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Cyclicity in Bergman-Type Spaces

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1 Introduction

Given a linear topological space X of analytic functions in the unit disk $\mathbb{D}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, closed under multiplication by the coordinate function z, we say that an element f of X is *cyclic* in X if functions of the form $q(z)f(z)$, where $q \in \mathcal{P}$ ($\mathcal{P}$ is the ring of polynomials), are dense in X. In this paper we consider, for $0 < p < +\infty$, the Banach space A^{-p}, and its separable subspace A_0^{-p}. A holomorphic function f on $\mathbb{D}$ is in A^{-p} if

$$\|f\|_{A^{-p}} = \sup\left\{(1-|z|)^p|f(z)| : \ z \in \mathbb{D}\right\} < +\infty,$$

and it belongs to A_0^{-p} if, in addition, $(1-|z|)^p|f(z)| \to 0$ as $|z| \to 1$. We are also interested in the Bergman space B^p, which consists of those holomorphic functions f on $\mathbb{D}$ that have

$$\|f\|_{B^p} = \left(\int_{\mathbb{D}} |f(z)|^p \, dS(z)\right)^{1/p} < +\infty,$$

where dS is the area measure on $\mathbb{C}$, normalized so that the area of $\mathbb{D}$ is 1. The Bergman space B^p is a Banach space for $1 \le p < +\infty$, a Hilbert space for $p = 2$, and a complete metric space for $0 < p < 1$. The space $A^{-\infty}$ is the union of all the spaces A^{-p}, $0 < p < +\infty$, supplied with the inductive limit topology. It can also be thought of as the union of all the spaces A_0^{-p}, $0 < p < +\infty$, or of all the Bergman spaces B^p, $0 < p < +\infty$, and the inductive limit topologies it gets in this way coincide with the earlier one.

If, as is the case for the spaces $X = A^{-\infty}, A_0^{-p}, A^{-p}, B^p$, with $0 < p < +\infty$, the point evaluation functional

$$f \mapsto f(z), \qquad |z| < 1,$$

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is continuous and nontrivial, then an obvious necessary condition for a function $f \in X$ to be cyclic is that

$$f(z) \neq 0, \qquad |z| < 1.$$

Boris Korenblum's factorization theory for the space $A^{-\infty}$ [10], [11] offers a description of the cyclic vectors in this space that uses the notion of so-called $\varkappa$-singular measure.

Theorem A. An element f in $A^{-\infty}$ is cyclic if and only if $f(z) \neq 0$, $|z| < 1$, and the $\varkappa$-singular measure associated to f is equal to 0. □

As regards cyclicity in A_0^{-p}, the following statement is proved in [3].

Theorem B. If $f \in A_0^{-p}$ and f is invertible (or just cyclic) in $A^{-\infty}$, then f is cyclic in every A_0^{-q}, $p < q < +\infty$. □

Note that f is invertible in $A^{-\infty}$ if and only if, for some positive constants C, N,

$$|f(z)| \geq C\,(1 - |z|)^N, \qquad z \in \mathbb{D}.$$

Clearly, cyclicity in A_0^{-p} implies cyclicity in $A^{-\infty}$ and in A_0^{-q}, for $p < q < +\infty$. A natural question arises: If $f \in A_0^{-p}$ and f is invertible (or cyclic) in $A^{-\infty}$, must then f be cyclic in A_0^{-p}? This question was posed by Leon Brown and Boris Korenblum in [3]. Here we answer this question in the negative.

Theorem 1.1. Fix two real numbers $p, q > 0$. Then there exists a function f in A_0^{-p} such that $1/f \in A^{-q}$, f is cyclic in every A_0^{-r}, $p < r < +\infty$, but f is not cyclic in A_0^{-p}. □

When equipped with the norm topology, A^{-p} is not separable, and thus cannot have any cyclic elements. Thus, to get meaningful results about cyclicity, it is necessary to weaken the topology somewhat. The Banach space A^{-p} has a natural predual (it is a quotient space of functions integrable in the disk with respect to a weight); equip A^{-p} with the corresponding weak-star topology. One can then show that a sequence $\{f_k\}_k$ $(k = 1, 2, 3, \ldots)$ of functions in A^{-p} converges weakly-star to $f \in A^{-p}$ if and only if $\sup_k \|f_k\|_{A^{-p}} < +\infty$, and $f_k(z) \to f(z)$ uniformly on compact subsets of $\mathbb{D}$. A set $Y \subset A^{-p}$ is said to be weakly-star sequentially closed if all weak-star limits of sequences of elements in Y also belong to Y. By the Krein-Shmulian theorem (see [4, p. 429]), a convex subset Y of A^{-p} is weakly-star closed if and only if it is weakly-star sequentially closed.

