above, offering support for Gurariĭ's conjecture. The model spaces $Q(\mathbf{C}_-, w)$ are in our opinion just as basic as the weighted L^1 spaces on $\mathbf{R}_+$, and they contain all the essential difficulties with quasianalyticity. Given a function in a weighted L^1 space on $\mathbf{R}_+$ or $\mathbf{R}$, it is difficult to tell its norm from its Fourier transform; this is the problem we have in extending our main theorem to the weighted L^1 case. In this respect, we expect to have more success in a Hilbert space context, involving weighted L^2 spaces on $\mathbf{R}_+$.

To prove the theorem stated above without assuming that the weight w is non-quasianalytic, a fundamentally new idea was required. In our proof, we initially follow the by now standard approach using a resolvent (analytic, Carleman-type; a dear child has many names, as a Swedish saying goes) transformation on the dual side, changing the problem to one concerning obtaining growth estimates that force a certain entire function to vanish identically. We derive a new estimate of this function which does the job also for quasianalytic weights w, where the traditional method based on the log–log theorem fails, and are thus able to prove our theorem without any additional assumption whatsoever on the weight w.

We believe that our results are interesting not only for harmonic analysis, but also for the general study of quasianalyticity.

The reader will find related material in [2, 7, 11, 18, 22, 25].

This paper is organized as follows. In Section 2, we introduce our weighted spaces $Q(\mathbf{C}_-, w)$ and discuss their relation to the classical

Beurling algebras $L^1(\mathbf{R}_+, \omega)$. In Section 3, basic properties of the spaces $Q(\mathbf{C}_-, w)$ are derived, and, in Section 4, we formulate our main theorem. In Section 5, we introduce the traditional resolvent transform of a bounded linear functional, and show that if the functional annihilates a closed ideal lacking common zeros in $\bar{\mathbf{C}}_-$, then its resolvent transform extends to an entire function. In Section 6, we derive the standard "quick" estimate of this entire function, and in Section 7, we obtain the new estimate that is needed to prove our main theorem in the quasianalytic case. In Section 8, we present the technical device that lets us find a sufficiently big set where the estimate from Section 7 applies. In Section 9, we finish the proof of our main theorem, and in the last section, Section 10, we prove a structure theorem for ideals lacking common zeros in $\bar{\mathbf{C}}_-$. We thank Vladimir Gurariĭ for reading and correcting this manuscript, and for helping us restructure it.

2. The Spaces $Q(\mathbf{C}_-, w)$ of Asymptotically Holomorphic Analytic Functions

The Fourier transform of an $L^1(\mathbf{R}_+)$ function is a continuous function in the closed lower half-plane, holomorphic in the interior, which is given by the expression

$$\mathscr{F}f(z) = \int_0^{+\infty} e^{-itz} f(t)\, dt, \qquad z \in \bar{\mathbf{C}}_-.$$

We denote by $A_0(\mathbf{C}_-)$ the space of continuous functions f on $\bar{\mathbf{C}}_-$ that are holomorphic on $\mathbf{C}_-$, and vanish at infinity; that is,

$$f(z) \to 0 \qquad \text{as} \quad \bar{\mathbf{C}}_- \ni z \to \infty.$$

Let ω be a weight function on $\mathbf{R}_+$, non-quasianalytic or quasianalytic (these concepts were defined in the Introduction). The Fourier image $\mathscr{F}L^1(\mathbf{R}_+, \omega)$ of the space $L^1(\mathbf{R}_+, \omega)$ is a subspace of $A_0(\mathbf{C}_-)$, and the functions in it have a certain smoothness property. The degree of this smoothness is, however, difficult to state in detail. For this reason, we study a closely related class of Banach algebras, where the smoothness is stated in a more concrete way, in terms of the functions having extensions to the whole complex plane with small $\bar{\partial}$-derivative near the real line, and mention later more explicitly how this class relates to the spaces $\mathscr{F}L^1(\mathbf{R}_+, \omega)$.

First, let $C_0(\mathbf{C})$ denote the space of restrictions to $\mathbf{C}$ of all continuous functions on the extended complex plane $\mathbf{C} \cup \{\infty\}$ that vanish at the point

∞. Let w be a bounded monotonically increasing continuous function on $\mathbf{R}_+$, having $w(0)=0$, which is positive on $\mathbf{R}_+\setminus\{0\}$, and introduce the space $C_0(\mathbf{C}_+, w)$, consisting of all continuous functions f on the upper half-plane,

$$\mathbf{C}_+ = \{z \in \mathbf{C} : \Im z > 0\},$$

for which the function

$$F_w(z) = f(z)/w(\Im z), \qquad z \in \mathbf{C}_+,$$

extends to an element in $C_0(\mathbf{C})$ that vanishes on the lower half-plane $\mathbf{C}_-$. The space of asymptotically holomorphic functions that we are interested in, denoted by $Q(\mathbf{C}, w)$, consists of all functions $f \in C_0(\mathbf{C})$ such that f is analytic on $\mathbf{C}_-$, and $\bar\partial f$ belongs to $C_0(\mathbf{C}_+, w)$ on $\mathbf{C}_+$ in the sense of distribution theory [24]. Here, as usual, $\bar\partial = \partial/\partial\bar z$. The norm in $Q(\mathbf{C}, w)$ is given by the expression

$$\|f\|_{Q(\mathbf{C},w)} = \sup_{\mathbf{C}} |f(z)| + \sup_{\mathbf{C}_+} \frac{|\bar\partial f(z)|}{w(\Im z)};$$

$Q(\mathbf{C}, w)$ is a commutative Banach algebra without unit if we define multiplication to be ordinary pointwise multiplication of functions. The subspace

$$J(w) = \{f \in Q(\mathbf{C}, w) : f|_{\mathbf{C}_-} \equiv 0\}$$

is a closed ideal in $Q(\mathbf{C}, w)$, and the related restriction algebra

$$Q(\mathbf{C}_-, w) = Q(\mathbf{C}, w)|_{\mathbf{C}_-} = Q(\mathbf{C}, w)/J(w)$$

is also a commutative Banach algebra if supplied with the standard quotient norm.

Algebras of this type on the unit disc were introduced by Evseĭ Dyn'kin in [10], where he related for a C^1-function on the closed unit disc $\bar{\mathbf{D}}$ the degree of decay of its $\bar\partial$-derivative near the unit circle $\mathbf{T}$ with the smoothness of its boundary values.

The critical condition of quasianalyticity for a weight w, which for weights ω associated with the spaces $L^1(\mathbf{R}_+, \omega)$ is the divergence of the integral

$$\int_0^\infty \frac{\log \omega(t)}{1+t^2}\, dt = \infty,$$

turns out to be for these spaces $Q(\mathbf{C}_-, w)$ the divergence of the integral

$$\int_0^\delta \log\log(1/w(t))\,dt = +\infty,$$

for some $\delta > 0$, chosen so that $w(\delta) \leqslant 1/e$, so as to make the integrand nonnegative. This condition should be thought of as requiring that the functions in $Q(\mathbf{C}_-, w)$ are extremely smooth on the real line $\mathbf{R}$, and we will say that they are quasianalytically smooth on $\mathbf{R}$. The reason for this terminology is as follows. Let $\tilde{w}(t) = w(|t|)$ for $t \in \mathbf{R}$, and consider the space $Q(\mathbf{R}, \tilde{w})$ consisting of all continuous functions f on $\mathbf{R}$ having continuous extensions $\tilde{f}$ to the whole complex plane with $\tilde{f} \in C_0(\mathbf{C})$, and $\bar{\partial}\tilde{f} \in C_0(\mathbf{C}\backslash\mathbf{R}, \tilde{w})$; here, $C_0(\mathbf{C}\backslash\mathbf{R}, \tilde{w})$ consists of all continuous functions g on $\mathbf{C}\backslash\mathbf{R}$ for which the function

$$G_w(z) = g(z)/w(|\Im z|), \qquad z \in \mathbf{C}\backslash\mathbf{R},$$

extends to an element in $C_0(\mathbf{C})$ that vanishes on the real line $\mathbf{R}$. Then the space $Q(\mathbf{R}, \tilde{w})$ contains nontrivial functions with compact support if and only if

$$\int_0^\delta \log\log(1/w(t))\,dt < +\infty,$$

for some $\delta > 0$, chosen so that $w(\delta) \leqslant 1/e$; see [4, 10].

We should indicate the relationship between the spaces $\mathscr{F}L^1(\mathbf{R}_+, \omega)$ and $Q(\mathbf{C}_-, w)$. As before ω is a weight function on $\mathbf{R}_+$, quasianalytic or non-quasianalytic, and let us make the additional requirement that $\log\omega$ be strictly concave on $\mathbf{R}_+$. Suppose f is a function belonging to the weighted space $L^1(\mathbf{R}_+, \omega)$. Following an idea of Domar [9], we consider a cut-off Fourier transform

$$\tilde{\mathscr{F}}f(z) = \int_0^{v(\Im z)} e^{-itz} f(t)\,dt,$$

where for $x > 0$, $v(x)$ is by definition the point t_0 where the infimum of the expression

$$w(x) = \inf\{e^{tx}/\omega(t) : t \in \mathbf{R}_+\}, \qquad x \in \mathbf{R}_+,$$

is attained, and for $x \leqslant 0$, we put $v(x) = +\infty$. Under inessential additional assumptions on the function f and the weight ω, this transform $\tilde{\mathscr{F}}f$ is an asymptotically holomorphic function, in the sense that

$$|\bar{\partial}(\tilde{\mathscr{F}}f)(z)| \leqslant C(f, \varepsilon)\, w((1+\varepsilon)\,\Im z), \qquad \Im z > 0,$$

holds for every $\varepsilon > 0$. On the other hand, given a weight w, associated with the spaces of type $Q(\mathbf{C}, w)$, and a function F belonging to $Q(\mathbf{C}_-, w)$, satisfying the size condition that some extension $\tilde{F} \in Q(\mathbf{C}, w)$ of F has $z^2\tilde{F}$ in $Q(\mathbf{C}, w)$, then by using Green's formula one can prove that F belongs to $\mathscr{F}L^1(\mathbf{R}_+, \omega_1)$, where the weight ω_1 is given by the formula $\omega_1(t) = \omega(t)/(1+t^2)$, and ω is related to w in the way previously indicated; a direct formula is as follows:

$$\omega(t) = \inf\{e^{tx}/w(x) : x > 0\}, \qquad t \in \mathbf{R}_+.$$

We do not want to spend a lot of energy here proving the above assertions, especially since they are not really essential to this work; they only serve to motivate it. Related results can be found in [3].

