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FINITENESS OF LIMIT CYCLES AND UNIQUENESS THEOREMS FOR ASYMPTOTICALLY HOLOMORPHIC FUNCTIONS

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Abstract. We present a finiteness theorem for limit cycles of quasianalytically smooth vector fields on the plane. We consider vector fields with only non-degenerate singular points. Our result can be viewed as an extension of [13] and [20]. Our assumptions make it possible to prove certain quasianalytic properties of a monodromy transformation. These properties follow from estimates on $\overline{\partial}$-derivative of extensions of monodromy transformations to certain complex domains. The second part of the article is devoted to functions satisfying such estimates on the $\overline{\partial}$-derivative (asymptotically holomorphic functions). Sharp uniqueness theorems for such functions are proved which allow us to complete the proof of the finiteness theorem.

1. Introduction.

1.1. The problem of finiteness of limit cycles has attracted attention for quite a long time (since [18]). In its initial setting, which is due to H. Poincaré, it amounts to proving that any polynomial vector field on the plane:

$$\begin{cases} \dot{x} = \alpha(x,y) \\ \dot{y} = \beta(x,y), \qquad (x,y) \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

(α, β are real polynomials) has only finite number of *limit cycles*.

H. Dulac made a basic contribution in [6], applying an asymptotic expansion of a monodromy transformation of *a polycycle*. He proved that if infinitely many *limit cycles* occur in a neighborhood of *a polycycle*, then this asymptotic expansion contains only the first term, that is the identity. He wrongly concluded that this means that the monodromy transformation is equal to the identity.

Reviews of the growing knowledge in this area can be found in [14] and [15]. We only mention here that there were several other attempts to solve this problem (see [14] for more references), but even for polynomials of degree 2 these attempts were not successful until recently. Now the problem is solved by Yu. S. Il'yashenko and independently, using different methods, by two groups of French mathematicians: by J. Ecalle and by J. Martinet–R. Moussu–J.-P. Ramis [10,11].

Before describing our contribution let us emphasize that there are actually two problems disguised as one. The first problem is to prove that the number of *limit cycles* of (1.1) is

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finite. The second is to estimate how many *limit cycles* may occur for given degrees of polynomials. The second problem is extremely difficult and its full solution lies in the future. The first problem has at least one transparent advantage: it is local, in the sense that to solve this problem it is sufficient to prove that

$$\text{the number of limit cycles close to a polycycle of (1.1) is finite.} \tag{1.2}$$

Assertion (1.2) was proved by Il'yashenko and Ecalle and Martinet–Moussu–Ramis under the assumption that α and β are real analytic in a neighborhood of *the polycycle.* The requirement that α and β are polynomials is superfluous here. Their proofs seem to be different although they use certain quasianalytic properties of monodromy transformations of *polycycles.* These properties become more and more involved when the order of degeneracy of singular points on our *polycycle* grows.

However, the quasianalytic nature of the proofs gives rise to a natural question:

$$\begin{gathered}\text{Is the number of limit cycles close to a polycycle of (1.1) finite,}\\ \text{if } \alpha \text{ and } \beta \text{ are quasianalytically smooth functions?}\end{gathered} \tag{1.3}$$

Standard methods of ODE show that for smooth non-quasianalytic vector fields the answer to (1.3) is "No".

We formulate a

Conjecture. If α and β in the right hand side of (1.1) belong to a quasianalytic Carleman class, then the number of *limit cycles* close to any *polycycle* is finite.

In our present work we prove this conjecture under the assumption that all singular points on our *polycycle* are *non-degenerate.* A previous result of this type was published in [20]. There the second author managed to treat only "very" quasianalytic Carleman classes (see Remark 4 after Theorem 2.2 below).

The case of *non-degenerate* singular points can be considered as the first (relatively simple) step in proving this conjecture. This step is parallel to the work of Il'yashenko [13] where he proved that (1.2) holds for real analytic vector fields with *non-degenerate* singular points.

1.2. Notations. We shall use the following terminology. A characteristic is an integral curve defined by the system (1.1). A closed characteristic is called *a cycle.* A cycle is called *a limit cycle* if its neighborhood contains no other cycles.

A finite connected union of *singular points* (points where $\alpha = \beta = 0$) and characteristics connecting these points one after another is called *a polycycle.*

A monodromy transformation of a cycle is a classical object which is introduced as follows. For a given cycle C and a transversal Γ we define $m : \Gamma \to \Gamma$ as the transformation of the first return along a characteristic. This transformation is well defined as a germ of maps $(\mathbb{R}, 0) \to (\mathbb{R}, 0)$ if there are no singular points in a neighborhood of C.

A monodromy transformation of a polycycle is defined as for ordinary cycles, only the transversals in the definition are replaced by semitransversals, that is, semi-intervals that are transverse to the polycycle. It is convenient to regard monodromy transformations as germs of maps $(\mathbb{R}_+, 0) \to (\mathbb{R}_+, 0)$.

However, we have to warn the reader that in general not every polycycle admits a monodromy transformation.

Now we introduce a way to measure the smoothness of the right-hand part in (1.1). A C^∞-smooth function will be called *flat on a set* if it vanishes along with all its derivatives on this set. For a given sequence $\{M_n\}$ the Carleman class $C\{M_n\}$ is defined by

$$C\{M_n\} = \{f \in C^\infty(\mathbb{R}^2) : \text{for some}\, B_f, C_f > 0,\ |f^{(n)}(x)| \leqslant B_f C_f^{|n|} M_{|n|}\}.$$

Here $n = (n_1, n_2)$, $|n| = n_1 + n_2$. We consider only Carleman classes with *regular* sequences $\{M_n\}$: namely, if $m_n = M_n/n!$, then we assume

1. $m_n^2 \leqslant m_{n-1} m_{n+1}$;
2. $\sup_n (m_{n+1}/m_n)^{1/n} < \infty$.
3. $m_n^{1/n} \to \infty$, $n \to \infty$.

Under these assumptions properties of $C\{M_n\}$ can be adequately expressed by the "weight function"

$$\rho(x) = \inf_{n \geqslant 0} x^n \frac{M_n}{n!}.$$

In case $M_n = n!$,

$$\rho(x) = 0, \qquad 0 < x < 1,$$

and $C\{M_n\}$ contains only real analytic functions. Furthermore, $C\{M_n\}$ is quasianalytic ($C\{M_n\} \in QA$, i.e., $C\{M_n\}$ does not contain any non-trivial function *flat* at a point) if and only if

$$\int_0 \log \log \frac{1}{\rho(x)}\, dx = \infty. \tag{1.4}$$

To see this, one combines the following two remarks. Let us denote $m = \log(1/\rho)$ and let p be the Legendre transform of m,

$$p(t) = \inf_{0<\zeta<1} (m(\zeta) + \zeta t).$$

The first remark is that (see [16, p.337])

$$\int_1^\infty \frac{p(t)}{1+t^2}\, dt = \infty \Longleftrightarrow \int_0^1 \log m(\zeta)\, d\zeta = \infty. \tag{1.5}$$

If $T(x) = \sup_{n \geqslant 0} x^n/M_n$, one can easily compute that $\exp p(t) \geqslant T(t)$. The second remark is that since the sequence $\{\log m_n\}$ is convex, an opposite inequality holds for large t:

$$T(2t) \geqslant \exp p(t). \tag{1.6}$$

Combining (1.5) and (1.6) we see that (1.4) guarantees that

$$\int_1^\infty \frac{\log T(t)}{1+t^2}\, dt = \infty.$$

By the classic Denjoy–Carleman theorem this condition implies that $C\{M_n\}$ is a QA class.

Along with (1.4) which is a condition on smallness of ρ, the following *regularity* condition will be imposed on ρ to prove our main result:

$$\lim_{x\to 0}\frac{\log\log\dfrac{1}{\rho(x)}}{\log\dfrac{1}{x}}=\infty. \tag{R}$$

Let us discuss this condition. In terms of M_n it means that $M_n \leqslant C_\varepsilon (n!)^{1+\varepsilon}$ for each positive ε. In other words (R) says that we consider only classes $C\{M_n\}$ lying in the intersection of all Gevrey classes. All Gevrey classes are non-quasianalytic. Thus (R) is a natural condition. However, there are quasianalytic classes which do not lie inside the intersection of all Gevrey classes. Thus, (R) is actually restrictive. We will need (R) once in the proof. We tried to get rid of (R) but could only weaken it slightly.

Classes $C\{M_n\}$ are defined for functions of one variable as well, and one may wonder whether the smoothness of monodromy transformations can be expressed in terms of these classes. More precisely, the following question arises naturally: is it true that monodromy transformations for QA vector fields are QA functions? In other words: if $\alpha, \beta \in C\{M_n\}$, (1.4) holds for the associated ρ, and m is a monodromy transformation for (1.1), is it true that $m \in C\{\widetilde{M}_n\}$ with associated $\widetilde{\rho}$ satisfying (1.4)?

If m is a monodromy transformation for a cycle, the answer is "Yes" and we can choose $\widetilde{M}_n = M_n$. This is manifest because the implicit function theorem holds in Carleman classes (see [9]).

If m is a monodromy transformation for a polycycle, the answer seems to be "No" in general, but still is "Yes" if certain conditions are imposed on the polycycle. The proof of this positive part is the essence of our article. However, it is very indirect. Namely, instead of estimating the n-th derivatives of monodromy transformations we are going to use the technique of asymptotically holomorphic functions (AHF). Let us describe what they are. They will be defined in domains of the type

$$\Omega = \Omega_\psi = \bigl\{\zeta = \xi + i\eta : \xi > 0, |\eta| < \psi(\xi)\bigr\},$$

where ψ is smooth positive (usually monotonic) function on $\mathbb{R}_+$. The classes of AHF in Ω will be determined by two positive functions $A : \Omega \to [0,1]$, $\rho : [0,1] \to [0,1]$ in the following manner ($\zeta = \xi + i\eta$)

$$f \in AH_{A,\rho}(\Omega) \Longleftrightarrow |\bar\partial f(\zeta)| \leqslant \rho(A(\xi + i\eta)). \tag{1.7}$$

The function ρ is always monotone, $\rho(0) = 0$, and ρ satisfies (1.4). So (1.7) shows that f becomes more and more "holomorphic" along any path where $A(\xi + i\eta) \to 0$. A number of papers exist (see [2,3,4]) which are concerned with the estimates of functions in $AH_{A,\rho}(\{z : \operatorname{Re} z > 0\})$. Usually one chooses

$$A(\xi + i\eta) = \eta \text{ or } A(\xi + i\eta) = \frac{1}{\xi},$$

which means, respectively, that the functions f, $f \in AH_{A,\rho}$, are getting more and more holomorphic towards the real half-axis $\mathbb{R}_+$ or towards ∞ uniformly on vertical lines.

