Beurling algebras and the generalized Fourier transform.

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Abstract. We investigate primary ideals at ∞ in Beurling-type Frechet algebras in the quasianalytic case. They are described by two parameters characterizing the rate of decay of their Fourier transforms at $\pm\infty$ (Theorem 9.6). We use the so called generalized Fourier transform to treat related convolution equations. A necessary and sufficient condition for the orthogonality of a functional with empty spectrum and an ideal generated by a function is given in terms of their Fourier transforms (Theorem 3.6). Furthermore, we describe all primary ideals at ∞ in Beurling-type algebras on the half-line (Theorem 9.7).

Beside that, the technique of asymptotically holomorphic functions helps us to describe asymptotics of quasianalytically smooth function (Theorem 9.9) and to prove an extension of Levinson's log-log theorem (Theorem 9.8).

1. Introduction.

Let B be a given commutative Banach or Frechet algebra having no unit. We extend it formally with a unit and consider in the algebra obtained (closed) primary ideals that lie in B. These ideals are called primary ideals at ∞ (see, for example, [**23,25**]).

The General Tauberian Theorem of Wiener states that there are no nontrivial primary ideals at ∞ in $L^1(\mathbb{R})$. We reformulate this theorem as a statement on the completeness of translates. For a function f in $L^1(\mathbb{R})$ define $\tau_t f(x) = f(x-t)$, $x \in \mathbb{R}$, $t \in \mathbb{R}$. Let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a family of elements of $L^1(\mathbb{R})$. Then Wiener's theorem claims that the system of translates $\{\tau_t f_\gamma\}$, $\gamma \in \Gamma$, $t \in \mathbb{R}$, is complete in $L^1(\mathbb{R})$ if and only if

$$\bigcap_{\gamma\in\Gamma} \bigl\{z \in \mathbb{R} : \mathcal{F}f_\gamma(z) = 0\bigr\} = \emptyset,$$

where

$$\mathcal{F}f(z) = \int_{\mathbb{R}} f(x)e^{ixz}dx.$$

In 1938 A. Beurling [**2**] extended this result to the so called Beurling algebras

$$L^1_p(\mathbb{R}) = L^1(\mathbb{R}, p(x)dx),$$

where

$$p(x) \ge p(0) = 1, \qquad p(x+y) \le p(x)p(y), \qquad p(tx) \ge p(x), \quad t \ge 1.$$

These properties imply the existence of finite limits

$$\alpha = \lim_{x\to\infty} \frac{\log p(x)}{x} \ge \beta = \lim_{x\to-\infty} \frac{\log p(x)}{x}.$$

A closed ideal I in $L^1_p(\mathbb{R})$ is primary at ∞ if and only if

$$\bigcap_{f\in I} \bigl\{z \in S : \mathcal{F}f(z) = 0\bigr\} = \emptyset,$$

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where $S = \{z \in \mathbb{C} : -\alpha \leq \operatorname{Im} z \leq -\beta\}$.

A. Beurling distinguished between three cases: $\alpha+\beta > 0$ — the analytic case, $\alpha+\beta = 0$ and the integral

$$\int_{-\infty}^{\infty} \frac{\log p(x) - \alpha x}{1+x^2}\, dx \tag{1.1}$$

diverges — the quasianalytic case, $\alpha + \beta = 0$ and the integral (1.1) converges — the non-quasianalytic case. He proved that in the non-quasianalytic case there are no primary ideals at ∞. (A modern proof is presented in [**23**]). Later, in 1950, B. Nyman [**37**] proved the existence of certain non-trivial ideals primary at ∞ in two cases: in the quasianalytic one, for $p(x) \sim \exp(|x|/\log|x|)$, and in the analytic one, for $p(x) = \exp|x|$. In the last case he proved also that every primary ideal at ∞ is contained in one of the primary ideals described by him. A complete description of primary ideals at ∞ in the case $p(x) = \exp|x|$ was obtained independently by B. I. Korenblum [**33**] in 1958.