In conclusion, we note that the spaces $Q(\mathbf{C}_-, w)$ and the Fourier images of the Beurling spaces $\mathscr{F}L^1(\mathbf{R}_+, \omega)$ are quite similar.

3. Basic Properties of the Spaces $Q(\mathbf{C}_-, w)$

We need to identify the maximal ideal space of the algebra $Q(\mathbf{C}_-, w)$; however, since this algebra lacks a unit, it is preferable to consider instead the unitization $Q_e(\mathbf{C}_-, w)$ of $Q(\mathbf{C}_-, w)$, which consists of all sums $\alpha + f$, where α is a $\mathbf{C}$-valued constant function on $\bar{\mathbf{C}}_-$, and $f \in Q(\mathbf{C}_-, w)$; normed suitably, this is a Banach algebra with unit. The following lemma will help us to show that the maximal ideal space of $Q_e(\mathbf{C}_-, w)$ coincides with $\bar{\mathbf{C}}_- \cup \{\infty\}$ in the obvious way. At this point in time, it is convenient also to introduce the unitization $Q_e(\mathbf{C}, w)$ of $Q(\mathbf{C}, w)$, defined analogously. Recall that $\mathbf{C}_+$ denotes the upper half-plane.

Lemma 3.1. *For $\lambda \in \mathbf{C}_+$, introduce the functions*

$$b_\lambda(z) = 1/(\lambda - z), \qquad z \in \bar{\mathbf{C}}_-.$$

The functions b_λ, with $\lambda \in \mathbf{C}_+$, belong to the algebra $Q(\mathbf{C}_-, w)$, and any function in $Q(\mathbf{C}_-, w)$ can be approximated in norm by finite linear combinations of these functions.

Proof. Let us first check that b_λ belongs to $Q(\mathbf{C}_-, w)$ for all $\lambda \in \mathbf{C}_+$. For s, t, $0 < s < t < 1$, let $\varphi_{s,t}$ be a C^∞ function on $\mathbf{R}_+$ such that $0 \leqslant \varphi_{s,t} \leqslant 1$ on $\mathbf{R}_+$, $\varphi_{s,t} = 0$ on $[0, s]$, and $\varphi_{s,t} = 1$ on $[t, \infty[$; note that it is possible to find a $\varphi_{s,t}$ which also meets the condition

$$|\varphi'_{s,t}(x)| \leqslant \frac{2}{t-s}, \qquad x \in \mathbf{R}_+. \tag{3.1}$$

Now, for each $\lambda \in \mathbf{C}_+$, and s, t, $0<s<t<1$, let us denote by $B_\lambda^{s,t}$ the function

$$B_\lambda^{s,t}(z)=\varphi_{s,t}\left(\frac{|z-\lambda|}{\Im\lambda}\right)\cdot\frac{1}{\lambda-z}, \qquad z\in\mathbf{C}\setminus\{\lambda\},$$

extended continuously to $\mathbf{C}$ by declaring $B_\lambda^{s,t}(\lambda)=0$. We note that the support of $\bar\partial B_\lambda^{s,t}$ is contained in the annulus

$$\{z\in\mathbf{C}: s\Im\lambda\leqslant|z-\lambda|\leqslant t\Im\lambda\},$$

and that $B_\lambda^{s,t}$ itself is supported in the region

$$\{z\in\mathbf{C}: s\Im\lambda\leqslant|z-\lambda|\}.$$

The functions $B_\lambda^{s,t}$ belong to $Q(\mathbf{C}, w)$, and by (3.1) they enjoy the estimate

$$\left|\frac{\bar\partial B_\lambda^{s,t}(z)}{w(\Im z)}\right|\leqslant\frac{1}{t-s}\cdot\frac{1}{\Im\lambda}\cdot\frac{1}{|\lambda-z|}\cdot\frac{1}{w(\Im z)}, \qquad z\in\mathbf{C}_+,$$

and since we know where $\bar\partial B_\lambda^{s,t}$ is supported, we get

$$\left|\frac{\bar\partial B_\lambda^{s,t}(z)}{w(\Im z)}\right|\leqslant\frac{1}{s(t-s)}\cdot\frac{1}{(\Im\lambda)^2}\cdot\frac{1}{w((1-t)\Im\lambda)}, \qquad z\in\mathbf{C}_+.$$

We also have the a priori estimate

$$|B_\lambda^{s,t}(z)|\leqslant\frac{1}{|\lambda-z|}, \qquad z\in\mathbf{C}\setminus\{\lambda\},$$

so that since we know where $B_\lambda^{s,t}$ is supported, we have

$$|B_\lambda^{s,t}(z)|\leqslant\frac{1}{s\Im\lambda}, \qquad z\in\mathbf{C}.$$

The above estimates lead to the norm control

$$\|B_\lambda^{s,t}\|_{Q(\mathbf{C},w)}\leqslant\frac{1}{s(t-s)}\cdot\frac{1}{(\Im\lambda)^2}\cdot\frac{1}{w((1-t)\Im z)}+\frac{1}{s\Im\lambda}. \tag{3.2}$$

Note that no matter what admissible choice of the parameters s and t is made, we always have $B_\lambda^{s,t}|_{\mathbf{C}_-}=b_\lambda$, so that if we choose $s=t/2$ in (3.2), we have the estimate

$$\|b_\lambda\|_{Q(\mathbf{C}_-,w)}\leqslant\frac{4}{(t\Im\lambda)^2}\cdot\frac{1}{w((1-t)\Im\lambda)}+\frac{2}{t\Im\lambda}<\infty, \tag{3.3}$$

for all $\lambda \in \mathbf{C}_+$ and all t, $0<t<1$. As a byproduct, we see that $b_\lambda \in Q(\mathbf{C}_-, w)$ for all $\lambda \in \mathbf{C}_+$.

Let us turn our attention to the assertion concerning approximation. As a first step, we observe that every function $f \in Q(\mathbf{C}, w)$ can be approximated in norm (as $\varepsilon \to +0$, $a \to +\infty$) by the functions $f_{\varepsilon, a}$, defined by the formula

$$f_{\varepsilon,a}(z) = f(z - i\varepsilon)(ia)^2 \, (B_{ia}^{1/3, 2/3}(z))^2, \qquad z \in \mathbf{C}.$$

The restrictions of these functions to $\mathbf{C}_- + i\varepsilon$ belong to $H^1(\mathbf{C}_- + i\varepsilon)$, and in $Q(\mathbf{C}_-, w)$ they can, in their turn, be approximated by finite linear combinations of b_λ, $\lambda \in \mathbf{R} + i\varepsilon$, because by the Cauchy formula we have, for a function $g \in H^1(\mathbf{C}_- + i\varepsilon)$,

$$g(z) = -\frac{1}{2\pi i} \int_{\mathbf{R}+i\varepsilon} \frac{g(\lambda)}{\lambda - z} \, d\lambda = -\frac{1}{2\pi i} \int_{\mathbf{R}+i\varepsilon} b_\lambda(z) \, g(\lambda) \, d\lambda, \qquad z \in \bar{\mathbf{C}}_-,$$

and this integral converges in $Q(\mathbf{C}_-, w)$, because of (3.3). The proof of Lemma 3.1 is complete. ∎

LEMMA 3.2. *The maximal ideal space of* $Q_e(\mathbf{C}_-, w)$ *coincides with* $\bar{\mathbf{C}}_- \cup \{\infty\}$, *in the sense that every nontrivial complex homomorphism* $\tau: Q_e(\mathbf{C}_-, w) \to \mathbf{C}$ *has the form*

$$\tau(f) = f(z_0), \qquad f \in Q_e(\mathbf{C}_-, w),$$

for some $z_0 \in \bar{\mathbf{C}}_- \cup \{\infty\}$.

Proof. Clearly, every point evaluation in $\bar{\mathbf{C}}_- \cup \{\infty\}$ gives rise to a nontrivial complex homomorphism, and that these homomorphisms are different for different points is easy to see, because we have a sufficiently large collection of functions in $Q_e(\mathbf{C}_-, w)$, by Lemma 3.1. Let τ be a nontrivial complex homomorphism on $Q_e(\mathbf{C}_-, w)$, and suppose that we have, for the value $\lambda = i$, $\tau(b_i) = 1/(i - z_0)$, for some number $z_0 \in \mathbf{C} \setminus \{i\} \cup \{\infty\}$, where we declare that $\tau(b_i) = 0$ if $z_0 = \infty$. The formula

$$b_\lambda(z) = (1 + (\lambda - i) \, b_i(z))^{-1} \, b_i(z), \qquad z \in \bar{\mathbf{C}}_-, \tag{3.4}$$

shows that for those $\lambda \in \mathbf{C}$ for which $1 + (\lambda - i) \, b_i(z)$ is invertible in $Q_e(\mathbf{C}_-, w)$, we have

$$\tau(b_\lambda) = 1/(\lambda - z_0). \tag{3.5}$$

On the other hand,

$$(1 + (\lambda - i) \, b_i(z))(1 - (\lambda - i) \, b_\lambda(z)) = 1, \qquad z \in \bar{\mathbf{C}}_-,$$

so that by Lemma 3.1, $1+(\lambda-i)\,b_i(z)$ is invertible in $Q_e(\mathbf{C}_-, w)$ for all $\lambda \in \mathbf{C}_+$. Since τ is not allowed to attain the value ∞ on $Q_e(\mathbf{C}_-, w)$, formula (3.5) requires z_0 to belong to the set $\bar{\mathbf{C}}_- \cup \{\infty\}$. But then τ coincides with the point evaluation at z_0 on the dense subset of $Q(\mathbf{C}_-, w)$ consisting of finite linear combinations of the functions b_λ, $\lambda \in \mathbf{C}_+$, so being continuous, τ must coincide with the point evaluation at z_0 on all of $Q(\mathbf{C}_-, w)$. Concerning the constant functions, it is clear that $\tau(1)=1$ (this is a general Banach algebra fact), so that τ is also the point evaluation at z_0 on $Q_e(\mathbf{C}_-, w)$. This completes the proof of the lemma. ∎

We shall now see that for $x \in \mathbf{R}_+$ and $f \in Q(\mathbf{C}_-, w)$, the function

$$M_x(f)(z) = e^{-ixz} \cdot f(z), \qquad z \in \bar{\mathbf{C}}_-,$$

belongs to $Q(\mathbf{C}_-, w)$. Let $\tilde{f}$ be a function in $Q(\mathbf{C}, w)$ such that $\tilde{f} = f$ on $\mathbf{C}_-$. Moreover, let ψ be a C^∞ function on $\mathbf{R}$, such that $0 \leqslant \psi(t) \leqslant 1$ for all $t \in \mathbf{R}$, $\psi(t)=1$ for all $t \leqslant 1$, and $\psi(t)=0$ for all $t \geqslant 2$. Then, for each $\varepsilon > 0$, the function

$$M_x^\varepsilon(\tilde{f})(z) = \psi(\varepsilon \Im z) \cdot e^{-ixz} \cdot \tilde{f}(z), \qquad z \in \mathbf{C}, \tag{3.6}$$

belongs to $Q(\mathbf{C}, w)$, and provides an extension of the function $M_x(f)$.