Here we deal with estimates and uniqueness theorems for functions in $AH_{A,\rho}$ with ρ satisfying (1.4) and with

$$A(\xi + i\eta) = e^{-\xi} \text{ or } A(\xi + i\eta) = \eta e^{-\xi}.$$

We need just these choices of asymptotic behavior of the $\overline{\partial}$-derivative because our main idea is to extend the germ of a monodromy transformation of a polycycle

$$m(e^{-\xi}) : (\mathbb{R}_+, \infty) \to (\mathbb{R}_+, \infty)$$

to a function from $AH_{A,\rho}(\Omega)$ with such A and ρ.

It is necessary to point out that the idea to extend monodromy transformations to a complex domain Ω is due to Il'yashenko [13]. In case of real analytic vector fields with non-degenerate singular points this extension is a holomorphic function, so its $\overline{\partial}$-derivative vanishes, and, by the way, the extension is unique.

In our case the $\overline{\partial}$-derivative does not vanish, but its smallness can be grasped from the smoothness of the right hand part of (1.1). The extension is certainly non-unique. That is why we have to choose one which is good enough.

We hope that in the future our technique will allow us to deal with degeneracies of singular points in both the real analytic and quasianalytic cases in a uniform way.

At last, let us survey the content of this article. Section 2 contains statements of the main results. In Section 3 the idea of the proof of our finiteness theorem is presented. Section 4 is devoted to a reduction to a local problem. In Section 5 we construct an asymptotically holomorphic extension of the monodromy transformation. Sections 6 and 7 contain uniqueness theorems for such functions. In Section 8, we combine the results of Sections 4–7 to complete the proof of our finiteness theorem.

A preliminary version of this work was published in [5].

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2. Statement of Results.

2.1. A supplementary result. In [6], Dulac was concerned only with real analytic fields, although he actually proved the following result:

Theorem A (Dulac). *Let C be a polycycle of a C^∞ vector field on $\mathbb{R}^2$ having finitely many singular points in a neighborhood of C. Assume that*

1. *all singular points on C are of finite multiplicity;*
2. *there is a sequence of limit cycles converging to C.*

Then

a) *C admits a monodromy transformation;*
b) *this transformation is the identity plus a flat germ.*

One can find Theorem A in this form in Il'yashenko's paper [14].

2.2. Main theorem.

First we formulate three basic assumptions.

A. From now on we assume (unless otherwise stated) that $\alpha, \beta \in C\{M_n\}$, M_n is a regular sequence (see 1.2),

$$\rho(x) = \inf_{n>0} x^n \frac{M_n}{n!},$$

ρ satisfies (R), and

$$\int_0^1 \log\log \frac{1}{\rho(x)}\, dx = \infty.$$

B. We assume that for every compact subset of $\mathbb{R}^2$ there is only finite number of points (x, y) belonging to it and such that

$$\alpha(x, y) = \beta(x, y) = 0$$

(These are called singular points.)

C. And our "for the sake of simplicity" assumption is that each singular point $z = x+iy$ is non-degenerate: that is, the matrix

$$\begin{pmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{pmatrix}$$

has two non-zero eigenvalues $\lambda_1(z), \lambda_2(z)$.

For singular points lying on polycycles this means that they are saddle points. In other words, for a polycycle P and a singular point z_0, $z_0 \in P$ we have

$$\lambda_1(z_0) \cdot \lambda_2(z_0) < 0.$$

Theorem 2.1. *Under assumptions* A,B,C *each polycycle has a neighborhood without limit cycles.*

In [14] the finiteness theorem for limit cycles was deduced from the conclusion of Theorem 2.1. The analyticity of the field was not used at all. Thus this part of [14] gives us our main result.

Theorem 2.2. *Under assumptions* A,B,C *given above the vector field* (1.1) *has only finite number of limit cycles in each finite part of the plane* $\mathbb{R}^2$.

Remarks. 1. If α and β are real analytic, that is $\alpha, \beta \in C\{n!\}$, we trivially have (1.4) since $\rho(x) \equiv 0$ for $x \in (0, 1)$.

2. We believe that assumption C is superfluous. We are working on the general version of Theorem 2.2. However, as in the real analytic case [15] the technique turns out to be very involved. Fortunately, the phase portraits near singular points are the same for real analytic and QA vector fields on the plane. This statement looks non-trivial because its proof is based upon the Lojasiewicz inequality for QA functions on $\mathbb{R}^2$ (see [5]) and an application of a result of Dumorthier [6].

3. It is worth noting that for QA vector fields B can be assumed without loss of generality. If two quasianalytic functions have an infinite set of common zeros, there is a common QA divisor.

4. Under the assumption

$$\rho(x) \leqslant \exp\Big(-\exp\frac{1}{x^{16}}\Big)$$

Theorem 2.1 was proved in [20].

2.3. Uniqueness theorems for a class of asymptotically holomorphic functions. To prove Theorem 2.1 we need two results on AHF.

Let $\Omega_\varepsilon = \{\xi + i\eta : \xi > 0, |\eta| < e^{\varepsilon\xi}\}$, where ε is a positive number. Let ρ be an increasing function on $\mathbb{R}_+$, $\rho(0) = 0$.

The first theorem is an analog of classical Phragmen–Lindelöf theorem for the class $AH_{\exp(-\varepsilon\xi),\rho}(\Omega_\varepsilon) \cap L^\infty(\Omega_\varepsilon)$.

Theorem 2.3. *Let ρ satisfy the condition*

$$\text{the function } \log\frac{1}{\rho(e^{-x})} \text{ is convex for large } x.$$

Given a bounded C^∞-smooth function f on Ω_ε such that

$$|\bar\partial f(\zeta)| \leqslant \rho(e^{-\varepsilon\xi}),$$

and

$$|f(\xi)| = O(e^{-\xi n}), \qquad \xi \to \infty, \text{ for every } n,$$

we have for large ξ

$$|f(\xi)| \leqslant \rho(A\, e^{-\varepsilon\xi}), \tag{2.1}$$

where A is an absolute constant.

The second theorem is our main result for AHF.

Let $\Omega_1 = \{\xi + i\eta : \xi > 0, |\eta| < 1\}$. We deal with the class $AH_{\eta\exp(-\varepsilon\xi),\rho}(\Omega_1) \cap L^\infty(\Omega_1)$.

Theorem 2.4. *Let f be a bounded C^∞-smooth function on Ω_1 such that*

$$|\bar\partial f(\zeta)| \leqslant \rho(|\eta| e^{-\varepsilon\xi}),$$

where

$$\int_0^1 \log\log\frac{1}{\rho(x)}\, dx = \infty,$$

and suppose that (2.1) *holds. Then*

$$f(\xi) = 0, \qquad \xi > 0.$$

Theorems 2.3 and 2.4 are partial cases of Corollary 6.2 and Theorem 7.1 respectively, which are proved in Sections 6 and 7 below.

3. The idea of the proof of Theorem 2.1.

Let C be a polycycle all of whose singular points are saddle points and which admits a sequence of limit cycles $\{C_n\}$ converging to C. Let Γ be a semitransversal to C with a local coordinate x, $0 < x < \delta$. Theorem A shows that the corresponding monodromy transformation m exists and satisfies the relation

$$m(x) - x = r(x), \qquad r(x) = o(x^n), \qquad x \to \infty, \text{ for every } n.$$

The next step is to extend $\xi \longmapsto r(e^{-\xi})$ to a function from $A_{\eta \exp(-c\xi),\rho}(\Omega_\varepsilon)$. This is done in Section 4 and 5, however one simple lemma concerning such an extension is formulated at the end of this section. The application of Theorems 2.3 and 2.4 shows that $r(e^{-\xi}) \equiv 0$ or $m(x) \equiv x$. This means that cycles C_n are not limit cycles and this contradiction proves Theorem 2.1.

Here is a simple lemma on how to obtain an AHF from a family of locally defined holomorphic functions. Similar arguments were employed in [9].

Let Ω be a planar domain and a be a positive Lipschitz function on Ω ($K \geqslant 1$):

$$|a(x,y) - a(\bar{x},\bar{y})| \leqslant K(|x-\bar{x}| + |y-\bar{y}|), \qquad (x,y),(\bar{x},\bar{y}) \in \Omega;$$
$$a|\partial\Omega = 0.$$

Let us decompose Ω into the union of squares with sides parallel to the coordinate axes $\mathcal{F} = \{Q\}$ with the following properties

$$Q', Q'' \in \mathcal{F} \Longrightarrow \operatorname{int} Q' \cap \operatorname{int} Q'' = \emptyset;$$
$$\frac{1}{1000K} a(C(Q)) \leqslant \operatorname{diam} Q \leqslant \frac{1}{50K} a(C(Q)), \qquad Q \in \mathcal{F}. \tag{3.1}$$

Here for a square Q we denote by $C(Q)$ its center, and by $2Q$ the square (with sides parallel to the coordinate axes) centered at $C(Q)$ and such that $\operatorname{diam} 2Q = 2 \operatorname{diam} Q$.