In 1985 H. Hedenmalm [**28**] described all primary ideals at ∞ in the following analytic-non-quasianalytic case:

$$\alpha+\beta > 0, \qquad \int^{\infty} \frac{\log p(x) - \alpha x}{1+x^2}\, dx < \infty, \qquad \int_{-\infty} \frac{\log p(x) - \beta x}{1+x^2}\, dx < \infty.$$

We should mention here also related results concerning the completeness of systems of translates in $L^1(\mathbb{R}_+)$ obtained by B. Nyman [**37**] and generalized for $L^1_p(\mathbb{R}_+)$ in the non-quasianalytic case by V. P. Gurariĭ and B. Ja. Levin [**27**]:

Let $\{f_\gamma\}_{\gamma\in\Gamma}$ be a set of elements of $L^1_p(\mathbb{R}_+)$, $\log p(x) = o(x)$, $x \to \infty$. The system of right translates $\{\tau_t f_\gamma\}$, $\gamma \in \Gamma$, $t \geq 0$, where

$$\tau_t f(x) := \begin{cases} f(x-t), & x \geq t, \\ 0, & x < t, \end{cases}$$

is complete in $L^1_p(\mathbb{R}_+)$ if and only if

$$\bigcap_{\gamma\in\Gamma} \left\{z \in \overline{\mathbb{C}}_+ : \mathcal{F} f_\gamma(z) = 0\right\} = \emptyset, \qquad 0 \in \operatorname{clos} \bigcup_{\gamma\in\Gamma} \operatorname{ess\,supp} f_\gamma.$$

The main steps in these works for $L^1_p(\mathbb{R})$, $L^1_p(\mathbb{R}_+)$ are the following ones: first, we prove the possibility of analytic continuation for the Carleman transform of a functional annihilating an ideal or a translation-invariant subspace, and then, use the log-log theorem of Levinson [**35,38**], that permits us to get precise estimates of this continuation.

In the quasianalytic case the first step can be made using the theory of commutative Banach algebras (see [**26**]). However, proper extensions of the second step have been unknown up to now.

That is why, in a number of papers by B. I. Korenblum [**34**], S. P. Geisberg [**19**], S. P. Geisberg and V. S. Konjuhovskiĭ [**21,22**], A. Vretblad [**39**], Y. Domar [**14**] only some examples of continual chains of primary ideals at ∞ are constructed.

Here we deal with the Frechet algebra $\mathcal{L}_p$ with even weight function p,

$$\mathcal{L}_p = \{f : \text{ for every } n \geq 0,\ f(x)x^n p(|x|) \in L^1(\mathbb{R})\}.$$

These algebras are contained in the corresponding Beurling algebras $L^1_p(\mathbb{R})$ and are sufficiently similar to them in the sense that there exist continuous chains of primary ideals at ∞ in $\mathcal{L}_p$ analogous to that discovered for $L^1_p(\mathbb{R})$ by B. I. Korenblum in [**34**]. Our typical p are $\exp\{|x|/\log(|x|+2)\}$ and $\exp\{|x|+|x|/\log(|x|+2)\}$.

We present (Section 9) a complete description of the primary ideals at ∞ in the quasianalytic case for the Frechet algebras $\mathcal{L}_p$. This description is similar to that in the analytic case for $p(x)=\exp|x|$ in [**33**] and in the analytic-non-quasianalytic case in [**28**]. We obtain it through an application of the generalized Fourier transform. Roughly speaking, we are able to extend the Fourier transforms of elements in $\mathcal{L}_p$ to functions in the whole complex plane with a certain control on their $\overline{\partial}$ derivative. Such functions are called asymptotically holomorphic functions. Technical tools to work with these functions were developed in [**5,6**]. The generalized Fourier transform is just the map associating to every element in $\mathcal{L}_p$ a suitable asymptotically holomorphic extension of its Fourier transform.

The reason to consider $\mathcal{L}_p$ instead of $L^1_p(\mathbb{R})$ is to make simpler the argument involved in the construction of this generalized Fourier transform. It looks plausible that sharper estimates on the generalized Fourier transform together with results of this paper could lead to a description of primary ideals in $L^q_p(\mathbb{R})$, at least for $q=2$.

The paper is organized as follows. The generalized Fourier transform for $\mathcal{L}_p$ is constructed in Section 2. In Section 3 we give a necessary and sufficient condition for the orthogonality of a functional φ with empty spectrum (that is for every z in S there exists f_z such that $\mathcal{F}f_z(z)\neq 0$, $f_z*\varphi=0$), and an ideal generated by a given function f, in terms of the Fourier transforms of φ and f. This result reduces the problem of orthogonality to that of describing the asymptotics of some entire functions and some quasianalytically smooth functions. We begin to discuss these questions in Section 4. Results on these asymptotics which seem to be of interest also in their own right are obtained in Sections 8 and 9. The proofs involve some auxiliary functions produced by the conformal mappings technique in Section 6. The case $l_p>0$ is taken care of in Section 7. Corresponding regularity questions are discussed in Section 5. These results on asymptotics lead to a description of primary ideals at ∞ in $\mathcal{L}_p$ in Section 9. Several remarks and unsolved problems are contained in Section 10.