Lemma 3.3. *Let $\mathscr{K}$ be a collection of elements in $Q(\mathbf{C}_-, w)$, and $\mathscr{T}_+(\mathscr{K})$ be the set of all (finite) linear combinations of functions of the type*

$$M_x f(z) = e^{-ixz} \cdot f(z), \qquad z \in \bar{\mathbf{C}}_-,$$

with $f \in \mathscr{K}$ and $x \in \mathbf{R}_+$. Denote by $I(\mathscr{K})$ the closure of $\mathscr{T}_+(\mathscr{K})$ in $Q(\mathbf{C}_-, w)$. Furthermore, let $\mathscr{S}_+(\mathscr{K})$ denote the set of finite linear combinations of functions of the type $b_\lambda f$, with $\lambda \in \mathbf{C}_+$ and $f \in \mathscr{K}$, and write $J(\mathscr{K})$ for the closure in $Q(\mathbf{C}_-, w)$ of $\mathscr{S}_+(\mathscr{K})$. Then $I(\mathscr{K})$ and $J(\mathscr{K})$ both coincide with the closure of the ideal in $Q(\mathbf{C}_-, w)$ generated by $\mathscr{K}$.

Proof. First note that by Lemma 3.1, $J(\mathscr{K})$ equals the closure of the ideal in $Q(\mathbf{C}_-, w)$ generated by $\mathscr{K}$. For an $f \in Q(\mathbf{C}_-, w)$ with extension $\tilde{f}$ in $Q(\mathbf{C}, w)$, $a > 0$, $x \in \mathbf{R}_+$, and $\varepsilon > 0$, the restriction to $\bar{\mathbf{C}}_-$ of the function

$$ia \cdot B_{ia}^{1/3,2/3}(z) \cdot M_x^\varepsilon(\tilde{f})(z) = (ia \cdot B_{ia}^{1/3,2/3}(z) \cdot \psi(\varepsilon \Im z) \cdot e^{-ixz}) \cdot f(z), \qquad z \in \mathbf{C},$$

where $B_\lambda^{s,t}$ is as in the proof of Lemma 3.1, belongs to the ideal generated by $\mathscr{K}$ in $Q(\mathbf{C}_-, w)$, and as $a \to +\infty$, it converges to $M_x(f)$ in the norm of $Q(\mathbf{C}_-, w)$. Consequently, $I(\mathscr{K})$ is contained in the closed ideal in $\mathscr{K}$

generated by $\mathscr{K}$. On the other hand, if we make a simple norm estimate of the function $M_x(f)$, using (3.6), we see that for $\lambda \in \mathbf{C}_+$, the integral

$$i\int_0^\infty M_x(f)(z)\cdot e^{ix\lambda}\,dx, \qquad z\in\bar{\mathbf{C}}_-,$$

is norm convergent, so that it defines an element of $I(\mathscr{K})$, and a trivial computation shows that it equals $b_\lambda f$, and consequently, $I(\mathscr{K})$ contains $J(\mathscr{K})$. This completes the proof of the lemma. ∎

4. The Approximation Theorem

We now state the Nyman–Gurariĭ–Levin-type approximation theorem for the asymptotically holomorphic space $Q(\mathbf{C}_-, w)$, mentioned in the introduction.

Theorem 4.1. *Let* $\mathscr{K}$ *be a collection of elements in* $Q(\mathbf{C}_-, w)$, *and* $\mathscr{T}_+(\mathscr{K})$ *be the set of all (finite) linear combinations of functions of the type*

$$M_x f(z) = e^{-ixz}\cdot f(z), \qquad z\in\bar{\mathbf{C}}_-,$$

with $f\in\mathscr{K}$ *and* $x\in\mathbf{R}_+$. *Then* $\mathscr{T}_+(\mathscr{K})$, *being a subset of* $Q(\mathbf{C}_-, w)$, *is dense in* $Q(\mathbf{C}_-, w)$ *if and only if*

(a) *the functions in* $\mathscr{K}$ *have no common zeros in* $\bar{\mathbf{C}}_-$, *and*

(b) *there is no* $\varepsilon > 0$, *such that*

$$\mathscr{K}\subset e^{-i\varepsilon z}\cdot H^\infty(\mathbf{C}_-).$$

For the spaces $Q(\mathbf{C}_-, w)$, the standard approach, using the Beurling–Levinson–Sjöberg log–log theorem [21, p. 376] as the main vehicle to carry out the proof of the denseness of the set $\mathscr{T}_+(\mathscr{K})$, fails utterly beyond the border of quasianalyticity, just as it does for the weighted L^1 spaces on $\mathbf{R}_+$. Our proof resembles to a substantial degree that of Nyman [23], and Gurariĭ and Levin [17], where the main objective is to prove that a certain entire function vanishes identically; what makes our proof work also in the quasianalytic case is the fact that we can use the asymptotically holomorphic extensions of the functions in $\mathscr{K}$ to the upper half-plane $\mathbf{C}_+$,

$$\mathbf{C}_+ = \{z\in\mathbf{C} : \Im z > 0\},$$

to produce a better estimate of the entire function in $\mathbf{C}_+$, which then allows us to prove that it actually has to vanish identically. The proof of Theorem 4.1 is given in Section 9.

The above result has the following consequence, the proof of which we will postpone until Section 10.

COROLLARY 4.2. *For every collection $\mathscr{K}$ of elemements in $Q(\mathbf{C}_-, w)$ lacking common zeros in $\bar{\mathbf{C}}_-$, the closure of $\mathscr{T}_+(\mathscr{K})$ in $Q(\mathbf{C}_-, w)$ coincides with the subspace*

$$e^{-i\varepsilon(\mathscr{K})z} \cdot Q(\mathbf{C}_-, w),$$

where $\varepsilon(\mathscr{K}) = \sup\{x \geqslant 0 : \mathscr{K} \subset e^{-ixz} \cdot H^\infty(\mathbf{C}_-)\}$.

We believe that our method of proof will generalize to certain classes of Hilbert spaces, which are also Banach algebras, of analytic functions on $\mathbf{C}_-$ with quasianalytic smoothness up to the boundary. These may then, via the Fourier transform, be viewed as weighted L^2 spaces on $\mathbf{R}_+$, so that the Gurariĭ–Levin theorem concerning weighted L^1 spaces on $\mathbf{R}_+$, which was stated in the introduction, should remain valid beyond the border of quasianalyticity in an L^2 setting.

5. THE RESOLVENT TRANSFORM OF A FUNCTIONAL

For a functional ϕ in the dual Banach space $Q(\mathbf{C}_-, w)^*$ to $Q(\mathbf{C}_-, w)$, let us associate with it what we shall call its resolvent transform,

$$\mathscr{R}[\phi](\lambda) = \langle b_\lambda, \phi \rangle, \qquad \lambda \in \mathbf{C}_+, \tag{5.1}$$

which is a holomorphic function on $\mathbf{C}_+$, because the functions b_λ vary analytically with $\lambda \in \mathbf{C}_+$. By the density of the linear span of the functions $\{b_\lambda : \lambda \in \mathbf{C}_+\}$ in $Q(\mathbf{C}_-, w)$ (Lemma 3.1), the functional ϕ and its resolvent transform are in a one-to-one correspondence. The idea to use such a transform to gather information about the structure of closed ideals in a Banach algebra goes back to Beurling, Carleman, and Gelfand [14]; as far as we know, Gelfand's paper is the earliest publication using this method, and he also has, in a simple special case, the elegant way of getting the analytic continuation which was later rediscovered and elaborated upon by Domar [7].

If I is a closed ideal in $Q_e(\mathbf{C}_-, w)$, then it is standard to identify the maximal ideal space of the quotient algebra $Q_e(\mathbf{C}_-, w)/I$ with the hull $\mathscr{Z}_\infty(I)$ of I [12, p. 12]:

$$\mathscr{Z}_\infty(I) = \{z \in \bar{\mathbf{C}}_- \cup \{\infty\} : f(z) = 0 \text{ for all } f \in I\};$$

note that here we use the identification of the maximal ideal space of $Q_e(\mathbf{C}_-, w)$ obtained in Lemma 3.2. Let $\mathscr{K}$ be a collection of functions in $Q(\mathbf{C}_-, w)$, and let $I(\mathscr{K})$ denote the closure of the ideal in $Q(\mathbf{C}_-, w)$

generated by $\mathscr{K}$. Being a closed ideal in $Q(\mathbf{C}_-, w)$, $I(\mathscr{K})$ is a closed ideal in $Q_e(\mathbf{C}_-, w)$ as well. Let us now make the assumption that the functions in $\mathscr{K}$ lack common zeros in $\bar{\mathbf{C}}_-$, that is, $\mathscr{Z}_\infty(I(\mathscr{K})) = \{\infty\}$. For $\lambda \in \mathbf{C}$, consider the element

$$1 + (\lambda - i)\, b_i + I(\mathscr{K})$$

in the quotient algebra $Q(\mathbf{C}_-, w)/I(\mathscr{K})$. Since $b_i(\infty) = 0$, we have for each $\lambda \in \mathbf{C}$ that the Gelfand transform of this element vanishes nowhere on the maximal ideal space $\{\infty\}$ of $Q(\mathbf{C}_-, w)/I(\mathscr{K})$, and hence it is invertible. Let us define

$$A_\lambda = (1 + (\lambda - i)\, b_i + I(\mathscr{K}))^{-1} \cdot (b_i + I(\mathscr{K})), \qquad \lambda \in \mathbf{C}, \tag{5.3}$$

as an element of $Q(\mathbf{C}_-, w)/I(\mathscr{K})$, and note that we have

$$A_\lambda = (1 + (\lambda - i)\, b_i)^{-1} \cdot b_i + I(\mathscr{K}), \qquad \lambda \in \mathbf{C}_+,$$

so that by (3.4), we have in fact

$$A_\lambda = b_\lambda + I(\mathscr{K}), \qquad \lambda \in \mathbf{C}_+. \tag{5.4}$$