Claim 3.1. *Let $Q \in \mathcal{F}$. Then $2Q \subset \Omega$. If $\xi', \xi'' \in 2Q$, then*

$$|a(\xi')/a(\xi'')| \leqslant \frac{3}{2}. \tag{3.2}$$

Proof. $|a(\xi') - a(C(Q))| \leqslant 2K \operatorname{diam} Q \leqslant \frac{1}{25} a(C(Q))$. The same is true for ξ'' and the claim follows. •

Let us call such a decomposition an a-decomposition. Suppose that ρ is a positive function such that for every n, $\rho(x) = 0(x^n)$, $x \to 0$. Assume now that along with an a-decomposition $\mathcal{F}$ we are given a family of holomorphic functions $\{h_Q\}_{Q \in \mathcal{F}}$ such that

$$h_Q \in \operatorname{Hol}(2Q); \tag{3.3}$$
$$|h_{Q'}(z) - h_{Q''}(z)| \leqslant C_1 \rho(a(z)) \text{ if } 2Q' \cap 2Q'' \neq \emptyset. \tag{3.4}$$

Lemma 3.2. *Given an a-decomposition $\mathcal{F}$ and a family $\{h_Q\}_{Q\in\mathcal{F}}$ satisfying* (3.3), (3.4), *there exists a function h such that*

$$1)\qquad |h(z)-h_Q(z)|\leqslant C_1\rho(a(z)),\qquad z\in Q,\ Q\in\mathcal{F};\tag{3.5}$$

$$2)\qquad |\overline{\partial}h(z)|\leqslant C_2(\rho(a(z)))^{1/2}.\tag{3.6}$$

Proof. Claim 3.1 and property (3.1) show that

$$\frac{1}{k}\leqslant\frac{\operatorname{diam}Q_1}{\operatorname{diam}Q_2}\leqslant k$$

for each pair of "neighbors" Q_1,Q_2 from $\mathcal{F}$, $2Q_1\cap 2Q_2\neq\emptyset$, and for some k. This allows us to construct a partition of unity of finite multiplicity based on $\{2Q\}_{Q\in\mathcal{F}}$. Let $\{\varphi_Q\}_{Q\in\mathcal{F}}$ be such a partition of unity having the following properties:

1) $\operatorname{supp}\varphi_Q\subset 2Q$;
2) $0\leqslant\varphi_Q\leqslant 1,\ \varphi_Q\in C^\infty$;
3) $|\nabla\varphi_Q|\leqslant C(k)(\operatorname{diam}Q)^{-1}$;
4) $\sup_{z\in\Omega}\operatorname{card}\{Q\in\mathcal{F}:\varphi_Q(z)\neq 0\}<\infty$.

Put $h(z)=\sum_{Q\in\mathcal{F}}h_Q(z)\varphi_Q(z)$. Clearly (3.4) implies (3.5). Combining (3.1)–(3.4) we can easily see

$$|\overline{\partial}h(z)|\leqslant C_2\frac{\rho(a(z))}{a(z)}.$$

This implies (3.6) because $\rho(x)=O(x^2)$. •

4. Localization.

In this section we repeat the reasoning from Section 2.6 of [13]. All the (semi-) transversals are assumed to be real analytic. Let C be a polycycle (admitting a monodromy transformation) such that all its vertices are saddles. Let Γ be a semitransversal to C and let

$$\delta_{C,\Gamma}:\Gamma\to\Gamma$$

denote a monodromy transformation.

Figure 1 shows how $\delta_{C,\Gamma}$ can be decomposed into the superposition of correspondence maps (for each saddle) and the mappings which can be extended to QA mappings of the whole transversals.

More specifically, let us enumerate the saddle points on C in a natural order. Let Γ_j,Γ_j' be two semitransversals such that characteristics arrive to the j-th sector through Γ_j and leave j-th sector through Γ_j'. The mapping

$$\delta_j:\Gamma_j\to\Gamma_j'$$

is called the correspondence map for this singular point.

Remark. In our Figure 1 we can choose Γ_j,Γ_j' as close to j-th singular point as we wish.

Let $f_j:\Gamma_j'\to\Gamma_{j+1}$ be the mapping along the characteristics. Denote by $\hat\Gamma_j,\hat\Gamma_j'$ the transversals containing semitransversals Γ_j,Γ_j'. Clearly the f_j are very good mappings, i.e. being considered in local coordinates on $\hat\Gamma_j,\hat\Gamma_j'$ they become of the same class of smoothness as the field.

FIGURE 1

Proposition 4.1. *If $\alpha, \beta \in C\{M_n\}$ and $\{M_n\}$ is a* regular *sequence, then $f_j \in C\{M_n\}$.*

The proof can be found in Dynkin [9].

Let $\Gamma = \Gamma_1$. We can write

$$\delta_{C,\Gamma} = f_n \circ \delta_n \circ \ldots \circ f_1 \circ \delta_1. \tag{4.1}$$

Let us fix local coordinates on Γ_j, Γ'_j with the domain of definition $(0, \varepsilon]$. We shall use the change of variables $e(\xi) = e^{-\xi}$, $\xi \geqslant \log(1/\varepsilon)$. Put

$$\Delta_i = e^{-1} \circ \delta_i \circ e, \ \ F_i = e^{-1} \circ f_i \circ e.$$

Then (4.1) can be rewritten in the form

$$\Delta_{C,\Gamma} \overset{\text{def}}{=} e^{-1} \circ \delta_{C,\Gamma} \circ e = F_n \circ \Delta_n \circ \ldots \circ F_1 \circ \Delta_1.$$

As mentioned, we are going to extend the Δ_i and F_i to the complex domain from the ray $[\log \frac{1}{\varepsilon}, \infty)$. Proposition 4.1 will take care of F_i: see Section 8 where we shall just apply Dynkin's extension [9] to get the estimates of $\overline{\partial} F_j$ we need.

It is more difficult to handle the Δ_i because generally speaking they can become less and less smooth as $\xi \to \infty$ since we are approaching the singular point of Figure 1. The analysis of Δ_i is a local problem around the i-th singular point. To analyze Δ_i it is useful to change variables in a neighborhood of the singular point to reduce our system (1.1) to a special form.

It turns out that the change of variables we need is of the same $C\{M_n\}$ class. The next proposition (and its proof) follow immediately from the reasoning in [6, pp.68–77].

Proposition 4.2. *Let* (1.1) *be a field of a regular* $C\{M_n\}$ *class with a saddle at the origin. Let* $\lambda = |\lambda_1(0)/\lambda_2(0)|$, *where* $\lambda_1(0), \lambda_2(0)$ *are eigenvalues of the linear part of the mapping* $(\alpha, \beta) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. *Then there exists a change of variables* $(x, y) \to (\bar{x}, \bar{y})$ *of the class* $C\{M_n\}$ *and a linear change of time which reduce* (1.1) *to the system*

$$\begin{cases} \dot{\bar{x}} = \bar{x} \\ \dot{\bar{y}} = -\bar{y}(\lambda + \bar{x}^k \bar{y} f(\bar{x}, \bar{y})), \end{cases} \tag{4.2}$$

where $f \in C\{M_n\}$ *and*

$$k \geqslant \lambda.$$

The change of coordinates can be absorbed by F_j (since they are in the same class $C\{M_n\}$ and $C\{M_n\}$ is invariant under superposition [9]). Therefore, according to Proposition 4.2 we can consider only correspondence maps given by systems (4.2).

5. An asymptotically holomorphic extension of the correspondence map.

Let

$$\begin{cases} \dot{x} = x \\ \dot{u} = -u(\lambda + x^k u f(x, u)) \end{cases} \tag{5.1}$$

be a system with $k \geqslant \lambda$ and $f \in C\{M_n\}$. We shall consider the case $\lambda > 1$. The opposite case is completely similar. Rescaling, we can assume that

$$\max_{|x| \leqslant 1, |u| \leqslant 1} |f(x, u)| \leqslant \frac{1}{4}.$$

The extension will be given in a series of steps.

5.1. According to [9] one can extend the function $f(x, u) \in C\{M_n\}$ to a C^∞ function $f(z, w)$ on $\mathbb{C}^2$ such that

$$|D\overline{\partial}_z f(z, w)| + |D\overline{\partial}_w f(z, w)| \leqslant \rho(C_f \max(|\operatorname{Im} z|, |\operatorname{Im} w|)).$$

Here D denotes any derivative up to the order 1.

Reciprocally $f(x,u)$ can be restored from $\overline{\partial}_z f(z,w)$, $\overline{\partial}_w f(z,w)$ by the Martinelli–Bochner formula (see e.g. [17,19]).

For a number a, $a \in (0,1)$, let us fix the domain

$$G^a = \{(z,w) : |z| < 2, |w| < 2, |\operatorname{Im} z| < a, |\operatorname{Im} w| < a\}$$

which is a neighborhood of the square $S = \{(x,u) \in \mathbb{R}^2 : |x| \leqslant 1, |u| \leqslant 1\}$.

The Martinelli–Bochner formula immediately gives us a function f^a holomorphic in G^a and such that

$$|f - f^a| \leqslant B_f \rho(C_f |a|) \text{ in } G^a. \tag{5.2}$$

The solution of (5.1) with initial data

$$\begin{cases} x(0) = e^{-\xi} \\ u(0) = 1 \end{cases} \tag{5.3}$$

is given by the solution of $(u(0) = 1)$)

$$\dot{u}(t) = -u(t)(\lambda + x(t)^k u(t) f(x(t), u(t))),$$

where $x(t) = e^{t-\xi}$. We stop to follow t as soon as $x(t) = 1$ (at $t = \xi$) and then $\Delta(\xi) = -\log u(\xi)$ is the correspondence map.

To extend Δ to a complex domain we fix a point $\zeta_0 = \xi_0 + i\eta_0$ and a square $Q(\zeta_0)$ centered at ζ_0 with

$$\operatorname{diam} Q(\zeta_0) = \frac{1}{100} \max\Big(\eta_0 e^{-\xi_0/2}, e^{-\lambda\xi_0/2}\Big).$$

After that we consider a new system, which approximates (5.1) on the square S but which has the advantage that it has a holomorphic right part. It will be convenient to consider the new system with complex time,

$$\begin{cases} \dot{z} = z \\ \dot{w} = -w(\lambda + z^k w f^a(z,w)), \qquad (z,w) \in G^a. \end{cases} \tag{5.4}$$

The number a will be fixed later according to the location of $Q(\xi_0)$.