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2. The generalized Fourier transform.

Let a C^2-smooth function $p:[0,\infty)\to[1,+\infty)$ satisfy the condition $p(0)=1$, with $\log p(x)$ strictly concave on $\mathbb{R}_+$. Set $p(x)=p(|x|)$, $x\in\mathbb{R}$. We call such function p a weight function or just a weight and define

$$\lim_{x\to\infty}\frac{\log p(x)}{x}=l_p.$$

Unlike Beurling's terminology, we call the weight p quasianalytic if

$$\int^{\infty} \frac{\log p(x) - l_p x}{x^2}\, dx = \infty. \tag{2.1}$$

Let us introduce some notations. Put

$$L_p^1(\mathbb{R}) = \{f : fp \in L^1(\mathbb{R})\}, \qquad L_p^\infty(\mathbb{R}) = \{f : f/p \in L^\infty(\mathbb{R})\},$$
$$\mathcal{L}_p = \Big\{f : \text{ for every } n \ge 0, \int_{\mathbb{R}} |f(x)x^n| p(|x|)\, dx < \infty\Big\},$$
$$\mathcal{L}_p^* = \{f : \text{ for some } n \ge 0,\ c > 0,\ |f(x)| \le c(|x|+1)^n p(|x|)\},$$
$$\mathcal{M}_p = \Big\{f : \text{ for some } n \ge 0, \int |f(x)|(|x|+1)^{-n} p(|x|)\, dx < \infty\Big\},$$
$$\mathcal{M}_p^* = \{f : \text{ for every } n \ge 0 \text{ there exists } c > 0 \text{ such that } |f(x)| \le c(|x|+1)^{-n} p(|x|)\}.$$

Given a domain Ω, denote by $A(\Omega)$ the set of the functions analytic in Ω. For a majorant M defined on $\mathbb{R} \setminus [-c_-, c_+]$ with $c_-, c_+ \in \mathbb{R} \cup \{-\infty, \infty\}$ we introduce related spaces of analytic functions

$$A_M^{\pm} = \{f \in A(\mathbb{C}_\pm \pm ic_\pm) : \text{there exists } c > 0,\ |f(z)| < cM(\operatorname{Im} z)\},$$
$$A_M = \{(f,g) : f \in A_M^+,\ g \in A_M^-\},$$

where $\mathbb{C}_\pm = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$. The Fourier transform is defined in the usual way,

$$\mathcal{F}f\,(z) = \int f(x) e^{ixz}\, dx.$$

For any function f on $\mathbb{R}$ put $f_\pm = f \cdot \chi_{\mathbb{R}_\pm}$, where $\mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x > 0\}$. In the case $l_p = 0$ if $\log x = o(\log p(x))$, $x \to \infty$, then

$$\mathcal{F}\big((L_p^1(\mathbb{R}))_+\big) \subset C_A^\infty(\mathbb{C}_+), \qquad \mathcal{F}(L_p^1(\mathbb{R})) \subset C^\infty(\mathbb{R}),$$

in the case $l_p > 0$ if $\log x = o(\log p(x) - l_p x)$, $x \to \infty$, then

$$\mathcal{F}\big((L_p^1(\mathbb{R}))_+\big) \subset C_A^\infty(\mathbb{C}_+ - l_p i), \qquad \mathcal{F}(L_p^1(\mathbb{R})) \subset C_A^\infty\big(\{z : |\operatorname{Im} z| < l_p\}\big),$$

where $C_A^\infty(\Omega)$ is the space of the functions analytic in Ω and infinitely differentiable up to the boundary of Ω.

Beside that, in both cases, if p satisfies (2.1), then $\mathcal{F}(L_p^1(\mathbb{R}))|\mathbb{R} \pm il_p$ are quasianalytic (Δ) classes (for a definition see [**36**]).