Let us now return to our functional $\phi \in Q(\mathbf{C}_-, w)^*$ which annihilated $I(\mathscr{K})$. Any such functional may be considered as a bounded linear functional on $Q(\mathbf{C}_-, w)/I(\mathscr{K})$, by standard functional analysis arguments. By (5.4), the resolvent transform $\mathscr{R}[\phi]$ of ϕ, defined by (5.1), can also be represented by the formula

$$\mathscr{R}[\phi](\lambda) = \langle A_\lambda, \phi \rangle, \qquad \lambda \in \mathbf{C}_+.$$

By (5.3), A_λ is defined for all $\lambda \in \mathbf{C}$ as an element in $Q(\mathbf{C}_-, w)/I(\mathscr{K})$, which permits us to extend the definition of $\mathscr{R}[\phi]$ to the whole complex plane:

$$\mathscr{R}[\phi](\lambda) = \langle A_\lambda, \phi \rangle, \qquad \lambda \in \mathbf{C}. \tag{5.5}$$

A standard Banach algebra technique shows that the element A_λ of $Q(\mathbf{C}_-, w)/I(\mathscr{K})$ varies analytically in $\lambda \in \mathbf{C}$, so that the function $\mathscr{R}[\phi]$, defined by (5.5), is an entire function.

6. The Quick Estimate of the Resolvent Transform of a Functional

Let ϕ be a bounded linear functional annihilating $I(\mathscr{K})$, just as in Section 5, and assume that the functions in $\mathscr{K}$ have no common zeros in $\bar{\mathbf{C}}_-$. This implies that the function $\mathscr{R}[\phi]$, initially defined only on $\mathbf{C}_+$, extends to an entire function, by (5.5). By (3.3), we have, for every s, $0<s<1$,

$$|\mathscr{R}[\phi](\lambda)| \leqslant \|\phi\|_{Q(\mathbf{C}_-, w)^*} \left(\frac{4}{(1-s)^2 (\Im\lambda)^2} \cdot \frac{1}{w(s\Im\lambda)} + \frac{2}{s\Im\lambda}\right), \qquad \lambda \in \mathbf{C}_+. \quad (6.1)$$

This estimate controls $\mathscr{R}[\phi]$ in the upper half-plane; to be able to deduce the desired conclusion $\mathscr{R}[\phi] \equiv 0$, we need some information about the behavior of $\mathscr{R}[\phi]$ in the lower half-plane. Let us look for concrete representatives in $Q(\mathbf{C}_-, w)$ for the cosets A_λ.

Let $f \in Q(\mathbf{C}, w)$ be such that $f|_{\mathbf{C}_-}$ belongs to the collection $\mathscr{K}$, and suppose also that f does not vanish identically on $\mathbf{C}_-$. Introduce the notation

$$\mathscr{Z}(f, \mathbf{C}_-) = \{z \in \mathbf{C}_- : f(z) = 0\},$$

and put

$$(T_\lambda f)(z) = \frac{f(\lambda) - f(z)}{\lambda - z}, \qquad z \in \mathbf{C} \backslash \{\lambda\},$$

for $\lambda \in \mathbf{C}_-$, and

$$H_\lambda(z) = \frac{i-z}{\lambda - z} \cdot (1 - f(z)/f(\lambda)), \qquad z \in \mathbf{C} \backslash \{\lambda\},$$

for $\lambda \in \mathbf{C}_- \backslash \mathscr{Z}(f, \mathbf{C}_-)$. To check that for $\lambda \in \mathbf{C}_-$, $T_\lambda f \in Q(\mathbf{C}, w)$, and that for $\lambda \in \mathbf{C}_- \backslash \mathscr{Z}(f, \mathbf{C}_-)$, $H_\lambda \in Q_e(\mathbf{C}, w)$, where $Q_e(\mathbf{C}, w)$ is the unitization of $Q(\mathbf{C}, w)$, it is sufficient to note that for $T_\lambda f$ we have $T_\lambda f \in C_0(\mathbf{C})$, $\bar{\partial} T_\lambda f \in C_0(\mathbf{C}_+, w)$, and the estimates

$$\|T_\lambda f\|_{L^\infty(\mathbf{C})} \leqslant 2\, \|f\|_{L^\infty(\mathbf{C})} / |\Im\lambda|, \qquad \lambda \in \mathbf{C}_-,$$

by the maximum principle, and

$$\sup_{z \in \mathbf{C}_+} \left|\frac{\bar{\partial} T_\lambda f(z)}{w(\Im z)}\right| \leqslant \left(\sup_{z \in \mathbf{C}_+} \left|\frac{\bar{\partial} f(z)}{w(\Im z)}\right|\right) \Big/ |\Im\lambda|, \qquad \lambda \in \mathbf{C}_- \backslash \mathscr{Z}(f, \mathbf{C}_-),$$

so that $T_\lambda f \in Q(\mathbf{C}, w)$, and

$$\|T_\lambda f\|_{Q(\mathbf{C}, w)} \leqslant 2\, \|f\|_{Q(\mathbf{C}, w)} / |\Im\lambda|, \qquad \lambda \in \mathbf{C}_-; \quad (6.2)$$

similar computations can be made for H_λ. The calculation

$$(1 + (\lambda - i)\, b_i(z))\, H_\lambda(z) = \frac{\lambda - z}{i - z} H_\lambda(z) = 1 - f(z)/f(\lambda)(\in 1 + I(\mathscr{K})), \quad z \in \bar{\mathbf{C}}_-,$$

shows that for $\lambda \in \mathbf{C}_{-}\mathscr{Z}(f, \mathbf{C}_{-})$, H_λ is an element of the coset

$$(1 + (\lambda - i)\, b_i + I(\mathscr{K}))^{-1},$$

and since $T_\lambda f|_{\mathbf{C}_-} = f(\lambda) \cdot b_i \cdot H_\lambda|_{\mathbf{C}_-}$, it follows that

$$A_\lambda = (T_\lambda f|_{\mathbf{C}_-})/f(\lambda) + I(\mathscr{K}), \qquad \lambda \in \mathbf{C}_- \backslash \mathscr{Z}(f, \mathbf{C}_-).$$

Hence we have, by (5.5),

$$\mathscr{R}[\phi](\lambda) = \langle T_\lambda f|_{\mathbf{C}_-}, \phi\rangle / f(\lambda), \qquad \lambda \in \mathbf{C}_- \backslash \mathscr{Z}(f, \mathbf{C}_-),$$

so that

$$|\mathscr{R}[\phi](\lambda)| \leqslant \|\phi\|_{Q(\mathbf{C}_-, w)^*} \cdot \|T_\lambda f|_{\mathbf{C}_-}\|_{Q(\mathbf{C}_-, w)} / |f(\lambda)| \qquad \lambda \in \mathbf{C}_- \backslash \mathscr{Z}(f, \mathbf{C}_-).$$

By (6.2), we have, by varying $f \in Q(\mathbf{C}, w)$ but keeping $f|_{\mathbf{C}_-}$ fixed,

$$|\mathscr{R}[\phi](\lambda)| \leqslant 2(\|\phi\|_{Q(\mathbf{C}_-, w)^*} \cdot \|f|_{\mathbf{C}_-}\|_{Q(\mathbf{C}_-, w)}) / (|f(\lambda)| \cdot |\Im\lambda|), \qquad \lambda \in \mathbf{C}_-,$$

and if we write $C(f, \phi) = 2\, \|\phi\|_{Q(\mathbf{C}_-, w)^*} \cdot \|f|_{\mathbf{C}_-}\|_{Q(\mathbf{C}_-, w)}$, this becomes

$$|\mathscr{R}[\phi](\lambda)| \leqslant \frac{C(f, \phi)}{|\Im\lambda| \cdot |f(\lambda)|}, \qquad \lambda \in \mathbf{C}_-, \tag{6.3}$$

so that the function $\mathscr{R}[\phi]$ belongs to the Nevanlinna class in $\mathbf{C}_- - i$. Choosing different functions f, and taking into account condition (b) of our theorem, we see that function $\mathscr{R}[\phi]$ belongs to the Smirnov class in the region $\mathbf{C}_- - i$.

7. The New Idea—Another Basic Estimate

We are now at the classical point where estimates (6.1) and (6.3) are sufficient to prove that $\mathscr{R}[\phi] \equiv 0$ in case w is a non-quasianalytic weight, that is,

$$\int_0^\delta \log\log(1/w(t))\, dt < \infty$$

holds, for some $\delta > 0$ with $w(\delta) \leqslant 1/e$, the main tool being Levinson's log–log theorem (and of course the Phragmén–Lindelöf principle), but these estimates are simply insufficient to handle the remaining quasianalytic case.