To investigate cyclicity in A^{-p} with the weak-star topology, it is not enough to look at sequential limits. The following example illustrates this point.

Theorem 1.2. Fix two real numbers $p, q > 0$. Then there exists an outer (in Beurling's sense) function f in the Nevanlinna class such that $f \in A^{-p}$, $1/f \in A^{-q}$, f is cyclic in every A_0^{-r}, $p < r < +\infty$, and the set of all weak-star sequential limits in A^{-p} of functions of the type $f(z)q(z)$, where $q(z)$ is a polynomial, does not contain the constant function 1. □

Note, however, that in the situation described in Theorem 1.2, the function 1 does belong to the weak-star closure of the set of polynomial multiples of f, and hence f is cyclic with respect to the weak-star topology. Recall that we say that $f \in A^{-p}$ is cyclic in A^{-p} with respect to the weak-star topology if its polynomial multiples are dense. It can be shown that the weak-star closures of the polynomial multiples and the H^∞ multiples of f coincide.

The following result shows that it is possible to get a genuine analog of Theorem 1.1.

Theorem 1.3. Fix two real numbers $p, q > 0$. Then there exists a function f in A^{-p} such that $1/f \in A^{-q}$, f is cyclic in every A_0^{-r}, $p < r < +\infty$, but f is not cyclic in A^{-p} (with respect to the weak-star topology). □

The first result analogous to Theorem B for the Bergman space B^p was proved by H. S. Shapiro [15], [16].

Theorem C. If $f \in B^p$ and $1/f \in A^{-\infty}$, then f is cyclic in every B^q, $0 < q < p$. □

Moreover, as was pointed out in [3], the following analog of Theorem B holds for the spaces B^p.

Theorem D. If $f \in B^p$ for some p, $0 < p < +\infty$, and f is cyclic in $A^{-\infty}$, then f is cyclic in every B^q, $0 < q < p$. □

The question of whether the conditions

(a) $f \in B^p$ and $1/f \in A^{-\infty}$, or

(b) $f \in B^p$ and f is cyclic in $A^{-\infty}$

imply that f is cyclic in B^p was first raised in [15], and later in [14, p. 93], [1, pp. 187, 190], [17, Question 25], [12, Conjectures 1, 2], [18, Conjecture 1], [19, Question 5], and [9, Problem 8.4]. A special case of this question is whether the conditions $f \in B^p$, $1/f \in B^q$ imply that f is cyclic in B^p. This question was raised in [14, p. 93], [17, Question 25′], and [18]. See also [7].

We construct an example answering all these questions in the negative.

Theorem 1.4. Fix two real numbers $p, q > 0$. Then there exists a function f in B^p such that $1/f \in A^{-q}$, f is cyclic in every B^r, $0 < r < p$, but f is not cyclic in B^p. □

The particular properties of the functions f mentioned in Theorems 1.1–1.4, which imply that they are noncyclic, are specified in Section 2. They are required to be, in a sense, extremally large in the given space X ($X = A_0^{-p}$, A^{-p}, B^p). More precisely, the set $E(f)$ of points in $\mathbb{D}$ where $|f|$ is "maximally" large is pretty massive: for Theorem 1.2, its closure contains the unit circle $\mathbb{T}$; for Theorem 1.3, $E(f)$ is a dominating set for H^∞ (that is, bounded analytic functions attain their supremum modulus on $E(f)$); and for Theorems 1.1 and 1.4, the harmonic measure of $\mathbb{T}$ in the region $\mathbb{D} \setminus E(f)$ equals 0. To see the relevance of extremal growth for noncyclicity, let us look at the setting of Theorem 1.3. Here, f is in A^{-p}, and the set

$$E(f) = \left\{z \in \mathbb{D} : (1 - |z|)^p |f(z)| \geq 1\right\}$$

is dominating for H^∞. Let g_n be a sequence of H^∞ functions, such that fg_n converges weakly-star to a function $h \in A^{-p}$. Weak-star convergence requires that $\|fg_n\|_{A^{-p}} \leq C$ for some constant C, and hence $\sup\{|g_n(z)| : z \in E(f)\} \leq C$. But then g_n is a bounded sequence in H^∞, and $h = fg$, with $g \in H^\infty$. Thus $f \cdot H^\infty$ is weakly-star closed in A^{-p}, and it cannot contain the constant function 1, because f is not in the Nevanlinna class. (It does not have finite nontangential boundary values.) We conclude that f is not cyclic in A^{-p}. Proofs of Theorems 1.1–1.4 are supplied in Section 3, based on Theorems 2.1–2.4 (which will be proved elsewhere). The constructions on which the proofs of Theorems 2.1–2.4 are based resemble an example given by Nikolaĭ Nikolskiĭ [14, p. 84, Theorem 2].