5.2. We are going "to solve" (5.4) with initial data

$$\begin{cases} z(0) = e^{-\nu}, \qquad \nu \in Q(\zeta_0), \\ w(0) = 1. \end{cases} \tag{5.5}$$

In other words, we are going to find a path $\gamma = \gamma_{\nu,0}$ connecting ν with 0 such that

$$(z(\zeta), w(\zeta)) \in G^a, \qquad \zeta \in \gamma, \tag{5.6}$$

$$(z(\zeta), w(\zeta))_{\zeta\in\gamma} \text{ satisfies (5.4).} \tag{5.7}$$

In the last statement we mean that the derivatives are taken with respect to ζ.

Taking for a moment, the existence of such a path for granted we consider the moment when $z = 1$ ($\zeta = \nu$, that is the end of our path), and then $\Delta^{\zeta_0}(\nu) \overset{\text{def}}{=} -\log w(\nu)$, $\nu \in Q(\zeta_0)$, will serve as a building block for our extension $\Delta(\nu)$ of $\Delta(\xi)$.

Remark 5.1. Actually the extension Δ will be constructed from the functions $\{-\log w(\nu)\}_{\nu \in Q(\zeta)}$ with the help of Lemma 3.2.

5.3 The choice of a. Clearly, the bigger a is, the easier it is to ensure (5.6). At the same time, the smaller a is, the better will be the estimate of

$$\Delta^{\zeta_1}(\nu) - \Delta^{\zeta_2}(\nu), \qquad \nu \in Q(\zeta_1) \cap Q(\zeta_2),$$

and so the better is the estimate of $\overline{\partial}\Delta(\zeta)$. So, we need to choose a as small as we can and also to ensure (5.6), (5.7).

The main difficulty is to ensure

$$|\operatorname{Im} z(\zeta)| \leqslant a, \qquad |\operatorname{Im} w(\zeta)| \leqslant a. \tag{5.8}$$

The estimate $|z(\zeta)| \leqslant 1$ is guaranteed by the requirement that

$$\operatorname{Re} \zeta \leqslant \operatorname{Re} \nu, \qquad \zeta \in \gamma_{\nu,0}. \tag{5.9}$$

The estimate $|w(\zeta)| \leqslant 1$ is ensured by the following *a-priori* estimate (see [13])

$$\left(\frac{1}{2}|w|^2\right)^{\cdot} = -|w|^2 \operatorname{Re}\left(\lambda + z^k w f^a(z,w)\right) < 0 \tag{5.10}$$

and the estimates $|z(\zeta)| \leqslant 1$, $|f^a| \leqslant \lambda/4$, $|w(0)| = 1$.

Let us consider the domains

$$\Omega^a = \{\zeta : |\eta| < a e^{\xi}\},$$
$$\mathcal{O}^a = \{\zeta : |\eta| < \frac{1}{2} a e^{\xi/2}\}.$$

If a is chosen in such a way that $Q(\zeta_0) \subset \Omega^a$ then *any* path $\gamma_{\nu,0}$ from $\nu \in Q(\zeta_0)$ to 0 lying in Ω^a will guarantee the first requirement from (5.8).

The second requirement is much more subtle. To justify the choice of the path $\gamma_{\nu,0}$ below let us remark that presumably $u = \lambda \log z + \log w$ is "almost the first integral of (5.4)" because $z^k w f^a(z,w)$ is supposed to be small. As we will see, in fact, the function

$$u(\zeta) \stackrel{\text{def}}{=} \lambda \log z(\zeta) + \log w(\zeta) \tag{5.11}$$

does not vary much along $\gamma_{\nu,0}$. In other words,

$$\left|\lambda \log \frac{z(0)}{z(\zeta)} - \log \frac{w(\zeta)}{w(0)}\right| \text{ is small}, \qquad \zeta \in \gamma_{\nu,0},$$

which means that along the path $\gamma_{\nu,0}$ the value of $-\log w(\zeta)$ is almost $\lambda(\zeta - \nu)$.

For any path $\gamma = \gamma_{\nu,0}$ let γ^i denote the inverse path $\{\lambda(\zeta - \nu)\}_{\zeta \in \gamma_{\nu,0}}$.

The conclusion from this consideration is that to ensure the second requirement of (5.8) we have to choose $\gamma = \gamma_{\nu,0}$ in such a way that it connects ν and 0 and

$$\gamma \subset \Omega^a, \qquad \gamma^i \subset \Omega^a. \tag{5.12}$$

The choice is not unique, but now we are going to choose a path which clearly will be practically optimal. Given ζ_0 let us choose the least $a = a(\zeta_0)$ such that

$$Q(\zeta_0) \subset \operatorname{clos} \mathcal{O}^a. \tag{5.13}$$

This is our choice of a.

Respectively $\gamma = \gamma_{\nu,0}$, $\nu \in Q(\zeta_0)$, $\nu = \mu + i\delta$, will be fixed in the following way: $\gamma = \gamma^I \cup \gamma^{II} \cup \gamma^{III}$, where

$$\begin{aligned} \gamma^I &= \{\mu + i\delta - t : 0 \leqslant t \leqslant \frac{\mu}{2}\}, \\ \gamma^{II} &= \{\frac{\mu}{2} + i\delta - it : 0 \leqslant t \leqslant \eta\}, \\ \gamma^{III} &= \{\frac{\mu}{2} - t : 0 \leqslant t \leqslant \frac{\mu}{2}\}. \end{aligned}$$

This path is pictured on Figure 2.

FIGURE 2

Clearly $\gamma \subset \Omega^a$ if $Q(\zeta_0) \subset \mathcal{O}^a$. Also, γ^i is the image of γ under the map $\zeta \longmapsto \lambda(\nu - \zeta)$ which is the superposition of the symmetry with center $\nu/2$ and the λ-dilation. Our picture shows that if $\gamma \subset \Omega^a$, $\operatorname{Re}\nu \geqslant 1$, then $\gamma^i \subset \Omega^a$. Thus, (5.13) $\Rightarrow$ (5.12). Note that (5.9) holds as well.

Remark 5.2. Clearly the line segment connecting ν and 0 is symmetric with respect to $\nu/2$. But it is not the path we need, as it does not lie in Ω^a.

5.4 Keeping track of the location of $-\log w(\zeta)$**.**

In this section, letters A with indices mean absolute constants. So, $-\log z(\zeta)$ goes along $\gamma = \gamma^I \cup \gamma^{II} \cup \gamma^{III}$. As to $-\log w(\zeta)$ we know only that it has a positive real part for $\zeta \in \gamma$ (see (5.10)). Now we are going to estimate the oscillation of $u(\zeta)$ from (5.11). This will show how far $-\log w(\zeta)$ is tilting from γ^i.

Using notation (5.11) and equations (5.4) we can write that for $\zeta \in \gamma$

$$|\dot{u}| = \left|\lambda \frac{\dot{z}}{z} + \frac{\dot{w}}{w}\right| = |\lambda - (\lambda + z^k w f^a(z, w))| \leqslant |z^k w| \leqslant |z^\lambda w| \leqslant |e^u|. \tag{5.14}$$

If $v = |u - u(0)|$, we deduce from (5.14) that

$$|\dot{v}| \leqslant e^{\operatorname{Re}(u - u(0)) + \operatorname{Re} u(0)} \leqslant e^{-\lambda\mu} e^v.$$

In other words, $|(e^{-v})^{\cdot}| \leqslant e^{-\lambda\mu}$, $v(0) = 0$. And

$$|1 - e^{-v(\zeta)}| \leqslant (|\mu| + \delta)e^{-\lambda\mu} \leqslant |\mu|e^{-\lambda\mu} + \frac{a}{2}e^{-\lambda\mu/2} \leqslant A_0 e^{-\lambda\mu/2}, \qquad \zeta \in \gamma_{\nu,0}.$$

Thus, for $\zeta \in \gamma_{\nu,0}$, $\nu = \mu + i\delta$,

$$\left|\lambda \log \frac{z(0)}{z(\zeta)} - \log \frac{w(\zeta)}{w(0)}\right| \leqslant v(\zeta) \leqslant A_1 e^{-\lambda\mu/2}. \tag{5.15}$$

Assertion (5.15) means that

$$\nu \in Q(\zeta_0) \Rightarrow \text{the set } \{-\log w(\zeta)\}_{\zeta \in \gamma_{\nu,0}} \text{ lies in the } A_2 e^{-\lambda \operatorname{Re}\zeta_0/2}\text{-neighborhood of } \gamma^i. \tag{5.16}$$

5.5 Asymptotically holomorphic approximation.

Theorem 5.1. *There exists a function* $\widetilde{\Delta}(\zeta)$, C^∞*-smooth in the domain* $\mathcal{O} \overset{\text{def}}{=} \{\zeta = \xi + i\eta : |\eta| < \frac{1}{2}e^{\xi/4}, \xi > 0\}$ *and such that*

$$|\bar{\partial}\widetilde{\Delta}(\zeta)| \leqslant B_f A_5 \sqrt[4]{\rho(C_f A_4 \max(A_6 e^{-\lambda \operatorname{Re}\zeta/2}, A_7 |\operatorname{Im}\zeta| e^{-\operatorname{Re}\zeta/2}))}, \tag{5.17}$$

and $\widetilde{\Delta}$ *is close to the correspondence map* $\Delta(\xi) = -\log w(\xi)$ *for* (5.1) *in the sense that*

$$|\Delta(\xi) - \widetilde{\Delta}(\xi)| \leqslant B_f A_5 \sqrt[4]{\rho(C_f A_4 A_6 e^{-\lambda\xi/2})}.$$

Proof. Let us fix $Q(\zeta_0)$ and recall that $(z(\zeta), w(\zeta))_{\zeta\in\gamma_{\nu,0}} = (z^{\zeta_0}(\zeta), w^{\zeta_0}(\zeta))_{\zeta\in\lambda_{\nu,0}}$ solves (5.4) and $\Delta^{\zeta_0}(\nu)$ denotes $-\log w(\nu)$ for $\nu \in Q(\zeta_0)$. We fix the number a as

$$a = a(\zeta_0) = \max(A_2 e^{-\lambda \operatorname{Re}\zeta_0/2}, A_3|\operatorname{Im}\zeta_0|e^{-\operatorname{Re}\zeta_0/2}). \tag{5.18}$$

The choice of A_2 and A_3 guarantees that

$$Q(\zeta_0) \subset \mathcal{O}^a.$$

Also (5.18) guarantees that (5.6), (5.7), (5.8) hold. In fact, the most difficult second part of (5.8) follows from (5.12), (5.16), (5.18) because (5.12) and (5.16) imply

$$\{-\log w(\zeta)\}_{\zeta\in\gamma_{\nu,0}} \subset \Omega^a,$$

if a complies with (5.18). And the last inclusion trivially gives $|\operatorname{Im} w(\zeta)| < a$.