To deal with the general case, we need more information, preferably in the form of growth control of the entire function $\mathscr{R}[\phi]$. This we get by constructing more ingenious representatives than b_λ of the coset A_λ in

$Q_e(\mathbf{C}_-, w)/I(\mathscr{K})$ for $\lambda \in \mathbf{C}_+$. The inspiration comes from the successful estimate we have obtained in the lower half-plane. Recall that f was a function in $Q(\mathbf{C}, w)$ such that $f|_{\mathbf{C}_-}$ belongs to the collection $\mathscr{K}$, and such that f does not vanish identically on $\mathbf{C}_-$. Let us begin by observing that the set $\mathbf{C}\backslash\mathscr{Z}(f, \mathbf{C})$ is open, where

$$\mathscr{Z}(f, \mathbf{C}) = \{z \in \mathbf{C} : f(z) = 0\},$$

and that the function $\bar{\partial} f/f$ is continuous on it. Considering that f is analytic on $\mathbf{C}_-$, and that f does not vanish identically on $\mathbf{C}_-$, we realize that we may extend $\bar{\partial} f/f$ continuously to the open region $(\mathbf{C}\backslash\mathscr{Z}(f, \mathbf{C})) \cup \mathbf{C}_-$ by declaring this function to vanish on all of $\mathbf{C}_-$. Let us fix a parameter $\alpha > 0$, and consider the open set

$$\Omega(f, \alpha) = \{z \in \mathbf{C} : |f(z)| > \alpha \cdot w(\Im z)\} \cup \mathbf{C}_- ;$$

here we have extended w to all of $\mathbf{R}$ by putting it equal to 0 on the interval $]-\infty, 0[$. Clearly, $\Omega(f, \alpha)$ is contained within the open set $(\mathbf{C}\backslash\mathscr{Z}(f, \mathbf{C})) \cup \mathbf{C}_-$. For $\lambda \in \Omega(f, \alpha)$, let $\rho(\lambda)$ be a real number such that $0 < \rho(\lambda) < \operatorname{dist}(\lambda, \mathbf{C}\backslash\Omega(f, \alpha))$. In what follows, we shall assume that the point λ belongs to the set $\Omega(f, \alpha)$. Let χ_λ be an infinitely differentiable compactly supported function on $\mathbf{C}$ with values between 0 and 1, which vanishes off a disk of radius $\rho(\lambda)$ centered at λ, and has value 1 on a disk of half that radius, also centered at λ. By construction, the function χ_λ is supported inside $\Omega(f, \alpha)$. Now let the function h_λ solve the $\bar{\partial}$ problem

$$\bar{\partial} h_\lambda(z) = -\chi_\lambda(z)\, \bar{\partial} f(z)/f(z), \qquad z \in \mathbf{C};$$

just put

$$h_\lambda(z) = -\int_{\mathbf{C}} \frac{\chi_\lambda(\zeta)\, \bar{\partial} f(\zeta)}{f(\zeta)(z-\zeta)}\, dm_2(\zeta)/\pi, \qquad z \in \mathbf{C},$$

where dm_2 denotes area measure on $\mathbf{C}$. Then h_λ belongs to $C_0(\mathbf{C})$, and h_λ is analytic on $\mathbf{C}_-$. Note that the closed disk-portion

$$\{z \in \mathbf{C} : |z - \lambda| \leqslant \rho(\lambda), \Im z \geqslant 0\}$$

is a compact subset of the set $\mathbf{C}\backslash\mathscr{Z}(f, \mathbf{C})$, so that there must be some $\varepsilon > 0$ such that $|f(z)| \geqslant \varepsilon$ on it. It is now clear from the definition of h_λ that because $\bar{\partial} f \in C_0(\mathbf{C}_+, w)$, we must also have $\bar{\partial} h_\lambda \in C_0(\mathbf{C}_+, w)$, so that, in fact, $h_\lambda \in Q(\mathbf{C}, w)$. Moreover, we have the estimate

$$\sup\{|h_\lambda(z)| : z \in \mathbf{C}, |z - \lambda| \leqslant \rho(\lambda)\} \leqslant (2/\alpha)\, \rho(\lambda) \sup_{z \in \mathbf{C}_+} (|\bar{\partial} f(z)|/w(\Im z)),$$

and we also have the same estimate of the sup-norm outside the disk $\{z\in\mathbf{C}: |z-\lambda| \leqslant \rho(\lambda)\}$, by the maximum principle, because $h_\lambda \in C_0(\mathbf{C})$ is holomorphic off the disk centered at λ with radius $\rho(\lambda)$, so that we get

$$\|h_\lambda\|_{L^\infty(\mathbf{C})} \leqslant (2/\alpha)\,\rho(\lambda) \sup_{z\in\mathbf{C}_+} (|\bar\partial f(z)|/w(\Im z)). \tag{7.1}$$

Let the function g_λ be defined by the relation

$$g_\lambda(z) = (1/f(\lambda)) \exp(h_\lambda(z) - h_\lambda(\lambda)), \qquad \lambda\in\mathbf{C}.$$

Since $h_\lambda \in Q(\mathbf{C}, w)$, we see that g_λ belongs to $Q_e(\mathbf{C}, w)$. Also, we have the estimate

$$\|g_\lambda\|_{L^\infty(\mathbf{C})} \leqslant (1/|f(\lambda)|)\cdot \exp(2\,\|h_\lambda\|_{L^\infty(\mathbf{C})}). \tag{7.2}$$

It is now time to consider the function F_λ given by

$$F_\lambda(z) = (1 - f(z)\, g_\lambda(z))/(\lambda - z), \qquad z\in\mathbf{C}\setminus\{\lambda\}.$$

By the way we constructed the function g_λ, we see that the function fg_λ has the property that

$$\bar\partial(fg_\lambda)(z) = g_\lambda(z)\,\bar\partial f(z)(1-\chi_\lambda(z)), \qquad z\in\mathbf{C},$$

and consequently, fg_λ is analytic on a disk of radius $\rho(\lambda)/2$ centered at λ. Also, $(fg_\lambda)(\lambda) = 1$, so that in the division, F_λ does not get a pole at λ, and we have $F_\lambda \in C_0(\mathbf{C})$. If we apply the $\bar\partial$-operator to the function F_λ, we get the expression

$$\bar\partial F_\lambda(z) = -g_\lambda(z)\,\bar\partial f(z)\,\frac{1-\chi_\lambda(z)}{\lambda - z}, \qquad z\in\mathbf{C}\setminus\{\lambda\}, \tag{7.3}$$

so that since $f\in Q(\mathbf{C}, w)$, the above formula shows that $\bar\partial F_\lambda \in C_0(\mathbf{C}_+, w)$, and consequently, $F_\lambda \in Q(\mathbf{C}, w)$. By the maximum principle,

$$\|F_\lambda\|_{L^\infty(\mathbf{C})} \leqslant \frac{2}{\rho(\lambda)}\cdot (1 + \|f\|_{L^\infty(\mathbf{C})}\,\|g_\lambda\|_{L^\infty(\mathbf{C})}), \tag{7.4}$$

and by formula (7.3), we have the estimate

$$\sup_{z\in\mathbf{C}_+} \frac{|\bar\partial F_\lambda(z)|}{w(\Im z)} \leqslant \frac{2}{\rho(\lambda)}\cdot \|g_\lambda\|_{L^\infty(\mathbf{C})}\cdot \sup_{z\in\mathbf{C}_+} \frac{|\bar\partial f(z)|}{w(\Im z)}. \tag{7.5}$$

From (7.4) and (7.5), together with the basic estimates (7.1) and (7.2), we obtain the norm estimate

$$\|F_\lambda\|_{Q(\mathbf{C},w)} \leqslant \frac{2}{\rho(\lambda)} \cdot \left(1 + \frac{\|f\|_{Q(\mathbf{C},w)}}{|f(\lambda)|} \cdot \exp(4\rho(\lambda)\, \|f\|_{Q(\mathbf{C},w)}/\alpha)\right). \quad (7.6)$$

We shall now see that

$$A_\lambda = F_\lambda + I(\mathscr{K}), \qquad \lambda \in \Omega(f, \alpha),$$

and this relation will give us the additional estimate we need on the function $\mathscr{R}[\phi]$. Consider the function

$$G_\lambda(z) = (i - z)\, F_\lambda(z) = \frac{i-z}{\lambda - z}(1 - f(z)\, g_\lambda(z)), \qquad z \in \mathbf{C},$$

which belongs to $Q_e(\mathbf{C}, w)$, because $G_\lambda - G_\lambda(\infty) = G_\lambda - 1 \in C_0(\mathbf{C})$, and because of the formula

$$\bar\partial G_\lambda(z) = -\frac{(i-z)(1-\chi_\lambda(z))}{\lambda - z}\, g_\lambda(z)\, \bar\partial f(z), \qquad z \in \mathbf{C}.$$

The calculation

$$(1 + (\lambda - i)\, b_i(z))\, G_\lambda(z) = \frac{\lambda - z}{i - z}\, G_\lambda(z) = 1 - f(z)\, g_\lambda(z) (\in 1 + I(\mathscr{K})), \quad z \in \bar{\mathbf{C}}_-,$$

shows that for $\lambda \in \Omega(f, \alpha)$, G_λ is an element of the coset

$$(1 + (\lambda - i)\, b_i + I(\mathscr{K}))^{-1},$$

and since $F_\lambda|_{\mathbf{C}_-} = b_i \cdot G_\lambda|_{\mathbf{C}_-}$, it follows that

$$A_\lambda = F_\lambda|_{\mathbf{C}_-} + I(\mathscr{K}), \qquad \lambda \in \Omega(f, \alpha).$$

Hence we have, by (5.5),

$$\mathscr{R}[\phi](\lambda) = \langle F_\lambda|_{\mathbf{C}_-}, \phi\rangle, \qquad \lambda \in \Omega(f, \alpha),$$

so that by (7.6), we have, for some constant $C(f, \phi)$,

$$|\mathscr{R}[\phi](\lambda)| \leqslant \frac{C(f,\phi)}{\rho(\lambda)\, |f(\lambda)|} \cdot \exp(4\rho(\lambda)\, \|f\|_{Q(\mathbf{C},w)}/\alpha), \qquad \lambda \in \Omega(f, \alpha). \quad (7.7)$$

Recall that in the above estimate, $\rho(\lambda)$ was an arbitrary real number with $0 < \rho(\lambda) < \operatorname{dist}(\lambda, \mathbf{C} \backslash \Omega(f, \alpha))$.

8. The Fundamental Technical Result

The following general interpolation-type result will prove valuable for the proof of Theorem 8.2.