Put

$$w(x) = \sup_{r \ge 0} p(r) \exp(-rx), \qquad x > l_p.$$

LEMMA 2.1. (a) *If $(1+x^2)f(x) \in (L_p^\infty(\mathbb{R}))_+$, then $\mathcal{F}f \in A_w^+$.*

(b) *If $F(z)(1+|\operatorname{Re} z|)^2 \in A_w^+$, then $F = \mathcal{F}f$ for some function f, $f \in (L_p^\infty(\mathbb{R}))_+$, which is determined by the equality*

$$\frac{1}{2\pi} \int_{\mathbb{R}+is} F(z) e^{-ixz}\, dz = f(x), \qquad x \in \mathbb{R}_+, \quad s > l_p. \tag{2.2}$$

THE PROOF uses the relation

$$\inf_{x>l_p} w(x)e^{sx} = p(s), \qquad s \ge 0.$$

This is a standard property of the Legendre transform (see [**36**]). The equality (2.2) is a form of the inversion formula for the Fourier transform. ∎

Let c_p be a fixed number, $l_p < c_p < p'(0)/p(0)$, and let the function r be defined (uniquely) by the equalities

$$w(x) = p(r(x))\exp(-xr(x)), \qquad l_p < x \le p'(0)/p(0),$$
$$r(x) = 0, \qquad x > p'(0)/p(0).$$

This function is not necessarily in C^1, so we redefine it on $[c_p, p'(0)/p(0)]$ not increasing it in such a way that the obtained function is in $C^1((l_p,\infty))$. Note, that $r(x)\to\infty$, $x\to l_p+0$,

$$p(s)\exp(-sx) \ge 1, \qquad 0\le s\le r(x). \tag{2.3}$$

To define the generalized Fourier transform we use the construction of restricted Fourier transform (cf. [**13**], see also [**17**]), unlike [**5,7,8**] where a variant of continuation with controlled $\overline{\partial}$ derivative proposed by E. M. Dyn'kin in [**16**] was applied.

For every $f\in\mathcal{K}_p$,

$$\mathcal{K}_p := \big\{f : x\mapsto xf(x)\in L^1_p(\mathbb{R}),\ f\in L^\infty_{1/p}(\mathbb{R})\big\},$$

we define a function $\widetilde{\mathcal{F}}f$ on $(\mathbb{C}_+ + il_p)\cup(\mathbb{C}_- - il_p)$ in the following way:

$$\widetilde{\mathcal{F}}f(z) = \int_{-r(\operatorname{Im} z)}^{\infty} f(x)e^{ixz}\,dx, \qquad \operatorname{Im} z > l_p,$$
$$\widetilde{\mathcal{F}}f(z) = \int_{-\infty}^{r(-\operatorname{Im} z)} f(x)e^{ixz}\,dx, \qquad \operatorname{Im} z < -l_p.$$

The function $\widetilde{\mathcal{F}}f$ is bounded because of (2.3) and analytic for $|\operatorname{Im} z| > p'(0)/p(0)$ by the definition of r. Furthermore, $\widetilde{\mathcal{F}}f$ is continuously differentiable, for some c,

$$\Big|\overline{\partial}(\widetilde{\mathcal{F}}f)(z)\Big| < c\frac{|r'(|\operatorname{Im} z|)|}{w(|\operatorname{Im} z|)}, \qquad \text{where } \overline{\partial} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big), \tag{2.4}$$

and the function $z\mapsto|\overline{\partial}(\widetilde{\mathcal{F}}f)(z)|w(|\operatorname{Im} z|)(1+|z|)^{-2}$ is summable on $\{z : |\operatorname{Im} z| > l_p\}$. This follows from the equality

$$\overline{\partial}(\widetilde{\mathcal{F}}f)(z) = \frac{i}{2}r'(\operatorname{Im} z)f(-r(\operatorname{Im} z))e^{-izr(\operatorname{Im} z)}, \qquad \operatorname{Im} z > l_p, \tag{2.5}$$

the estimate

$$\int_{\operatorname{Im} z>l_p} \left|\bar\partial(\widetilde{\mathcal{F}}f)(z)\right| w(\operatorname{Im} z)(1+|z|)^{-2}\, dm_2(z) \le$$
$$c(p)\int_{l_p} |r'(x)||f(-r(x))|p(r(x))\, dx = c(p)\int^{\infty} |f(-x)|p(x)\, dx < \infty, \tag{2.6}$$

where $c(p)$ depends only on p, and analogous relations in the lower half-plane.