Now, (5.6)–(5.8) guarantee that $(z(\zeta), w(\zeta))$ stays in the domain of holomorphicity of the right hand part of (5.4), and so it depends holomorphically on the initial data (5.6).

Finally, we get

$$\Delta^{\zeta_0}(\nu) \in \operatorname{Hol}(Q(\zeta_0)). \tag{5.19}$$

Clearly we are going to paste all $\{\Delta^{\zeta_0}\}_{\zeta_0\in\mathcal{O}}$ to one C^∞ function as Lemma 3.2 indicates. For this purpose we have to estimate $\Delta^{\zeta_0}(\nu) - \Delta^{\zeta_1}(\nu)$ for $\nu \in Q(\zeta_0) \cap Q(\zeta_1)$.

Let $\hat a = \max(a(\zeta_0), a(\zeta_1))$. Clearly it follows from (5.2) that

$$|f^{a(\zeta_i)} - f^{\hat a}| \leqslant B_f \rho(C_f \hat a). \tag{5.20}$$

Let $(\hat z(\zeta), \hat w(\zeta))_{\zeta\in\gamma_{\nu,0}}$ be the solution of (5.4) with f^a replaced by $f^{\hat a}$ and with initial data (5.5). Let $\hat\Delta(\nu) = -\log \hat w(\nu)$. Using (5.4), (5.20) we get

$$\big|(\log w^{\zeta_0})^{\boldsymbol\cdot} - (\log \hat w)^{\boldsymbol\cdot}\big| \leqslant B'_f \rho(C_f \hat a) + K'_f |w^{\zeta_0} - \hat w|,$$

and $w^{\zeta_0}(0) = \hat w(0)$. Thus, using Gronwall's lemma (taking into consideration that the length of $\gamma_{\nu,0}$ does not exceed $2|\nu|$), we get

$$|\Delta^{\zeta_0}(\nu) - \hat\Delta(\nu)| \leqslant B_f \rho(C_f \hat a) e^{L_f|\nu|} \leqslant B_f \rho(C_f\cdot \hat a)\cdot \exp(L_f \cdot \hat a^{-2}). \tag{5.21}$$

The last inequality follows from rough estimates

$$\nu \in Q(\zeta_i) \subset \mathcal{O}^{a(\zeta_i)},\ i = 0, 1 \Rightarrow |\nu| \leqslant A_5 + A_6 e^{\operatorname{Re}\zeta_i/2},$$
$$\zeta \in \mathcal{O} \Rightarrow \hat a \leqslant A_3 \max(e^{-\operatorname{Re}\zeta_0/4}, e^{-\operatorname{Re}\zeta_1/4}).$$

An analog of (5.21) holds with ζ_0 replaced by ζ_1 and therefore,

$$\nu \in Q(\zeta_0) \cap Q(\zeta_1) \Rightarrow |\Delta^{\zeta_0}(\nu) - \Delta^{\zeta_1}(\nu)| \leqslant B''_f \sqrt{\rho(C_f \hat a)}$$

if $\zeta_0, \zeta_1 \in \mathcal{O}$. (Here we use (R) for the only time in the proof). Furthermore, for $\nu \in Q(\zeta_0) \cap Q(\zeta_1)$ we have

$$\hat a \leqslant A_4 \max(A_2 e^{-\lambda\operatorname{Re}\nu/2}, A_3|\operatorname{Im}\nu|e^{-\operatorname{Re}\nu/2}).$$

And finally, we get

$$\zeta_0, \zeta_1 \in \mathcal{O}, \quad \nu \in Q(\zeta_0) \cap Q(\zeta_1) \Rightarrow$$
$$|\Delta^{\zeta_0}(\nu) - \Delta^{\zeta_1}(\nu)| \leqslant B''_f \sqrt{\rho(C_f A_4 \max(A_2 e^{-\lambda\operatorname{Re}\nu/2}, A_3|\operatorname{Im}\nu|e^{-\operatorname{Re}\nu/2}))}.$$

Now (5.19) allows us to use Lemma 3.2, and an application of this lemma proves (5.17). The last statement of the theorem follows easily when we estimate $|\Delta^\xi(\xi) - \Delta(\xi)|$, $\xi \in \mathbb{R}_+$. •

5.6 An improvement of the $\overline{\partial}$-estimate.

We would like to show that one can replace the function $a(\zeta_0)$ in (5.18) by

$$b = b(\zeta_0) = A_9 |\operatorname{Im}\zeta_0| e^{-\operatorname{Re}\zeta_0/2}, \tag{5.22}$$

and still be able to repeat all the reasoning of Theorem 5.1. Now we consider $Q(\zeta_0)$ centered at ζ_0 with diameter $(1/100)\eta_0 e^{-\xi_0/2}$. The choice (5.22) of b still guarantees $Q(\zeta_0) \subset \mathcal{O}^b$ and thus (5.12) hold here:

$$\gamma, \gamma^i \subset \Omega^b.$$

The next lemma gives a much more precise version of (5.16).

Lemma 5.2. *If $\zeta_0 \in \mathcal{O}$, $\nu \in Q(\zeta_0)$, $\zeta \in \gamma^I_{\nu,0}$, then the points $\{-\log w(\zeta)\}$ lie in the $A_9 \min\{\exp(-(\lambda/2)\operatorname{Re}\zeta_0), |\operatorname{Im}\zeta_0| \exp(-(1/2)\operatorname{Re}\zeta_0)\}$-neighborhood of γ^i.*

Assuming this lemma, we can now easily prove

Theorem 5.3. *The correspondence map $\Delta(\xi) = -\log w(\xi)$ for (5.1) can be extended to a C^∞ function $\Delta(\zeta)$ in the domain $\mathcal{O}$ in such a way that*

$$|\overline{\partial}\Delta(\zeta)| \leqslant B_f A_{10} \sqrt[4]{\rho(C_f A_{11} |\operatorname{Im}\zeta| e^{-\operatorname{Re}\zeta/2})}.$$

Proof. Let $\nu \in Q(\zeta_0)$. Lemma 5.2 and (5.12) guarantee that

$$-\log w(\zeta) \in \Omega^b, \qquad \zeta \in \gamma^I_{\nu,0}, \tag{5.23}$$

if $b = A_9 |\operatorname{Im}\zeta_0| e^{-\frac{1}{2}\operatorname{Re}\zeta_0}$. At the same time

$$-\log w(\zeta) \text{ lies in the } A_2\text{-neighborhood of } (\gamma^{II})^i \cup (\gamma^{III})^i \text{ if } \zeta \in \gamma^{II}_{\nu,0} \cup \gamma^{III}_{\nu,0}.$$

If A_9 is chosen to be large enough, this A_2-neighborhood of $(\gamma^{II})^i \cup (\gamma^{III})^i$ lies in Ω^b. Combining this assertion with (5.23) we see that

$$-\log w(\zeta) \in \Omega^b, \qquad \zeta \in \gamma_{\nu,0}, \tag{5.24}$$

if b is taken as in (5.22) with A_9 large enough. Now (5.24) guarantees (5.6), (5.7), (5.8). We continue exactly as in the proof of Theorem 5.1 replacing a's by b's. This gives us an asymptotically holomorphic function in $\mathcal{O} \setminus \mathbb{R}_+$. Finally, to verify that this function is a C^∞-smooth continuation of $\Delta(\xi)$ we note that the above construction can be extended as follows.

(A) If $Q(\zeta_0) \subset \mathcal{O}^a$, then $Q'(\zeta_0) \overset{\text{def}}{=} \operatorname{conv}\{Q(\zeta_0), \overline{Q(\zeta_0)}\} \subset \mathcal{O}^a$, $\Delta^{\zeta_0} \in \operatorname{Hol}(Q'(\zeta_0))$.