THEOREM 8.1. *Let N be a positive integer, and let h, ε, δ, b be four real parameters with $0<h, \varepsilon, \delta \leqslant 1$, and $\varepsilon < b \leqslant 1$. Suppose we have a finite sequence $\zeta_0, ..., \zeta_N$ of points in the upper half-plane $\mathbf{C}_+$, with the property that $\varepsilon \leqslant \Im\zeta_j \leqslant b$, and*

$$2jh \leqslant \Re\zeta_j \leqslant (2j+1)h, \qquad j=0, ..., N.$$

Write $a=(2N+1)h$, and introduce the rectangle

$$R(\varepsilon, a, b)=\{z\in\mathbf{C}_+ : 0\leqslant \Re z\leqslant a,\ \varepsilon\leqslant\Im z\leqslant b\}.$$

There exists an absolute constant C such that the following holds: if f is an analytic function on $\mathbf{C}_+$ with $|f(z)|\leqslant 1$ on $\mathbf{C}_+$, and $|f(\zeta_j)|\leqslant\delta$ for all $j=0, ..., N$, then if $A=1+\min(a, b)/(2h)$ and

$$M_0 = CA\exp((37+40\log 1/h)A),$$

we have the estimate

$$|f(z)|\leqslant(1+\delta M_0)\exp\left(-\frac{2(N+1)\varepsilon}{4b^2+a^2}\cdot\Im z\right)+\delta M_0, \qquad z\in R(\varepsilon, a, b).$$

Proof. To begin with, let us consider an analytic function g on $\mathbf{C}_+$ with $|g(z)|\leqslant 1$ on $\mathbf{C}_+$, and $g(\zeta_j)=0$ for all $j=0, ..., N$. Let B be the finite Blaschke product corresponding to the sequence $\{\zeta_j\}_j$,

$$B(z)=\prod_j\frac{z-\zeta_j}{z-\bar\zeta_j}, \qquad z\in\mathbf{C}\setminus\bigcup_j\{\bar\zeta_j\}.$$

Then, by the well-known factoring theory for H^p spaces, $|g(z)|\leqslant|B(z)|$ on $\mathbf{C}_+$, and by Lemma VII.1.2 [13, p. 288], we have the estimate

$$|B(z)|\leqslant\exp\left(-2\sum_j\frac{\Im z\cdot\Im\zeta_j}{|z-\bar\zeta_j|^2}\right), \qquad z\in\mathbf{C}_+.$$

For $z\in R(\varepsilon, a, b)$, we have

$$|z-\bar\zeta_j|^2\leqslant 4b^2+a^2,$$

so that

$$\sum_j \frac{\Im z \cdot \Im \zeta_j}{|z-\bar{\zeta}_j|^2} \geqslant \frac{(N+1)\varepsilon}{4b^2+a^2} \cdot \Im z, \qquad z \in R(\varepsilon, a, b),$$

and, consequently,

$$|g(z)| \leqslant |B(z)| \leqslant \exp\left(-\frac{2(N+1)\varepsilon}{4b^2+a^2} \cdot \Im z\right), \qquad z \in R(\varepsilon, a, b). \tag{8.1}$$

Unfortunately, we do not know that the function f vanishes on the sequence $\zeta_0, ..., \zeta_N$, so we will try to subtract from f a function that interpolates the values of f on this sequence, and if we are lucky, this interpolating function can be chosen quite small. We shall use Carleson's interpolation theorem (see [13, p. 287]) to obtain an interpolating function.

For two points z, ζ in the upper half-plane $\mathbf{C}_+$, introduce the pseudohyperbolic metric $\rho(\cdot, \cdot)$:

$$\rho(z, \zeta) = |(z-\zeta)/(z-\bar{\zeta})|.$$

By the properties of the sequence $\{\zeta_j\}_j$, we have, for j and k with $j \neq k$,

$$\rho(\zeta_j, \zeta_k) \geqslant \eta = h \cdot (4b^2+h^2)^{-1/2} \geqslant h/\sqrt{5}, \tag{8.2}$$

and the last inequality holds because we assume $0 < b,\ h \leqslant 1$. If $S(r)$ is a (Carleson) square with side length r lying in the upper half-plane with one side supported on the real axis, one can check that

$$\sum_{j : \zeta_j \in S(r)} \Im \zeta_j \leqslant Ar,$$

where the constant A is given by the expression

$$A = 1 + \min(a, b)/(2h).$$

By [13, pp. 288, 289], we have

$$\inf_k \prod_{j : j \neq k} \rho(\zeta_j, \zeta_k) \geqslant \xi = \exp(-20A(1 + 2 \log 1/\eta)),$$

and by [13, pp. 292, 293], the constant of interpolation, denoted by M, is controlled by the expression

$$M \leqslant 4\pi CA/\xi,$$

where C is an absolute constant. If we use (8.2) and the notation $C' = 4\pi C$, we get

$$M \leqslant M_0 = C'A \exp((37 + 40 \log 1/h)\, A),$$

which makes it possible for us to find an analytic function h on $\mathbf{C}_+$ such that $h(\zeta_j) = f(\zeta_j)$ for all $j = 0, ..., N$, and

$$|h(z)| \leqslant \delta M_0, \qquad z \in \mathbf{C}_+.$$

If we now put $g = f - h$, we get a bounded analytic function in $\mathbf{C}_+$ which vanishes on the sequence $\zeta_0, ..., \zeta_N$, and the bound we have on it allows us to use (8.1) to conclude that

$$|g(z)| \leqslant (1 + \delta M_0) \exp\left(- \frac{2(N+1)\varepsilon}{4b^2 + a^2} \cdot \Im z \right), \qquad z \in R(\varepsilon, a, b).$$

Finally, since $f = g + h$, we get

$$|f(z)| \leqslant (1 + \delta M_0) \exp\left(- \frac{2(N+1)\varepsilon}{4b^2 + a^2} \cdot \Im z \right) + \delta M_0, \qquad z \in R(\varepsilon, a, b),$$

as asserted. ∎

To be able to apply our estimate (7.7), we should study the sets in $\mathbf{C}$ on which the values of a function f in $Q(\mathbf{C}, w)$, which does not vanish identically on $\mathbf{C}_-$, are separated away from zero.

For big positive integers k, say $k \geqslant k_0(w)$, it is always possible to find real numbers a_k, $0 < a_k \leqslant \frac{1}{2}$, such that

$$w(a_k) = \exp(-e^{2k}),$$

because the weight w is continuous and increasing on $\mathbf{R}_+ = [0, \infty[$, with $w(0) = 0$ and $w(t) > 0$ for all $t > 0$; the numbers a_k are uniquely determined if we make the additional inessential assumption that w is strictly increasing on $\mathbf{R}_+$. Note that the sequence $\{a_k\}_k$ is strictly decreasing down to 0, and that the decrease gets slower as the smoothness increases for the spaces $Q(\mathbf{C}_-, w)$. If $k \geqslant k_0(w)$, then introduce, for $0 \leqslant s \leqslant 3^k - 1$, the thin strips

$$D(s, k) = \{z \in \mathbf{C} : k + s3^{-k} < \Re z < k + (s+1)\, 3^{-k},\ -1/2 < \Im z < a_k - 2^{-k}\}.$$

THEOREM 8.2. *Let $f \in Q(\mathbf{C}, w)$ be such that f does not vanish identically on $\mathbf{C}_-$. Then, for all big k, say $k \geqslant k_1(f, w)$, having $a_k \geqslant (\frac{5}{6})^k$, there exists an s_k, $0 \leqslant s_k \leqslant 3^k - 1$, such that*

$$|f(z)| \geqslant w(a_k) = \exp(-e^{2k}), \qquad z \in D(s_k, k).$$

As a consequence,

$$D(s_k, k) \subset \Omega(f, 1) = \{z \in \mathbf{C} : |f(z)| > w(\Im z)\} \cup \mathbf{C}_- .$$

Proof. By a scaling argument, we may as well assume that f has norm $\leqslant 1$ in $Q(\mathbf{C}, w)$. Also, we see that we may assume, without loss of generality, that

$$|\bar{\partial} f(z)| \leqslant C(1+|z|)^{-3}\, w(\Im z), \qquad z \in \mathbf{C}_+, \tag{8.3}$$

for some constant C, because if f does not meet this condition, we simply replace it by the function

$$g(z) = (B_i^{1/3, 2/3}(z))^3 \cdot f(z), \qquad z \in \mathbf{C},$$

where $B_i^{1/3, 2/3}(z)$ is as in the proof of Lemma 3.1, which does meet the above condition, and it is even smaller than f for big values of the argument z.

To carry out the proof of Theorem 8.2, let us first suppose that the assertion does not hold, that is, suppose there exist points $\xi(s, k) \in D(s, k)$, such that

$$|f(\xi(s, k))| < w(a_k) = \exp(-e^{2k}), \qquad 0 \leqslant s \leqslant 3^k - 1,$$

and try then to show that this is not compatible with other available information about the function f. Since f is a bounded holomorphic function on $\mathbf{C}_-$, and f does not vanish identically on $\mathbf{C}_-$, we have

$$\int_{-\infty}^{\infty} \frac{\log |f(t)|}{1+t^2}\, dt > -\infty,$$

and hence, for all sufficiently large k, say $k \geqslant k_2(f)$, we are able to pick a point $x_k \in [k+\frac{1}{3}, k+\frac{2}{3}]$ such that

$$|f(x_k)| > 2e^{-k^2}.$$

Let S_k be the infinite strip

$$S_k = \{z \in \mathbf{C} : 0 < \Im z < a_k\},$$

and consider the function

$$f_k(z) = f(z) - \int_{S_k} \frac{\bar{\partial} f(\xi)}{z - \xi}\, dm_2(\xi)/\pi, \qquad z \in \mathbf{C},$$

which is holomorphic in the half-plane

$$\mathbf{C}_- + ia_k = \{z \in \mathbf{C} : \Im z < a_k\},$$

by the Cauchy–Green formula [19, p. 3]. If we use our assumption (8.3), the integral involved in defining f_k may be estimated as

$$\left|\int_{S_k} \frac{\bar\partial f(\xi)}{z-\xi}\, dm_2(\xi)/\pi\right| \leqslant Cw(a_k) \int_{S_k} |z-\xi|^{-1} (1+|z|)^{-3}\, dm_2(\xi)/\pi$$
$$\leqslant w(a_k), \qquad z \in \mathbf{C},$$

if k is sufficiently big, say $k \geqslant k_3(f, w)$ (here we choose $k_3(f, w)$ bigger than or equal to 2, $k_2(f)$, and $k_0(w)$), because the width a_k of the strip S_k converges to 0 as $k \to \infty$. If we recall that $w(a_k) < 1$ for $k \geqslant k_0(w)$, and note that

$$\exp(-e^{2k}) \leqslant e^{-k^2}, \qquad k = 0, 1, 2, \ldots,$$

we obtain, for $k \geqslant k_3(f, w)$, the estimates

$$\left.\begin{aligned} |f_k(z)| &\leqslant 2, & z &\in \mathbf{C}, \\ |f_k(\xi(s,k))| &\leqslant 2w(a_k), & 0 &\leqslant s \leqslant 3^k - 1, \\ |f_k(x_k)| &> e^{-k^2}. \end{aligned}\right\} \tag{8.4}$$