2 Technical constructions

Here we produce four functions satisfying certain growth conditions in the disk $\mathbb{D}$. The details of the constructions will appear elsewhere.

Theorem 2.1. Given a real number $\alpha > 0$, there is an outer function F in the Nevanlinna class such that

(a) $\sup\{(1 - |z|)|F(z)| : z \in \mathbb{D}\} < +\infty$,

(b) $|F(z)| \geq (1 - |z|)^\alpha$, $z \in \mathbb{D}$,

(c) $\inf\{|F(z)| : z \in \mathbb{D}\} = 0$,

(d) $\mathbb{T}$ is contained in the closure of the set $E(F)$, $E(F) = \{z \in \mathbb{D} : (1 - |z|)|F(z)| \geq 1\}$. □

A subset E of $\mathbb{D}$ is said to be dominating for H^∞ provided that $\sup\{|f(z)| : z \in E\} = \sup\{|f(z)| : z \in \mathbb{D}\}$ holds for all $f \in H^\infty$. It is well known that E is dominating if and only if, to almost every $w \in \mathbb{T}$, there corresponds a sequence of points in E approaching w nontangentially.

Theorem 2.2. Given a real number $\alpha > 0$, there is a function F analytic in $\mathbb{D}$ such that

(a) $\sup\{(1-|z|)|F(z)| : z \in \mathbb{D}\} < +\infty$,

(b) $|F(z)| \geq (1-|z|)^\alpha$, $z \in \mathbb{D}$,

(c) The set $E(F) = \{z \in \mathbb{D} : (1-|z|)|F(z)| \geq 1\}$ is dominating for H^∞. □

Theorem 2.3. Given a real number $\alpha > 0$, there exist a function F analytic in $\mathbb{D}$ and two increasing functions $U, V : [0,1[\to \mathbb{R}_+$, with $U(t) \to +\infty$ and $V(t) \to +\infty$ as $t \to 1$, such that

(a) $\sup\{(1-|z|)U(|z|)|F(z)| : z \in \mathbb{D}\} < +\infty$,

(b) $|F(z)| \geq (1-|z|)^\alpha$, $z \in \mathbb{D}$,

(c) $\inf\{V(|z|)|F(z)| : z \in \mathbb{D}\} = 0$, and

(d) the set $E(F, U) = \{z \in \mathbb{D} : (1-|z|)U(|z|)|F(z)| \geq 1\}$ has the following property: if φ is a function bounded from above and subharmonic in $\mathbb{D}$, and

$$\exp\varphi(z) \leq U(|z|), \qquad z \in E(F, U),$$

then

$$\exp\varphi(z) \leq V(|z|), \qquad z \in \mathbb{D}.$$ □

For real numbers c (or real-valued functions), write $c^+ = \max\{0, c\}$ and $c^- = \max\{0, -c\}$.

Theorem 2.4. Given a real number $\alpha > 0$, there exist a function F analytic in $\mathbb{D}$ and an increasing function $V : [0,1[\to \mathbb{R}_+$, with $V(t) \to +\infty$ as $t \to 1$, such that

(a) $\int_{\mathbb{D}} |F(z)|\, dS(z) < \infty$,

(b) $|F(z)| \geq (1-|z|)^\alpha$, $z \in \mathbb{D}$,

(c) $\limsup_{r\to 1}\left\{\int_{-\pi}^{\pi} \log^-|F(re^{i\theta})|\, d\theta - V(r)\right\} = +\infty$,

and the following additional property holds:

(d) if φ is a function bounded from above and subharmonic in $\mathbb{D}$, and

$$\int_{\mathbb{D}} |F(z)| \exp\varphi(z)\, dS(z) \leq 1,$$

then

$$\int_{-\pi}^{\pi} \varphi^+(re^{i\theta})\, d\theta \leq V(r), \qquad 0 \leq r < 1.$$ □

3 Deductions of the main theorems

Proof of Theorem 1.2. Put $f = F^p$, where F is as in Theorem 2.1 with $\alpha = q/p$. By (a) of that theorem, $f \in A^{-p}$; by (b), $1/f \in A^{-q}$; and by Theorem B, the function f is cyclic in every