(B) If $Q(\zeta_0), Q(\zeta_1) \subset \mathcal{O}$, $\nu \in Q'(\zeta_0) \cap Q'(\zeta_1)$, then for every $n \geqslant 0$,

$$|\Delta^{\zeta_0}(\nu)^{(n)} - \Delta^{\zeta_1}(\nu)^{(n)}| \leqslant B'_f \sqrt{\rho(C'_f(n) A_{12} \max(|\operatorname{Im}\zeta_0| e^{-\operatorname{Re}\zeta_0/2}, |\operatorname{Im}\zeta_1| e^{-\operatorname{Re}\zeta_1/2}))}.$$

(C) If $Q(\zeta) \subset \mathcal{O}$, $\nu \in Q'(\zeta) \cap \mathbb{R}_+$, then

$$|\Delta(\nu) - \Delta^\zeta(\nu)| \leqslant B'_f \sqrt{\rho(C'_f(n) A_{12} |\operatorname{Im}\zeta| e^{-\operatorname{Re}\zeta/2})}.$$

This proves Theorem 5.3. $\bullet$

5.7 Proof of Lemma 5.2. Let $\nu \in Q(\zeta_0)$. When $\zeta \in \gamma^I_{\nu,0}$ we have $|z(\zeta)| \leqslant Ae^{-\operatorname{Re}\zeta_0/2}$,

$$|\operatorname{Im} z(\zeta)| \leqslant Ae^{-\operatorname{Re}\zeta_0/2}|\operatorname{Im}\zeta_0|.$$

Let us use the notation $w^{\zeta_0}(\zeta) = w(\zeta) = |w(\zeta)|e^{i\alpha(\zeta)}$. Then by (5.4) we have

$$\lambda + \frac{\dot{w}}{w} = -z^k w f^b(z, w).$$

And therefore,

$$\dot{\alpha} = -\operatorname{Im}(z^k|w|e^{i\alpha} f^b(z, |w|e^{i\alpha})). \tag{5.25}$$

From (5.2) we clearly get (because $f(x, u)$ is real on S)

$$|\operatorname{Im} f^b(z, |w|e^{i\alpha})| \leqslant B_f\rho(C_f b) + B'_f(|\operatorname{Im} z| + |\alpha|).$$

Then (5.25) gives us

$$\begin{aligned}|\dot{\alpha}| &\leqslant |\alpha| + e^{-(k-1)\operatorname{Re}\zeta_0/2}B_f(|\operatorname{Im}\zeta_0|e^{-\operatorname{Re}\zeta_0/2} + \rho(C_f A_9|\operatorname{Im}\zeta_0|e^{-\operatorname{Re}\zeta_0/2}))\\ &\leqslant |\alpha| + B_f|\operatorname{Im}\zeta_0|e^{-k\operatorname{Re}\zeta_0/2}; \qquad \alpha(0) = 0.\end{aligned}$$

Solving this differential inequality we obtain

$$\zeta \in \gamma^I_{\nu,0} \Rightarrow |\alpha(\zeta)| \leqslant AB_f|\operatorname{Im}\zeta_0|e^{-k\operatorname{Re}\zeta_0/2}e^{\operatorname{Re}\zeta_0/2} \leqslant AB_f|\operatorname{Im}\zeta_0|e^{-(k-1)\operatorname{Re}\zeta_0/2}. \tag{5.26}$$

Let us recall that (5.10) shows that $|w(\zeta)| \leqslant 1$ along the whole path γ. Along with (5.26) this shows that

$$\nu \in Q(\zeta_0),\ \zeta \in \gamma^I_{\nu,0} \Rightarrow |\operatorname{Im} w(\zeta)| \leqslant A|\operatorname{Im}\zeta_0|e^{-\operatorname{Re}\zeta_0/2}. \tag{5.27}$$

Here, it is sufficient to consider only the case $k \geqslant 3$, because k can be chosen arbitrarily big subject only to the condition $k \geqslant \lambda$. The combination of (5.16) and (5.27) proves Lemma 5.2. •

6. A preparation theorem on asymptotically holomorphic functions.

Here we are going to state a uniqueness theorem for function f defined on the right half-plane $\mathbb{C}_+$ (or on its sufficiently massive subdomain) whose $\bar{\partial}$-derivative decays rapidly along the real axis,

$$|\bar{\partial} f(z)| < \omega(\operatorname{Re} z).$$

If such a function decays on $\mathbb{R}_+$ more rapidly than any exponential function, then we can make a stronger claim that it is no greater than $\omega(cx)$ on $\mathbb{R}_+$ for some $c > 0$.

This result includes a similar one proved earlier in [20, Theorem 3]. The proof uses the usual machinery of estimations of asymptotically holomorphic functions (see [1,4]).

Let ω be a positive decreasing function on $\mathbb{R}_+$ such that

$$\text{the function } \log\frac{1}{\omega(x)} \text{ is convex.} \tag{6.1}$$

Remark. We shall use later $\omega(x) = \rho(e^{-cx})$, where ρ is as given at the beginning of Section 2.2. So ρ is logarithmically convex which means exactly the convexity of $\log(1/\omega)$.

Theorem 6.1. *If* $f \in (C^1 \cap L^\infty)(\mathbb{C}_+)$,

$$|\overline{\partial} f(z)| < \omega(\operatorname{Re} z),$$
$$|f(x)| = O(\exp(-nx)), \qquad x \to \infty, \text{ for every } n,$$

then

$$|f(x)| < \omega(Kx) \qquad \text{for large } x$$

and for an absolute constant K.

Proof. First, let us remark that without loss of generality we can assume that

$$|\overline{\partial} f(z)| < (1+|z|)^{-4}\omega(\operatorname{Re} z), \tag{6.2}$$

and

$$f(z) = \frac{1}{\pi} \iint\limits_{\zeta \in \mathbb{C}_+} \frac{\overline{\partial} f(\zeta)}{z-\zeta}\, dm_2(\zeta). \tag{6.3}$$

Really, we can divide f by $(1+z)^6$ and then redefine it on the half-plane $\{\operatorname{Re} z < 1\}$ in such a way that $f|\{z : \operatorname{Re} z < 1/2\} = 0$ and

$$\iint_{\zeta \in \mathbb{C}_+} \overline{\partial} f(\zeta)\, dm_2(\zeta) = 0,$$
$$\iint_{\zeta \in \mathbb{C}_+} \zeta \overline{\partial} f(\zeta)\, dm_2(\zeta) = 0.$$

Dividing our new f by a large constant we obtain (6.2). To prove (6.3) let us consider the difference

$$a_f(z) = f(z) - \frac{1}{\pi} \iint \frac{\overline{\partial} f(\zeta)}{z-\zeta}\, dm_2(\zeta).$$

Now, a_f is an entire function, and it belongs to $H^1(\mathbb{C}_+)$, because

$$\left| z^2 \iint \frac{\overline{\partial} f(\zeta)}{z-\zeta}\, dm_2(\zeta) \right| = \left| \iint \frac{\zeta^2 \overline{\partial} f(\zeta)}{z-\zeta}\, dm_2(\zeta) \right| \leqslant c.$$

The equality follows from the fact that two first moments of $\overline{\partial} f$ vanish. The last inequality trivially follows from (6.2). Therefore,

$$f_0(z) \overset{\text{def}}{=} \frac{1}{\pi} \iint \frac{\overline{\partial} f(\zeta)}{z-\zeta}\, dm_2(\zeta) + \frac{1}{2\pi} \iint\limits_{0 < \operatorname{Re} \zeta < 1} \frac{a_f(\zeta)}{z-\zeta}\, dm_2(\zeta) = f(z), \qquad \operatorname{Re} z > 1.$$

It is evident that being divided by a large constant the function f_0 satisfies conditions (6.2)–(6.3) and has the same asymptotics as f does. From now on f satisfies (6.2), (6.3).

For every $t \geqslant 0$ let us consider the function $F(z) = f(z)e^{tz}$. It is bounded on the imaginary axis and on the real half-axis by the conditions of the theorem. It is of exponential growth in the right half-plane. Moreover, $\overline{\partial} F(z)$ is bounded and summable in the right half-plane (because ω decreases faster than any exponential function and because of (6.2)). Therefore, the function a_F,

$$a_F(z) = F(z) - \frac{1}{\pi} \iint\limits_{\zeta \in \mathbb{C}_+} \frac{\overline{\partial} F(\zeta)}{z - \zeta} \, dm_2(\zeta),$$

is analytic in both the first and the fourth quadrants, it is of exponential growth there, and it is bounded on their boundaries. The Phragmen–Lindelöf theorem for angles implies that a_F and F are bounded in the right half-plane. As a result, we have

$$|f(z)| = O(\exp(-n \operatorname{Re} z)) \text{ for every } n \text{ uniformly in } \operatorname{Im} z. \tag{6.4}$$

Furthermore, the functions

$$f_x(z) = \frac{1}{\pi} \iint\limits_{0 < \operatorname{Re} \zeta < x} \frac{\overline{\partial} f(\zeta)}{z - \zeta} \, dm_2(\zeta)$$

are analytic in the half-planes $\operatorname{Re} z > x$, and the difference $f - f_x$ can be estimated from above as follows:

$$|f(z) - f_x(z)| = \Bigl| \frac{1}{\pi} \iint\limits_{\operatorname{Re} \zeta > x} \frac{\overline{\partial} f(\zeta)}{z - \zeta} \, dm_2(\zeta) \Bigr| \leqslant \frac{\omega(x)}{\pi} \iint \frac{(1 + |\zeta|)^{-4}}{|z - \zeta|} \, dm_2(\zeta) < K\omega(x),$$

where K is an absolute constant. Let $A = \{n \in \mathbb{N} : \sup_{\operatorname{Re} z \geqslant \exp n} |f(z)| < \omega(\exp(n-2))\}$, $B = \mathbb{N} \setminus A$. Our local goal is to verify that A is infinite. Put

$$a(x) = -\exp(-x) \log \sup_{\operatorname{Re} z \geqslant \exp x} |f(z)|.$$

Condition (6.4) means that

$$a(n) \to \infty \text{ as } n \to \infty. \tag{6.5}$$

Given $n \in B$, consider the function $f_{\exp(n-2)}$ bounded and analytic in $\Omega_n = \{z : \operatorname{Re} z > \exp(n-2)\}$ and continuous up to the boundary. We have:

$$\sup_{\operatorname{Re} z = \exp(n-2)} |f_{\exp(n-2)}(z)| \leqslant K \exp[-a(n-2)e^{n-2}],$$
$$\sup_{\operatorname{Re} z = \exp n} |f_{\exp(n-2)}(z)| \leqslant K \exp[-a(n)e^{n}],$$

where K is an absolute constant (we use that $\sup_{\operatorname{Re} z \geqslant \exp n} |f(z)| \geqslant \omega(e^{n-2})$).