The function f_k is analytic in the half-plane $\mathbf{C}_- + ia_k$, and it is small on a rather numerous set of point. We are now in a situation where we may apply Theorem 8.1 to the function

$$F(z) = f_k(k+1-z+ia_k)/2, \qquad z \in \mathbf{C}_+,$$

with $a = 1$, $b = a_k + \frac{1}{2}$, $h = 3^{-k}$, $N = (3^k - 1)/2$, $\varepsilon = 2^{-k}$, and $\delta = \exp(-e^{2k})$. If we carry out the necessary computations, we get, for $k \geqslant k_3(f, w)$ ($\geqslant 2$), that $A \leqslant 3^k$, and

$$M_0 \leqslant C \cdot 3^k \exp((37 + 44k) \cdot 3^k),$$

where C is the absolute constant mentioned in Theorem 8.1, so that if k is sufficiently big, say $k \geqslant k_4(f, w)$, where we choose $k_4(f, w) \geqslant k_3(f, w)$, we get

$$M_0 \leqslant \exp(e^{2k}/2).$$

It follows that

$$\delta M_0 \leqslant \exp(-e^{2k}/2), \qquad k \geqslant k_4(f, w),$$

so, by Theorem 8.1 applied at the point $z_k = k + 1 - x_k + ia_k$,

$$|F(z_k)| \leqslant 2\exp(-(3/2)^k\, a_k/5) + \exp(-e^{2k}/2), \qquad k \geqslant k_4(f, w). \tag{8.5}$$

By the definition of F, $f_k(x_k)=2F(z_k)$, so that if we assume that $k \geqslant k_4(f, w)$ and $a_k \geqslant (\frac{5}{6})^k$, we have from (8.5)

$$|f_k(x_k)| \leqslant 6 \exp(-(5/4)^k/5).$$

For $k \geqslant 40$, this is in contradiction with (8.4). This completes the proof of Theorem 8.2. ∎

Later on, we shall need a simple assertion concerning functions analytic in $\mathbf{C}_+$.

LEMMA 8.3. *Let f and g be analytic in $\mathbf{C}_+$, with $f \not\equiv 0$, let f be bounded in $\mathbf{C}_+$, and suppose*

$$|f(z)\, g(z)| \leqslant 1/\Im z, \qquad z \in \mathbf{C}_+.$$

Then, for some constant $C(f) > 0$,

$$\log |g(z)| \leqslant C(f) \cdot (1+|z|^2)/\Im z \leqslant C(f) \cdot (1+|z|^4+(\Im z)^{-2}), \qquad z \in \mathbf{C}_+.$$

Proof. This statement was proved in [17]. For the sake of completeness, we present here the corresponding arguments. Given a $\xi > 0$, we estimate the function g in the shifted upper half-plane $\mathbf{C}_+ + i\xi$ in the following way (see [20]):

$$\log |g(z+i\xi)| \leqslant \log 1/\xi + \alpha(f)\, \Im z + \frac{\Im z}{\pi} \int_{-\infty}^{\infty} \frac{\log 1/|f(t+i\xi)|}{(\Re z - t)^2 + (\Im z)^2}\, dt, \quad z \in \mathbf{C}_+,$$

where $\alpha(f)$ denotes the constant

$$\alpha(f) = \liminf_{y \to +\infty} \log 1/|f(iy)| \in [0, +\infty[.$$

Moreover, since f is bounded and nonidentically vanishing in $\mathbf{C}_+$, it follows that

$$\sup_{0<\xi\leqslant 1} \int_{-\infty}^{\infty} \frac{\log 1/|f(t+i\xi)|}{1+t^2}\, dt < \infty,$$

so that we have for some constant $C_1(f)$, independent of ξ, $0 < \xi \leqslant 1$, that

$$\frac{\Im z}{\pi} \int_{-\infty}^{\infty} \frac{\log 1/|f(t+i\xi)|}{(\Re z - t)^2 + (\Im z)^2}\, dt \leqslant C_1(f) \cdot (|z|^2+1)/\Im z, \qquad z \in \mathbf{C}_+.$$

If we now pick $\xi = \min\{\Im z/2, 1\}$ in the above estimate for g, the desired assertion

$$\log|g(z)| \leqslant C(f)\,\frac{1+|z|^2}{\Im z}, \qquad z \in \mathbf{C}_+,$$

immediately follows. The final touch comes from the simple inequality

$$(1+|z|^2)/\Im z \leqslant 1 + |z|^4 + (\Im z)^{-2}, \qquad z \in \mathbf{C}_+. \quad \blacksquare$$

9. The Conclusion of the Proof of Theorem 4.1

The necessity of condition (a) is clear. That (b) is necessary as well follows from the fact that for $\varepsilon > 0$, $e^{-i\varepsilon z} \cdot H^\infty(\mathbf{C}_-)$ is a closed subspace (in fact, an ideal) in $H^\infty(\mathbf{C}_-)$. Consequently, we shall concentrate on proving the sufficiency of these two conditions.

As indicated previously, the method of our proof involves duality arguments. Let us assume that the functional ϕ in $Q(\mathbf{C}_-, w)^*$ is orthogonal to $I(\mathscr{K})$; the plot of our proof is to prove that every such ϕ equals the 0 functional, so that the desired assertion is a consequence of the Hahn–Banach theorem. In fact, we shall prove that $\mathscr{R}[\phi] = 0$, so we need the fact that ϕ is uniquely determined by its transform $\mathscr{R}[\phi]$, which, as has been pointed out already, follows from Lemma 3.1. By (5.5), $\mathscr{R}[\phi]$ extends to an entire function. Let $f \in \mathscr{K}$ be a function which does not vanish identically on $\mathbf{C}_-$. Then, by (6.1), (6.3), and (7.7), we have the estimates $(0 < s < 1)$

$$|\mathscr{R}[\phi](\lambda)| \leqslant C(\phi)\left(\frac{1}{(1-s)^2(\Im\lambda)^2}\cdot\frac{1}{w(s\Im\lambda)} + \frac{2}{s\Im\lambda}\right), \qquad \lambda \in \mathbf{C}_+, \tag{9.1}$$

$$|\mathscr{R}[\phi](\lambda)| \leqslant \frac{C(f,\phi)}{|\Im\lambda|\,|f(\lambda)|}, \qquad \lambda \in \mathbf{C}_-, \tag{9.2}$$

$$|\mathscr{R}[\phi](\lambda)| \leqslant \frac{C(f,\phi)}{\rho(\lambda)\,|f(\lambda)|}\cdot\exp(C(f)\,\rho(\lambda)/\alpha), \qquad \lambda \in \Omega(f,\alpha). \tag{9.3}$$

Introduce three sets of positive integers,

$$Z(f,w) = \{k \in \mathbf{Z} : k \geqslant k_1(f,w)\},$$
$$X = \{k \in Z(f,w) : a_k \geqslant (5/6)^k\},$$

and

$$Y = \{k \in Z(f,w) : a_k - 3\cdot 2^{-k} \geqslant a_{2k}\},$$

where $k_1(f, w)$ is the positive integer that appears in Theorem 8.2, which we will find advantageous to assume $\geqslant 4$. For $k \in X$, consider the line segments

$$I_k = \{z \in \mathbf{C}: \Re z = k + (s_k + \tfrac{1}{2})\, 3^{-k},\ -\tfrac{1}{4} \leqslant \Im z \leqslant a_k - 2^{1-k}\},$$

where $0 \leqslant s_k \leqslant 3^k - 1$ is as in the formulation of Theorem 8.2. Then, by Theorem 8.2, $I_k \subset D(k, s_k) \subset \Omega(f, 1)$ for all $k \in X$, and since every point of I_k is at least $(\tfrac{1}{2})\, 3^{-k}$ distance units away from the complement of $\Omega(f, 1)$, we conclude from (9.3) that

$$|\mathscr{R}[\phi](\lambda)| \leqslant C(f, \phi) \exp(2e^{2k}), \qquad \lambda \in I_k, \tag{9.4}$$

for some finite positive constant $C(f, \phi)$, provided that $k \in X$. For $k \in X$, let J_k be the line segment

$$J_k = \{z \in \mathbf{C}: \Re z = k + (s_k + \tfrac{1}{2})\, 3^{-k},\ a_k - 2^{1-k} \leqslant \Im z \leqslant \tfrac{1}{4}\},$$

and observe that if we apply (9.1) with the parameter value $s = (a_k - 3 \cdot 2^{-k})/\Im\lambda$, we obtain for $k \in X \cap Y$ the estimate

$$\begin{aligned} |\mathscr{R}[\phi](\lambda)| &\leqslant C(\phi) \cdot (2^{2k}/w(a_k - 3 \cdot 2^{-k}) + 1/(a_k - 3 \cdot 2^{-k})) \\ &\leqslant C(\phi) \cdot \exp(2e^{4k}), \qquad \lambda \in J_k, \end{aligned} \tag{9.5}$$

where we use the facts that $w(a_k - 3 \cdot 2^{-k}) \geqslant w(a_{2k}) = \exp(-e^{4k})$ for $k \in Y$, and $a_k - 3 \cdot 2^{-k} \geqslant 2^{-k}$ for $k \in X$. If we combine our estimates (9.4) and (9.5), we get

$$|\mathscr{R}[\phi](z)| \leqslant C(f, \phi) \cdot \exp(2e^{4k}), \qquad z \in K_k, \qquad k \in X \cap Y, \tag{9.6}$$

where K_k denotes the union of the line segments I_k and J_k:

$$K_k = I_k \cup J_k = \{z \in \mathbf{C}: \Re z = k + (s_k + \tfrac{1}{2})\, 3^{-k},\ -\tfrac{1}{4} \leqslant \Im z \leqslant \tfrac{1}{4}\}.$$

Let us gather the information we have built up about the entire function $\mathscr{R}[\phi]$: it satisfies (9.1), (9.2), and (9.6), with f an arbitrary nonidentically vanishing function in the collection $\mathscr{K}$; the fact that $\mathscr{K}$ has condition (b) of Theorem 4.1 entails, as has been noted before, that $\mathscr{R}[\phi]$ belongs to the Smirnov class on the shifted half-space $\mathbf{C}_- - i$. By Lemma 8.3, applicable by (9.2), we have

$$\begin{aligned} |\mathscr{R}[\phi](\lambda)| &\leqslant \exp(C(f, \phi) \cdot (1 + |\lambda|^2)/|\Im\lambda|) \\ &\leqslant \exp(C(f, \phi) \cdot (1 + |\lambda|^4 + |\Im\lambda|^{-2})), \qquad \lambda \in \mathbf{C}_-. \end{aligned} \tag{9.7}$$