Let $z_0 \in il_p + \mathbb{R}$, $z \in il_p + \mathbb{C}_+$, $z \to z_0$. Then

$$\left|\widetilde{\mathcal{F}}f(z) - \mathcal{F}f(z_0)\right| \le \left|\int_0^\infty f(x)e^{ixz_0}\left(e^{ix(z-z_0)}-1\right)dx\right| +$$
$$\left|\int_{-r(\operatorname{Im} z)}^0 f(x)e^{ixz}\left[1-e^{ix(z_0-z)}\right]dx\right| + \left|\int_{-\infty}^{-r(\operatorname{Im} z)} f(x)e^{ixz_0}\,dx\right|.$$

The first and the third terms tend to 0 because the function $x \mapsto f(x)\exp(l_p|x|)$ is summable. Furthermore, by (2.3),

$$\int_{-r(\operatorname{Im} z)}^0 \left|f(x)\left[1-e^{ix(z_0-z)}\right]\right| e^{-x\operatorname{Im} z}\,dx \le$$
$$\sup_{0\le t\le r(\operatorname{Im} z)} \frac{\exp(t\operatorname{Im} z)}{p(t)} \cdot \int_{-\infty}^0 \left|f(x)\left[1-e^{ix(z_0-z)}\right]\right| p(x)\,dx \le$$
$$\int_{-\infty}^0 |f(x)|\left|1-e^{ix(z_0-z)}\right| p(x)\,dx \to 0 \qquad \text{for } z \to z_0.$$

Thus, we can extend the function $\widetilde{\mathcal{F}}f$ continuously to the set $\{z : -l_p \le \operatorname{Im} z \le l_p\}$, by defining it there to be equal to $\mathcal{F}f$.

Now we impose a more restrictive condition on p: for sufficiently big t,

$$|(\log p)''(t)| \ge \frac{3}{t^2}. \tag{2.7}$$

Then for some $c > 0$, and for $l_p < x \le c_p$,

$$w(x) = \exp\left[\log p(r(x)) - r(x)(\log p)'(r(x))\right] = c\exp\left[-\int_0^{r(x)} s(\log p)''(s)ds\right] \ge cr^3(x),$$
$$|r'(x)| = \frac{1}{|(\log p)''(r(x))|} \le \frac{(r(x))^2}{3}, \qquad x \to l_p + 0, \tag{2.8}$$
$$\frac{|r'(x)|}{w(x)} = O\Big(\frac{1}{r(x)}\Big), \qquad x \to l_p + 0. \tag{2.9}$$

Taking into account estimate (2.4) we obtain that

$$\left|\bar\partial(\widetilde{\mathcal{F}}f)(z)\right| = o(1), \quad \operatorname{Im} z \to l_p + 0. \tag{2.10}$$

Analogously,

$$\left|\partial(\widetilde{\mathcal{F}}f)(z) - \int_{-r(\operatorname{Im} z)}^{\infty} ixf(x)e^{ixz}\,dx\right| = o(1), \qquad \operatorname{Im} z \to l_p + 0.$$

Finally, if $\operatorname{Im} z_0 = l_p$, then by (2.3) and Lebesgue bounded convergence theorem,

$$\lim_{\substack{z\to z_0\\ \operatorname{Im} z > l_p}} \int_{-r(\operatorname{Im} z)}^{\infty} ixf(x)e^{ixz}\,dx = \int_{-\infty}^{\infty} ixf(x)e^{ixz_0}\,dx.$$

Thus, $\widetilde{\mathcal{F}}f \in C^1(\mathbb{C})$. The function $\widetilde{\mathcal{F}}f$ is called *the generalized Fourier transform* of f. Of course, it depends on the choice of the weight function p. We indicate what weight is taken to construct $\widetilde{\mathcal{F}}f$ when this is not clear from the context.

We can reconstruct f from $\widetilde{\mathcal{F}}f$ as follows. For every $u \in C_0^\infty(\mathbb{R})$, $f \in \mathcal{K}_p$, if $F = \widetilde{\mathcal{F}}f \cdot \mathcal{F}u$, then

$$(\widetilde{\mathcal{F}}^{-1}F)(x) := \lim_{s\to l_p+0} \frac{1}{2\pi}\int_{\mathbb{R}+is} F(z)e^{-ixz}\,dz = (f*u)(x), \qquad x\in\mathbb{R}, \tag{2.11}$$

and an analogous equality holds for $s \to -l_p - 0$. This statement follows from the relation

$$\frac{1}{2\pi}\int_{\mathbb{R}+is} F(z)e^{-ixz}\,dz = \int_{-r(s)}^{\infty} f(t)u(x-t)\,dt \to (f*u)(x), \qquad s\to l_p+0.$$

Let us verify the multiplicative property of the generalized Fourier transform: it maps convolutions into products.