By *the two constants theorem* (see, for instance, [16, p.257])

$$\sup_{\operatorname{Re} z \geqslant \exp(n-1)} |f_{\exp(n-2)}(z)| \leqslant C \exp[-\frac{e}{e+1} a(n-2)e^{n-2} - \frac{1}{e+1} a(n)e^n].$$

As a result,

$$a(n-1) \geqslant -Ke^{-n} + \frac{a(n-2)+ea(n)}{1+e}. \qquad (6.6)$$

Therefore,

$$a(n) - a(n-1) \leqslant Ke^{-n} + \frac{1}{e}\big(a(n-1) - a(n-2)\big). \qquad (6.7)$$

If for some n_0 the set B contains all $n \geqslant n_0$, then inequalities (6.7) imply that

$$a(n) \leqslant K$$

and this is impossible by (6.5). Therefore, there exist arbitrarily large n, $n \in A$.

Let $n_1, n_2 \in A$ be numbers such that for every natural n, $n_1 < n < n_2$, we have $n \in B$, and $n_2 - n_1 > 1$. A simple two constants like estimate implies that

$$a(n_2 - 1) \geqslant -K \frac{\log \omega(\exp(n_2 - 2))}{\exp(n_2 - 2)}.$$

Beside that,

$$a(n_1) \geqslant -\frac{\log \omega(\exp(n_1 - 2))}{\exp n_1}.$$

Now, inequalities (6.6) employed for $n \in [n_1 + 1, n_2 - 1]$ imply that the function $G(x) = xa(\log x)$ is "almost concave" in the sense that for every $x \in [e^{n_1}, e^{n_2}]$

$$G(x) \geqslant K_1 \frac{G(e^{n_1})(e^{n_2-1} - x) + G(e^{n_2-1})(x - e^{n_1})}{e^{n_2-1} - e^{n_1}} + K_2,$$

where K_1 and K_2 are absolute constants. Since the function $\log(1/\omega)$ is convex, we get

$$G(x) \geqslant \log \frac{1}{\omega(Kx)}, \qquad e^{n_1} < x < e^{n_2},$$

for large n_1 and an absolute constant K. This proves our theorem. •

The assertion of Theorem 6.1 remains valid when the right half-plane is replaced by a smaller but still massive domain Ω_ε,

$$\Omega_\varepsilon = \{z \in \mathbb{C}_+ : |\operatorname{Im} z| < e^{\varepsilon \operatorname{Re} z}\},$$

where ε is a fixed number, $\varepsilon > 0$.

Corollary 6.2. *If* $f \in (C^1 \cap L^\infty)(\Omega_\varepsilon)$,

$$|\bar\partial f(z)| < \omega(\operatorname{Re} z),$$

and

$$|f(x)| = O(\exp(-nx)), \qquad x \to \infty, \text{ for every } n,$$

then

$$|f(x)| \leqslant \omega(Kx) \text{ for large } x,$$

where K is an absolute constant.

Proof. Let ψ be the conformal map from $\mathbb{C}_+$ onto Ω_ε such that

$$\psi(\infty) = \infty,\ \psi(\mathbb{R}_+) = \mathbb{R}_+,\ \psi'(\infty) = 1.$$

Standard estimates of the asymptotic behavior of conformal maps (see, for example, [12,22]) give that

$$\operatorname{Re}\psi(z) \geqslant \operatorname{Re} z,\ z \in \mathbb{C}_+, \qquad \psi(x) \leqslant x + C,\ x \in \mathbb{R}_+,$$

for sufficiently large x, and

$$|\psi'(z)| \leqslant c,\ z \in \mathbb{C}_+,$$

for some $c = c(\varepsilon) > 0$.

Now the function $F(z) = f(\psi(z))/c$ satisfies the conditions of Theorem 6.1, because by the chain rule

$$|\bar\partial F(z)| = \frac{1}{c}|\bar\partial f(\psi(z))||\psi'(z)| \leqslant \omega(\operatorname{Re}\psi(z))) \leqslant \omega(\operatorname{Re} z),$$

$$|F(x)| = \frac{1}{c}|f(\psi(x))| = O(\exp(-n\psi(x))) = O(\exp(-nx)), \qquad x \to \infty.$$

Therefore, for large x

$$|f(x)| = |F(\psi^{-1}(x))| \leqslant \omega(K\psi^{-1}(x)) \leqslant \omega(K_1 x). \quad \bullet$$

7. A uniqueness theorem for asymptotically holomorphic functions.

We deal here with asymptotically holomorphic functions f of the following special kind:

$$|\bar\partial f(z)| \leqslant \rho(\varphi(\operatorname{Re} z)|\operatorname{Im} z|),$$

where ρ is quasianalytic, that is

$$\rho \text{ is positive, increases, } \int_0^1 \log\log\frac{1}{\rho(x)}\,dx = \infty, \tag{7.1}$$

and φ is bounded. Functions having the above estimate of the $\bar{\partial}$-derivative are quasianalytically smooth on $\mathbb{R}_+$. If $\varphi(x) \equiv 1$, then the maximal possible rate of decay of these functions at ∞ depends on ρ and varies from $\exp(-\exp(\pi x/(2\varepsilon)))$ when ρ is equal to 0 on the interval $[0,\varepsilon]$, to arbitrarily rapid decay when ρ is sufficiently close to the boundary of quasianalyticity, that is when the integral in (7.1) diverges sufficiently slowly. Therefore, conditions like

$$|f(x)| < \rho\Big(\frac{1}{x}\Big) \qquad \text{or} \qquad |f(x)| < \rho(\exp(-x))$$

can never ensure that $f|\mathbb{R}_+ = 0$ unless ρ is sufficiently far from the boundary of quasianalyticity. (See, for example [20, Theorem 4]). When φ is not constant, but the integral

$$\int_0^\infty \varphi(x)\,dx$$

still diverges the situation is similar. However, if this integral converges, then the condition

$$|f(x)| < \rho(\varphi(x))$$

implies that $f|\mathbb{R}_+ = 0$.

In the next theorem S denotes the strip $\{z \in \mathbb{C}_+ : |\operatorname{Im} z| < 1\}$.

Theorem 7.1. *Let a function ρ satisfy condition* (7.1), *let a function φ decrease on $\mathbb{R}_+$, and*

$$\int_0^\infty \varphi(x)\,dx < \infty.$$

If $f \in (C^1 \cap L^\infty)(S)$,

$$|\bar{\partial} f(z)| < \rho(\varphi(\operatorname{Re} z)|\operatorname{Im} z|),$$

and

$$|f(x)| < \rho(\varphi(x)), \qquad x > 0, \tag{7.2}$$

then $f|\mathbb{R}_+ = 0$.

Proof. Put

$$s(t) = \rho^{-1}(\sup_{x \geqslant t} |f(x)|),$$
$$M = \log\log\frac{1}{\rho},$$

where ρ^{-1} is the inverse function to ρ. It is evident that s does not increase, and $s(t) < \varphi(t)$ (see (7.2)). We are going to prove that $s(t) \equiv 0$.

First, we verify that

$$M(s(t)) - M\Big(s\Big(t - \frac{s(t)}{\varphi(t-2)}\Big)\Big) \leqslant A, \tag{7.3}$$

for sufficiently large t, where A is an absolute constant. To prove this, consider a domain

$$P = \{z : t - 2\frac{s(t)}{\varphi(t-2)} \leqslant \operatorname{Re} z \leqslant t+1, \, |\operatorname{Im} z| \leqslant \frac{s(t)}{\varphi(t-2)}\}.$$

The function f_P,

$$f_P(z) = f(z) - \frac{1}{\pi} \iint\limits_{\zeta \in P} \frac{\overline{\partial} f(\zeta)}{\zeta - z} \, dm_2(\zeta),$$

is analytic in P, and for sufficiently large t we have

$$|f_P(x)| \leqslant 2 \exp(-\exp M(s(t))), \qquad x \in [t, t+1].$$

By *the two constants theorem*, for sufficiently large t we obtain

$$\Big| f_P\Big(t - \frac{s(t)}{\varphi(t-2)}\Big)\Big| \leqslant \exp(-\exp(M(s(t)) - A)),$$
$$\Big| f\Big(t - \frac{s(t)}{\varphi(t-2)}\Big)\Big| \leqslant \exp(-\exp(M(s(t)) - A_1)),$$

where A, A_1 are absolute constants. This proves (7.3).

Without loss of generality assume that ρ and φ are continuous. Then the function s is also continuous. Construct an infinite sequence of points t_k in such a way that $t_0 = 2$,

$$t_{k+1} - \frac{s(t_{k+1})}{\varphi(t_{k+1} - 2)} = t_k. \tag{7.4}$$

Such a sequence exists because if we consider $r(t) = t - s(t)/\varphi(t-2)$, then clearly $r(t_k) < t_k$ and $r(t) \geqslant t - 1$. Thus, t_{k+1} can be taken as the smallest solution of $r(t_{k+1}) = t_k$. Clearly $t_k \to \infty$. Now we get

$$M(s(t_{k+1})) - M(s(t_k)) \leqslant A$$

for some A which does not depend on k. Since

$$\int\limits_0^1 M(t) \, dt = \infty,$$

and $s(t) \to 0$ as $t \to \infty$, we have that

$$\sum_{k=0}^{\infty} s(t_k) = \sum_{k=1}^{\infty} k(s(t_{k-1}) - s(t_k)) \geqslant \sum_{k=1}^{\infty} \frac{M(s(t_k)) - M(s(t_0))}{A} (s(t_{k-1}) - s(t_k)) \geqslant$$
$$\geqslant \frac{1}{A} \sum_{k=1}^{\infty} \int\limits_{s(t_k)}^{s(t_{k-1})} M(x) \, dx = \infty$$

unless $s \equiv 0$. Furthermore, by (7.4)

$$\sum_{k=0}^{\infty} s(t_k) = s(t_0) + \sum_{k=1}^{\infty} \varphi(t_k - 2)(t_k - t_{k-1}) \leqslant s(t_0) + \sum_{k=1}^{\infty} \int_{t_{k-1}}^{t_k} \varphi(x-2)\,dx < \infty.$$

This contradiction shows that $s \equiv 0$. Theorem is proved. •

When φ is not summable, this result, generally speaking, does not hold. However, we can prove the following weaker statement.