A_0^{-r}, $p < r < +\infty$. The function f is in the Nevanlinna class because F is. Suppose $\{p_k\}_k$ is a sequence of polynomials, such that fp_k converges weakly-star in A^{-p} to some function $h \in A^{-p}$. Then the norm of fp_k in A^{-p} is bounded by some constant C, so that

$$\sup\{|p_k(z)| : z \in E(F)\} \le C,$$

and by property (d) of Theorem 2.1, together with the maximum principle,

$$\sup\{|p_k(z)| : z \in \mathbb{D}\} \le C.$$

Hence, by property (c) of Theorem 2.1, the infimum of the limit function $|h(z)|$ on $\mathbb{D}$ is 0. Thus we cannot get the constant function 1 as $h(z)$. The proof is complete. ■

Proof of Theorem 1.3. We argue as in the previous proof, with Theorem 2.1 replaced by Theorem 2.2. Put $f = F^p$, where F is as in Theorem 2.2 with $\alpha = q/p$. By (a) of that theorem, $f \in A^{-p}$; by (b), $1/f \in A^{-q}$; and by Theorem B, the function f is cyclic in every A_0^{-r}, $p < r < +\infty$. Let $\{g_k\}_k$ be a sequence of functions in H^∞ such that $fg_k \to fg$ weakly-star in A^{-p} as $k \to +\infty$, where $fg \in A^{-p}$. Then the norm of fg_k in A^{-p} is bounded by some constant C, so that

$$\sup\{|g_k(z)| : z \in E(F)\} \le C,$$

and by property (d) of Theorem 2.2,

$$\sup\{|g_k(z)| : z \in \mathbb{D}\} \le C.$$

Since $fg_k \to fg$ uniformly on compact subsets of $\mathbb{D}$, $g_k \to g$, again uniformly on compact subsets of $\mathbb{D}$, and hence $g \in H^\infty$. We conclude that $f \cdot H^\infty$ is weakly-star sequentially closed in A^{-p}, and hence weakly-star closed. The constant function 1 cannot belong to $f \cdot H^\infty$, for the following reason: if $1 = fg$ with $g \in H^\infty$, then f is in the Nevanlinna class, and so f has finite nontangential boundary values almost everywhere. But this is clearly not the case, given the rapid growth of $|F(z)|$ on the dominating set $E(F)$. ■

Proof of Theorem 1.1. We proceed as in the proof of Theorem 1.2, with Theorem 2.1 replaced by Theorem 2.3. Put $f = F^p$, where F is as in Theorem 2.3, with $\alpha = q/p$. By (a) of that theorem, $f \in A_0^{-p}$; by (b), $1/f \in A^{-q}$; and by Theorem B, the function f is cyclic in every A_0^{-r}, $p < r < +\infty$. We argue by contradiction, supposing that there exists a sequence $\{g_k\}_k$ of functions in H^∞ such that $fg_k \to 1$ in norm in A_0^{-p} as $k \to +\infty$. Without loss of generality, we may assume that the norm of fg_k in A_0^{-p} is bounded by 2, so that by the definition of the set $E(F, U)$,

$$|g_k(z)| \le 2\,U(|z|)^p, \qquad z \in E(F, U).$$

By property (d) of Theorem 2.3,

$$|g_k(z)| \le 2\,V(|z|)^p, \qquad z \in \mathbb{D},$$

for all k. However, by property (c), there is a point $w \in \mathbb{D}$ such that $|f(w)|\,V(|w|)^p < 1/2$, which contradicts the requirement that $f(w)g_k(w) \to 1$ as $k \to +\infty$. The proof is complete. ∎

Proof of Theorem 1.4. Put $f = F^{1/p}$, where F is as in Theorem 2.4, with $\alpha = pq$. By property (a) of Theorem 2.4, $f \in B^p$; by (b), $1/f \in A^{-q}$; and by Theorem C, the function f is cyclic in every B^r, $0 < r < p$. Suppose $\{g_k\}_k$ is a sequence of functions in H^∞ such that $fg_k \to 1$ in norm in B^p as $k \to +\infty$. We may assume, without loss of generality, that

$$\int_{\mathbb{D}} |f(z)|^p |g_k(z)|^p \, dS(z) \le 2, \qquad k = 1, 2, 3, \dots .$$

By (d) of Theorem 2.4,

$$\int_{-\pi}^{\pi} \log^+ \left(|g_k(re^{i\theta})|^p/2\right) \, d\theta \le V(r), \qquad 0 \le r < 1.$$

However, by (c) there exists an s, $0 < s < 1$, such that

$$\int_{-\pi}^{\pi} \log^- (|f(se^{i\theta})|^p) \, d\theta \ge V(s) + 10.$$

The uniform convergence of fg_k to 1 on $s\mathbb{T}$ requires that

$$\log^+ \left(|g_k(z)|^p/2\right) - \log^-(2|f(z)|^p) \to 0 \quad \text{as } k \to +\infty,$$

uniformly in $z \in s\mathbb{T}$, which is incompatible with the above two estimates. The proof is complete. ∎

4 Remarks

Theorems 1.1–1.4 suggest the following question: Is there a critical rate of decay of non-cyclic functions lacking zeros in the spaces considered here? Compare with [19, Question 6].