LEMMA 2.2. *If $f \in \mathcal{K}_p$, $g \in (L_s^\infty(\mathbb{R}))_+$, where $s(x) = p(x)(1+|x|)^{-4}$, $u \in C_0^\infty(\mathbb{R})$, $F = \widetilde{\mathcal{F}}f\cdot\mathcal{F}u\cdot\mathcal{F}g$, then*

(a) *the function $G : z \mapsto (1+|z|)^2\bar\partial F(z)$ is summable and bounded in the half-plane $\mathbb{C}_+ + il_p$.*

(b) $(\widetilde{\mathcal{F}}^{-1}F)(x) = \lim_{s\to l_p+0}\frac{1}{2\pi}\int_{\mathbb{R}+is}F(z)e^{-ixz}\,dz = (f*u*g)(x)$.

PROOF: (a) The summability of G is a consequence of (2.6) and the estimate

$$\int_0^\infty p(x)(1+x)^{-4}e^{-tx}\,dx < w(t).$$

Moreover, we can obtain that for some $c>0$

$$\int_0^\infty p(x)(1+x)^{-4}e^{-tx}\,dx < c\frac{w(t)}{(r(t))^2}, \qquad l_p < t \le c_p. \tag{2.12}$$

To get it, we verify that the expression

$$\sup_{0<s<\infty} \exp\bigl[-\log p(r(t)) + tr(t) + 2\log r(t) + \log p(s) - ts - 2\log(1+s)\bigr]$$

is bounded uniformly in t, $l_p < t \le c_p$, and this follows from (2.7).

Now we get that G is bounded as a consequence of (2.4), (2.8) and (2.12).

(b) This assertion is verified in the same way as (2.11). It is sufficient to indicate that

$$f_x \overset{L_p^1(\mathbb{R})}{\longrightarrow} f, \quad x\to-\infty, \qquad \text{where} \quad f_x(t) = \begin{cases} f(t), & t\ge x,\\ 0, & t<x.\end{cases} \qquad \blacksquare$$

Let us show that under some mild conditions every asymptotically holomorphic function is uniquely determined by its inverse (generalized) Fourier transform.

Lemma 2.3. *If $F \in C^1(\mathbb{C}_+)$, $(1+|z|^2)\overline{\partial}F(z) \in L^\infty(\Omega)\cap L^1(\Omega)$, where $\Omega = \{z : 0 < \operatorname{Im} z < 1\}$, and for every $s > 0$*

$$(1+|z|^2)F(z) \in L^\infty\big(\Omega\cap(\mathbb{C}_+ + is)\big),$$

then the limit

$$\lim_{s\to+0}\int_{\mathbb{R}+is} F(z)e^{-ixz}\,dz. \tag{2.13}$$

exists and is finite for every $x \in \mathbb{R}$.

Moreover, if these limits are equal to zero for all x, then F extends to $\mathbb{R}$ by 0 continuously, and the extended function F satisfies the conditions

$$F \in C(\overline{\mathbb{C}}_+), \qquad F(z)(1+|z|^2) \in L^\infty(\Omega).$$

Proof: Put

$$F_1(z) = \frac{1}{\pi}\int_\Omega \frac{\overline{\partial}F(\xi)}{z-\xi}\,dm_2(\xi) - \frac{1}{\pi(z-i)}\int_\Omega \overline{\partial}F(\xi)\,dm_2(\xi), \qquad z \in \mathbb{C}\setminus\{i\},$$
$$F_2(z) = F(z) - F_1(z), \qquad z \in \mathbb{C}_+\setminus\{i\}.$$

These functions are well defined and continuous because of the conditions on F and $\overline{\partial}F$. Analogously we obtain that the integrals $\int_{\mathbb{R}+is} F_j(z)e^{-ixz}\,dz$ converge for $j = 1$, $s \in \mathbb{R}$, and for $j = 2$, $0 < s < 1$. Moreover,

$$\lim_{s\to+0}\int_{\mathbb{R}+is} F_1(z)e^{-ixz}\,dz = \int_{\mathbb{R}} F_1(z)e^{-ixz}\,dz,$$

because

$$\Big(\int_{\mathbb{R}+is} - \int_{\mathbb{R}}\Big)F_1(z)e^{-ixz}\,dz = -2i\int_{0<\operatorname{Im} z<s}\overline{\partial}F_1(z)e^{-ixz}\,dm_2(z), \qquad 0 < s < 1.$$

The integrals $\int_{\mathbb{R}+is} F_2(z)e^{-ixz}\,dz$ do not depend on s, $0 < s < 1$, since F_2 is analytic in Ω. Thus, the existence of the limits (2.13) is proved.