Theorem 7.2. *Let φ be a positive decreasing function on $\mathbb{R}_+$,*

$$\lim_{x\to\infty} \varphi(x) = 0.$$

Then there exists an increasing function ρ on $\mathbb{R}_+$ such that the following uniqueness property holds:

If a function f, $f \in (C^1 \cap L^\infty)(S)$, satisfies the conditions

$$|\overline{\partial} f(z)| < \rho(\varphi(\operatorname{Re} z)|\operatorname{Im} z|), \qquad z \in S,$$
$$|f(x)| < \rho(\varphi(x)), \qquad x \in \mathbb{R}_+,$$

then

$$f|\mathbb{R}_+ = 0.$$

The proof of this theorem is similar to that of the preceding one. We shall construct ρ in such a way that

$$\rho(x) < \rho_1(x) = \exp\Bigl(-\exp\frac{1}{x}\Bigr).$$

Then for every f satisfying conditions of Theorem we have

$$|\overline{\partial} f(z)| < \exp\Bigl(-\exp\frac{1}{|\operatorname{Im} z|}\Bigr).$$

If $f|\mathbb{R}_+ \neq 0$, put

$$s(x) = \rho_1^{-1}(\sup_{t\geqslant x} |f(t)|),$$
$$M_1(x) = \log\log\frac{1}{\rho_1(x)} = \frac{1}{x}.$$

Construct a sequence of points t_k in such a way that $t_0 = 0$,

$$t_{k+1} - s(t_{k+1}) = t_k, \qquad k \geqslant 0.$$

We have again

$$M_1(s(t_{k+1})) - M_1(s(t_k)) \leqslant A, \qquad k \geqslant 0,$$

for some absolute constant A. Therefore,

$$s(t_{k+1}) \geqslant \frac{A'}{k},$$
$$t_{k+1} \geqslant A'' \cdot \log k, \qquad k \geqslant 1.$$

As a result, we obtain

$$\lim_{k\to\infty} t_k = \infty,$$
$$s(t_k) \geqslant \exp(-A''' \cdot t_k),$$
$$\sup_{t\geqslant t_k} |f(t)| \geqslant \exp(-\exp\exp(A''' \cdot t_k)).$$

Choose ρ in such a way that

$$\rho(x) < \exp\Bigl(-\exp\frac{1}{x}\Bigr),$$
$$\rho(\varphi(t_k)) < \exp(-\exp\exp(2A''' \cdot t_k)).$$

We get a contradiction to the assumption $|f(x)| < \rho(\varphi(x))$. Therefore $f|\mathbb{R}_+ = 0$. •

The sharpness of these results can be illustrated in the following way.

Example 7.3. *Let a function ρ increase on $\mathbb{R}_+$, and*

$$\int_0 \log\log\frac{1}{\rho(x)}\,dx < \infty.$$

For every two positive functions φ and ψ decreasing on $\mathbb{R}_+$ there exists a function $f \in (C^1 \cap L^\infty)(\mathbb{C}_+)$ such that

$$\begin{cases} |\bar\partial f(z)| < \rho(\varphi(\operatorname{Re} z)|\operatorname{Im} z|), \\ |f(x)| < \psi(x), \\ \text{and } \operatorname{supp}(f|\mathbb{R}_+) \text{ is unbounded.} \end{cases} \tag{7.5}$$

Proof. By a theorem of Dyn'kin [8] for every $c > 0$ there exists a C^1-smooth function F_c such that

$$\operatorname{supp} F_c \subset \{z : |\operatorname{Im} z| < 1,\ 0 < \operatorname{Re} z < 1\},$$

and, at the same time,

$$|F_c(z)| < 1,$$
$$|\bar\partial F_c(z)| < \rho(c|\operatorname{Im} z|),$$
$$F_c|\mathbb{R} \neq 0.$$

Put

$$f(z) = \sum_{k=0}^{\infty} \psi(k+1) F_{\varphi(k+1)}(z-k).$$

It is easily seen that f enjoys all the properties (7.5). •

Example 7.4. *Let a positive decreasing function φ on $\mathbb{R}_+$ satisfy the condition*

$$\int_0^\infty \varphi(x)\,dx = \infty. \tag{7.6}$$

Then there exists a function ρ on $\mathbb{R}_+$ satisfying condition (7.1), *and a function $f \in (C^1 \cap L^\infty)(\mathbb{C}_+)$ such that*

$$\begin{cases} |\bar\partial f(z)| < \rho(\varphi(\operatorname{Re} z)|\operatorname{Im} z|), & z \in \mathbb{C}_+, \\ |f(x)| < \rho(\varphi(x)), & x \in \mathbb{R}_+, \end{cases} \tag{7.7}$$

and supp $(f|\mathbb{R}_+)$ *is unbounded.*

Proof. Without loss of generality assume φ to be continuous. Put

$$f_0(z) = \begin{cases} \exp(-\exp z), & |\operatorname{Im} z| < 1 \\ 0, & |\operatorname{Im} z| \geqslant 1, \end{cases}$$

$$f(z) = \frac{4}{\pi^2} \iint_{|\zeta - z| < \frac{1}{2}} f_0(\zeta)\,dm_2(\zeta).$$

Then

$$\begin{cases} |\bar\partial f(z)| = 0, & |\operatorname{Im} z| < \dfrac{1}{2} \text{ or } |\operatorname{Im} z| > \dfrac{3}{2}, \\ |\bar\partial f(z)| < \exp(-\exp(\operatorname{Re} z/2)) \text{ for large } \operatorname{Re} z, \\ |f(x)| = \exp(-\exp x), & x \in \mathbb{R}_+. \end{cases} \tag{7.8}$$

Now, define

$$\rho(x) = \exp[-\exp(\varphi^{-1}(2x)/2)],$$

where φ^{-1} is the inverse function to φ. Property (7.1) follows from condition (7.6). Properties (7.7) of f follow from (7.8). •

We display our results on Figure 3, where ρ varies from 0 through the quasianalytic zone, and then through the non-quasianalytic zone (the boundary is the point (*)), and φ varies from $\varphi(x) \equiv 1$ through the zone where the integral

$$\int_0^\infty \varphi(x)\,dx$$

diverges, and then through the zone where it converges (the boundary is the point (**)). It is the shaded domain where our uniqueness theorem holds.

FIGURE 3

8. THE END OF THE PROOF OF THEOREM 2.1.

Let $\lambda^j = |\lambda_1^j/\lambda_2^j|$ be the characteristic number of the j-th saddle of the polycycle C. Assertion (5.15) says that

$$\Delta_j(\zeta) = \lambda_j \zeta + O(1). \tag{8.1}$$

Proposition 4.1 shows that the functions f_j (see Section 4 for the notations) are quasianalytic. This fact combined with an obvious estimate

$$F_j(\zeta) = \zeta + O(1) \tag{8.2}$$

shows that ($\zeta = \xi + i\eta \in \mathbb{C}_+$)

$$|\bar{\partial} F_j(\zeta)| \leqslant \rho(\eta e^{-\xi}). \tag{8.3}$$

Let us recall that $\mathcal{O} \overset{\text{def}}{=} \{\zeta : |\eta| < \frac{1}{2} e^{\xi/4}\}$ and that Theorem 5.3 implies that

$$|\bar{\partial} \Delta_j(\zeta)| \leqslant AB_f(\rho(AC_f \eta e^{-\xi/2}))^{1/4}, \qquad \zeta \in \mathcal{O}. \tag{8.4}$$

Let us denote $\rho_0(t) = CB_f(\rho(CC_f t))^{1/4}$, where C is a large constant depending only on $\{\lambda_j\}$ and on constants A in (8.4). Now the combination of (8.1)–(8.4) shows that the function $\Delta_{c,\Gamma} = F_n \circ \Delta_n \circ \cdots \circ F_1 \circ \Delta_1$ satisfies the estimate ($\Omega_\varepsilon = \{\zeta : |\eta| < e^{+\varepsilon\xi}\}$):

$$|\bar{\partial} \Delta_{c,\Gamma}(\zeta)| \leqslant \rho_0(e^{-\varepsilon\xi}), \qquad \zeta \in \Omega_\varepsilon.$$

Using the notation of Section 4 let us denote $f(\zeta) = e^{-\Delta_{c,\Gamma}(\zeta)} - e^{-\zeta} = \delta_{c,\Gamma}(e^{-\zeta}) - e^{-\zeta}$. The previous estimate and Theorem A (see Section 2) show that this function satisfies the conditions of Theorem 2.3 and we conclude that

$$|\delta_{c,\Gamma}(e^{-\zeta}) - e^{-\zeta}| \leqslant \rho_0(Ae^{-\varepsilon\xi}) \tag{8.5}$$

for sufficiently large ξ.

Assertions (8.1) and (8.4) imply that Δ_j is uniformly Lipschitz in the strip $S = \{\zeta \in \mathbb{C}_+ : |\operatorname{Im}\zeta| < 1\}$. Clearly, Δ_j maps $\mathbb{R}_+$ into $\mathbb{R}_+$ and thus

$$|\operatorname{Im}\Delta_j(\zeta)| \leqslant C_j|\operatorname{Im}\zeta|. \tag{8.6}$$

The analog of (8.6) for F_j follows from (8.2), (8.3):

$$|\operatorname{Im}F_j(\zeta)| \leqslant C|\operatorname{Im}\zeta|. \tag{8.7}$$

Combining (8.1)–(8.4) with (8.6) and (8.7) we obtain the estimate

$$|\bar{\partial}\Delta_{c,\Gamma}(\zeta)| \leqslant \rho_0(C\eta e^{-\varepsilon\xi}), \qquad \zeta \in S. \tag{8.8}$$

Considering again $\delta_{c,\Gamma}(e^{-\zeta}) - e^{-\zeta} = e^{-\Delta_{c,\Gamma}(\zeta)} - e^{-\zeta}$ we see that this function satisfies the conditions of Theorem 2.4 (see (8.5), (8.8)). The conclusion is that

$$\delta_{c,\Gamma}(e^{-\xi}) \equiv e^{-\xi}$$

and the finiteness theorem (Theorem 2.1) is completely proved. •

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