Consider the auxiliary entire function

$$\Phi_1(z) = \exp(-z^6) \cdot \mathscr{R}[\phi](z), \qquad z \in \mathbf{C},$$

and the infinite strip

$$S = \{z \in \mathbf{C} : -\tfrac{1}{4} < \Im z < \tfrac{1}{4}\}.$$

We plan to show that Φ_1 is bounded in the strip S. Elementary computation shows that we have the estimate

$$\tfrac{15}{16}(\Re z)^6 - \tfrac{1}{16} \leqslant \Re(z^6) \leqslant \tfrac{17}{16}(\Re z)^6 + \tfrac{1}{16}, \qquad z \in S, \tag{9.8}$$

so that by (9.1) and (9.7), the function Φ_1 is bounded on the boundary ∂S of S, and has the estimate

$$|\Phi_1(z)| \leqslant \exp(C(f, \phi) \cdot (1 + |\Im \lambda|^{-2})), \qquad z \in S \cap \mathbf{C}_-. \tag{9.9}$$

In addition, we have, by (9.6),

$$|\Phi_1(z)| \leqslant C(f, \phi) \cdot \exp(2e^{4k}), \qquad z \in K_k, \qquad k \in X \cap Y,$$

and the growth that this estimate permits is too small for the strip S; the critical growth in the strip S is in fact of the order of magnitude $\exp(\exp(2\pi z))$. More precisely, if the set $X \cap Y$ is not finite, we may apply the Phragmén–Lindelöf principle in the form that appears in, for instance, [18, pp. 145–146], to obtain that, in fact, Φ_1 is bounded on the portion of S that lies in the right half-plane.

This is all very well if $X \cap Y$ should happen to be infinite, but what if it is actually finite?

Claim. If $X \cap Y$ is finite, then X is also finite.

Since $X \cap Y$ is assumed finite, there exists an integer $l \in X \cap Y$ such that $k \leqslant l$ for all (other) $k \in X \cap Y$. If X contains an integer k which is bigger than l, then k cannot belong to Y, and hence

$$a_{2k} > a_k - 3 \cdot 2^{-k} \geqslant (\tfrac{5}{6})^k - 3 \cdot 2^{-k} > (\tfrac{5}{6})^{2k},$$

because $k \geqslant 4$. We see that $2k$ also belongs to X, and $2k$ being bigger than l, we must have $2k \in X \setminus Y$. If we continue inductively, it follows that for $n = 0, 1, 2, \ldots$, we have $2^n k \in X \setminus Y$, and we also get the inequality

$$a_{2^n k} > a_k - 3 \cdot (2^{-k} + 2^{-2k} + \cdots + 2^{-2^n k}) > a_k - 3/(2^k - 1).$$

Since $k \geqslant 4$, we have

$$a_k - 3/(2^k - 1) \geqslant (\tfrac{5}{6})^k - 3/(2^k - 1) > 0,$$

but this is impossible, because we already know that $a_j \to 0$ as $j \to \infty$. This contradiction demonstrates that our initial assumption concerning the existence of a $k \in X$ bigger than l must be wrong. ▮

If the set X is finite, then $a_k < (\frac{5}{6})^k$, and consequently

$$w((\tfrac{5}{6})^k) \geqslant w(a_k) = \exp(-e^{2k}),$$

must hold for all but finitely many indices $k \in Z(f, w)$. If we play around with this inequality for a little while, we see that

$$w(t) \geqslant \exp(-8/t^{11}), \qquad 0 < t < \varepsilon_0,$$

holds for some small but positive number ε_0. But this weight w is of non-quasianalytic type, so by the estimates (9.1), with parameter value s chosen to equal $\frac{1}{2}$, and (9.9), the Levinson log-log theorem [21, p. 376] applies to the function Φ_1, and proves that it is bounded in the whole strip S.

We conclude that the function Φ_1 is bounded at least on the portion of the strip S that lies in the right half-plane, no matter whether $X \cap Y$ is finite or not. However, the right half-plane plays no special role here, and we should note that Theorem 8.2 applies to the function $\bar{f}(-\bar{z})$ as well, allowing us to conclude that $|f|$ is reasonably big on a thin set (possibly) going off to infinity in the left half-plane. All the computations we have made for the right half-plane carry through analogously, and demonstrate that Φ_1 is bounded on the portion of S that lies in the left half-plane, and consequently, on the whole infinite strip S.

Let us again recapitulate the information we have concerning the entire function $\mathscr{R}[\phi]$: it is bounded in the shifted upper half-space $\mathbf{C}_+ + (\frac{1}{4})i$, and by (9.8) and the boundedness of Φ_1 on the strip S, it has the estimate

$$\mathscr{R}[\phi](z) \leqslant C(f, \phi) \cdot \exp(2(\Re z)^6), \qquad z \in S;$$

also, in the lower half-plane, we have the estimate

$$\mathscr{R}[\phi](z) \leqslant \exp(C(f, \phi) \cdot (1 + |z|^2)/\Im z), \qquad z \in \mathbf{C}_-,$$

and we know that $\mathscr{R}[\phi]$ is of slow growth, that is, for every $\varepsilon > 0$, we have

$$|\mathscr{R}[\phi](z)| = O(\exp(\varepsilon\, |z|)),$$

as z tends to infinity along rays $\alpha \mathbf{R}_+$, with $\alpha \in \mathbf{C}_-$, because $\mathscr{R}[\phi]$ belongs to the Smirnov class in $\mathbf{C}_- - i$. This information permits us to apply the Phragmén–Lindelöf theorem for angles to obtain that, as a first step, we must have, for each fixed $\varepsilon > 0$,

$$|\mathscr{R}[\phi](z)| = O(\exp(\varepsilon\, |z|)), \qquad |z| \to \infty,$$

and as a second step, that $\mathcal{R}[\phi]$ is bounded on the whole complex plane. Finally, Liouville's theorem then claims that $\mathcal{R}[\phi]$ must be constant, and since, by (9.1), $\mathcal{R}[\phi](iy) \to 0$ as $y \to +\infty$, this constant must equal 0. The proof of Theorem 4.1 is now complete. ▌

10. The Proof of Corollary 4.2

Recall from the proof of Theorem 4.1 the notation

$$M_x(f)(z) = e^{-ixz} \cdot f(z), \qquad z \in \bar{\mathbf{C}}_-,$$

for $x \in \mathbf{R}$ and $f \in Q(\mathbf{C}_-, w)$, and if $\tilde{f}$ denotes an extension in $Q(\mathbf{C}, w)$ of f, the notation

$$M_x^1(\tilde{f})(z) = \psi(\Im z) \cdot e^{-ixz} \cdot \tilde{f}(z), \qquad z \in \mathbf{C},$$

where ψ is a fixed C^∞ function on $\mathbf{R}$, such that $0 \leqslant \psi(t) \leqslant 1$ for all $t \in \mathbf{R}$, $\psi(t) = 1$ for all $t \leqslant 1$, and $\psi(t) = 0$ for all $t \geqslant 2$. In that proof, we indicated that for $x \geqslant 0$, $M_x(f)$ belongs to $Q(\mathbf{C}_-, w)$ provided that f belongs to $Q(\mathbf{C}_-, w)$, and it is not difficult to see that

$$\|M_x(f)\|_{Q(\mathbf{C}_-, w)} \leqslant C(x) \cdot \|f\|_{Q(\mathbf{C}_-, w)}, \qquad x \in \mathbf{R}_+, f \in Q(\mathbf{C}_-, w).$$

On the other hand, given the above concrete formulas for M_{-x}, which is the operator inverse to M_x, and for M^1_{-x}, we see that

$$\|f\|_{Q(\mathbf{C}_-, w)} = \|M_{-x} M_x f\|_{Q(\mathbf{C}_-, w)} \leqslant C(x) \cdot \|M_x f\|_{Q(\mathbf{C}_-, w)},$$
$$x \in \mathbf{R}_+, f \in (\mathbf{C}_-, w),$$

so that the norms $\|f\|_{Q(\mathbf{C}_-, w)}$ and $\|M_x f\|_{Q(\mathbf{C}_-, w)}$ are equivalent for each $x \geqslant 0$, with compatibility constants depending on x. This shows that the subspace

$$M_{\varepsilon(\mathcal{K})}(Q(\mathbf{C}_-, w)) = e^{-i\varepsilon(\mathcal{K})z} \cdot Q(\mathbf{C}_-, w)$$

is closed in $Q(\mathbf{C}_-, w)$. We need one more piece of information to be able to bring the proof of Corollary 4.2 to an end. If $f \in H^\infty(\mathbf{C}_-)$ belongs to the closed ideals $e^{-i\beta z} \cdot H^\infty(\mathbf{C}_-)$ for every $\beta < \varepsilon(\mathcal{K})$, then it also belongs to $e^{-i\varepsilon(\mathcal{K})z} \cdot A_0(\mathbf{C}_-)$, and if in addition $f \in Q(\mathbf{C}_-, w)$, then the concrete formula for $M^1_{-\varepsilon(\mathcal{K})}(\tilde{f})$ shows that f belongs to the closed subspace $e^{-i\varepsilon(\mathcal{K})z} \cdot Q(\mathbf{C}_-, w)$ of $Q(\mathbf{C}_-, w)$. Now, by assumption, $\mathcal{T}_+(\mathcal{K})$ is a subspace of $e^{-i\beta z} \cdot H^\infty(\mathbf{C}_-)$ for every $\beta < \varepsilon(\mathcal{K})$, so by the above remarks, we see that $\mathcal{T}_+(\mathcal{K})$ is also a subspace of $e^{-i\varepsilon(\mathcal{K})z} \cdot Q(\mathbf{C}_-, w)$. However, by the equivalence of norms we obtained earlier, it follows that $\mathcal{T}_+(\mathcal{K})$ is dense in $e^{-i\varepsilon(\mathscr{K})z} \cdot Q(\mathbf{C}_-, w)$ if and only if $M_{-\varepsilon(\mathscr{K})}(\mathscr{T}_+(\mathscr{K}))$ is dense in $Q(\mathbf{C}_-, w)$, and this last statement is an immediate consequence of Theorem 4.1.

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