If these limits are identically equal to zero, then

$$\int_{\mathbb{R}+is} F_2(z)e^{-ixz}\,dz = -\int_{\mathbb{R}} F_1(z)e^{-ixz}\,dz, \qquad 0 < s < 1.$$

Since $F_1 \in C(\overline{\mathbb{C}}_-)\cap H^\infty(\mathbb{C}_-)\cap L^1(\mathbb{R})$, we have

$$\int_{\mathbb{R}} F_1(z)e^{-ixz}\,dz = 0, \qquad \int_{\mathbb{R}+i/2} F_2(z)e^{-ixz}\,dz = 0, \qquad x \ge 0.$$

As a consequence, the function F_2, analytic in Ω, extends analytically across $\mathbb{R}$ to a function in $H^\infty(\mathbb{C}_- + i/2)$. Beside that, $F_2(z)(1+|z|^2) \in L^\infty(\mathbb{R}+i/2)$. As a result,

$$F \in C(\overline{\mathbb{C}}_+), \qquad F(z)(1+|z|^2) \in L^\infty(\Omega),$$
$$\int_{\mathbb{R}} F(z)e^{-ixz}\,dz = \lim_{s\to+0}\int_{\mathbb{R}+is} F(z)e^{-ixz}\,dz = 0, \qquad x \in \mathbb{R},$$

therefore $F|\mathbb{R} \equiv 0$. ∎

We complete this section with a statement associating the asymptotics of the convolution of two functions and boundedness of the product of their Fourier transforms.

THEOREM 2.4. *Let* $f \in \mathcal{K}_p$, $g \in (\mathcal{M}_p^*)_+$.

(a) *If*

$$(f * g)(x) = O\big(|x|^{-7} \exp(l_p x)\big), \qquad |x| \to \infty,$$

then for every $u \in C_0^\infty(\mathbb{R})$,

$$\widetilde{\mathcal{F}} f \cdot \mathcal{F} g \cdot \mathcal{F} u \in L^\infty\big(\{z : l_p < \operatorname{Im} z < l_p + 1\}\big) \cap C(\overline{\mathbb{C}}_+ + il_p),$$
$$(\widetilde{\mathcal{F}} f \cdot \mathcal{F} g)(z) = \mathcal{F}(f * g)(z), \qquad z \in \mathbb{R} + il_p.$$

(b) *If* $\widetilde{\mathcal{F}} f \cdot \mathcal{F} g \in L^\infty(\mathbb{C}_+ + il_p) \cap C(\overline{\mathbb{C}}_+ + il_p)$, *then for every* $u \in C_0^\infty(\mathbb{R})$,

$$\frac{1}{2\pi} \int_{\mathbb{R}+il_p} (\widetilde{\mathcal{F}} f \cdot \mathcal{F} g \cdot \mathcal{F} u)(z) e^{-ixz}\, dz = (f * g * u)(x).$$

PROOF: (a) Let us denote by $F_1(z)$ the generalized Fourier transform of the function $(f * g)(x)\exp(-l_p x)$ for the weight $p_0(x) = (1 + |x|)^4$ (which satisfies (2.7)), by $F(z)$ the function

$$(F_1(z - il_p) - (\widetilde{\mathcal{F}} f \cdot \mathcal{F} g)(z)) \cdot \mathcal{F} u(z).$$

Then (2.11) implies the existence of the limits

$$\lim_{s \to +0} \frac{1}{2\pi} \int_{\mathbb{R}+is} F_1(z) \mathcal{F} u(z + il_p) e^{-ixz}\, dz = (f * g * u)(x) \exp(-l_p x), \qquad x \in \mathbb{R}.$$

Lemma 2.2 (b) implies the existence of the limits

$$\lim_{s \to +0} \frac{1}{2\pi} \int_{\mathbb{R}+is} (\widetilde{\mathcal{F}} f \cdot \mathcal{F} g)(z + il_p) \mathcal{F} u(z + il_p) e^{-ixz}\, dz = (f * g * u)(x) \exp(-l_p x), \qquad x \in \mathbb{R}.$$

Thus,

$$\lim_{s \to +0} \frac{1}{2\pi} \int_{\mathbb{R}+is} F(z + il_p) e^{-ixz}\, dz = 0, \qquad x \in \mathbb{R}.$$