It is interesting to consider the example given in Theorem 1.4 in connection with the questions posed by B. I. Korenblum in [13, Questions 1 and 2]. He introduced the notion of an outer function in B^p in terms of domination, and proved that a cyclic function necessarily is outer. Recent work by A. Aleman, S. Richter, and C. Sundberg [2] has shown that the outer functions are precisely the cyclic functions. Thus Theorem 1.4 answers Question 2 in [13] in the negative.

One more application concerns the factorization problem for the Bergman spaces. Fix the parameter p, $0 < p < +\infty$. Let $\mathcal{M}$ be a proper closed subspace of B^p, invariant under multiplication by z, and let m be the order of the common zero at the origin of the functions in $\mathcal{M}$. We consider the extremal problem

$$\sup\left\{\operatorname{Re} f^{(m)}(0) : \ f \in \mathcal{M}, \ \|f\|_{B^p} = 1\right\}. \tag{4.1}$$

Suppose $\mathcal{M}$ is singly generated, that is, $\mathcal{M} = [F] = \operatorname{clos}_{B^p}\{qF : \ q \in \mathcal{P}\}$, the invariant subspace generated by F, for some $F \in B^p$. Then the above problem has a unique solution, which we denote by $G_{\mathcal{M}}$ and call the extremal function (or canonical divisor) for $\mathcal{M}$. It is proved in [2] (see also [5], [6], [8], [9]) that $\mathcal{M} = [G_{\mathcal{M}}]$, and that $G_{\mathcal{M}}$ is a contractive divisor on $\mathcal{M}$:

$$\|f/G_{\mathcal{M}}\|_{B^p} \le \|f\|_{B^p}, \qquad f \in \mathcal{M}. \tag{4.2}$$

A natural question arises: Is it true that $G_{\mathcal{M}}$ is an expansive multiplier,

$$\|f\|_{B^p} \le \|fG_{\mathcal{M}}\|_{B^p}, \qquad f \in B^p? \tag{4.3}$$

A closely related question is the following one. Is it true that

$$G_{\mathcal{M}}B^p \cap B^p = [G_{\mathcal{M}}] = \mathcal{M}?$$

To answer these two questions in the negative, we use as F the function f mentioned in Theorem 1.4, for some positive value of the parameter q, say $q = 1$.

Let $\mathcal{M} = [F](\neq B^p)$, and write $G = G_{\mathcal{M}}$. By (4.2), G has no zeros in $\mathbb{D}$, and $F/G \in B^p$. Therefore, for some $r, C > 0$,

$$|G(z)| > C(1 - |z|)^r, \qquad z \in \mathbb{D}. \tag{4.4}$$

As a result, we have $\varphi = G^{-\varepsilon} \in B^p$ for some ε, $0 < \varepsilon < 1$. Now, $\varphi G = G^{1-\varepsilon} \in B^q$ for $q = p/(1-\varepsilon) > p$, so that φG is cyclic in B^p according to Theorem C. It follows that φG cannot be in $\mathcal{M}$, although $\varphi G \in GB^p \cap B^p$. This answers the second question in the negative.

To deal with the first question, we argue by contradiction, supposing that (4.3) holds. It follows that

$$\|G^{-t}\|_{B^p} \le \|G^{1-t}\|_{B^p}, \qquad 0 \le t < \varepsilon,$$

that is,

$$\psi(t) = \int_{\mathbb{D}} \left(|G(z)|^p - 1\right) |G(z)|^{-t} dS(z) \ge 0, \qquad 0 \le t < \varepsilon p.$$

Since G solves (4.1), we have $\psi(0) = 0$. On the other hand, the integrand

$$\big(|G(z)|^p - 1\big)\,|G(z)|^{-t}$$

decreases strictly in the variable t ($t \in [0, +\infty[$) for those values of z where $|G(z)| \neq 1$, and vanishes identically for $|G(z)| = 1$. It follows that since $|G(z)|$ is not constant, the function $\psi(t)$ must decrease strictly on the interval $[0, \varepsilon p[$. This leads to a contradiction, which shows that (4.3) fails.

An interesting question, which we cannot answer, is the following. Given an arbitrary $r > 0$, is there an extremal function G, other than the constant one, such that (4.4) holds?

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