The function $F(\cdot + il_p)$ satisfies the conditions of Lemma 2.3, because both the functions $z \mapsto F_1(z - il_p) \cdot \mathcal{F} u(z)$ and $\widetilde{\mathcal{F}} f \cdot \mathcal{F} g \cdot \mathcal{F} u$ satisfy them as a consequence of estimate (2.10) and Lemma 2.2 (a) respectively. Therefore,

$$F \in C(\overline{\mathbb{C}}_+ + il_p), \qquad F\big|\mathbb{R} + il_p \equiv 0, \qquad F \in L^\infty\big(\{z : l_p < \operatorname{Im} z < l_p + 1\}\big).$$

Since $F_1 \in C^1(\mathbb{C})$, $F_1\big|\mathbb{R} = \mathcal{F}(f * g)(\cdot + il_p)\big|\mathbb{R}$, we get our assertion.

(b) By Lemma 2.2 (b), if $F = \widetilde{\mathcal{F}} f \cdot \mathcal{F} g \cdot \mathcal{F} u$, then the following equality holds:

$$\lim_{s \to l_p + 0} \frac{1}{2\pi} \int_{\mathbb{R}+is} F(z) e^{-ixz}\, dz = (f * g * u)(x).$$

Lemma 2.2 (a) and the conditions of the theorem imply that the limit in the left-hand side is equal to

$$\frac{1}{2\pi} \int_{\mathbb{R}+il_p} F(z) e^{-ixz}\, dz.$$

The theorem is proved. ∎

3. Criteria of orthogonality.

An element f of $\mathcal{L}_p$ or $\mathcal{M}_p$ is said to be *mean periodic* if $f * g = 0$ for some non-zero functional g. An immediate implication of Theorem 2.4 is the following result associating the property of mean periodicity with the behavior of the Fourier transforms.

Here we suppose that for every $c > 0$ there exists x_c such that

$$|(\log p)''(x)| > \frac{c}{x^2}, \qquad x \geq x_c. \tag{3.1}$$

Under this condition we obtain by an argument similar to that used in the proof of relation (2.9) that

$$\frac{1}{|r(x+l_p)|} = o(x), \qquad x \to +0,$$

and for every $n \geq 0$,

$$\frac{|r^2(x+l_p)|}{w(x+l_p)} = o(x^n), \qquad x \to +0. \tag{3.2}$$

Theorem 3.1. *Let the weight* p *satisfy conditions* (2.1) *and* (3.1). *If* $f \in \mathcal{K}_p \setminus \{0\}$, $g \in \mathcal{M}_p^*$, *then* $f * g = 0$ *if and only if the function* $\mathcal{F}g$ *which coincides with* $\mathcal{F}g_+$ *and* $-\mathcal{F}g_-$ *on* $\mathbb{C}_+ + il_p$ *and* $\mathbb{C}_- - il_p$ *respectively, can be meromorphically extended to* $\mathbb{C}$, *the divisor of its poles is subordinated to the divisor of zeros of* $\mathcal{F}f$, *and*

$$\mathcal{F}g \cdot \widetilde{\mathcal{F}}f \in L^\infty(\mathbb{C}).$$

In this situation

$$\mathcal{F}(f * g_+)(z) = -\mathcal{F}(f * g_-)(z) = \mathcal{F}f \cdot \mathcal{F}g(z), \qquad z \in \mathbb{R} + [-l_p, l_p]i.$$

This meromorphic function $\mathcal{F}g$ is called *the Carleman transform of* g.

In the proof of Theorem 3.1 and later on we use extensively the following statement. We consider bounded simply connected domains D.

Theorem (on harmonic estimation). *Let* f *be a function analytic and bounded in* D *and continuous up to* ∂D. *For* $z \in D$,

$$\log|f(z)| \leq \int_{\partial D} \log|f(\zeta)|\, \omega(z, d\zeta, D),$$

where $\omega(z, d\zeta, D)$ *is the harmonic measure in* D *as seen from* z.

A corollary of this theorem claims that under the same conditions if $|f(\zeta)| \leq M_1$ on a Borel set $E \subset \partial D$, $|f(\zeta)| \leq M_2$ on $\partial D \setminus E$, then

$$|f(z)| \leq M_1^{\omega(z,E,D)} M_2^{1-\omega(z,E,D)}.$$

We shall refer to this result as the theorem on two constants (see, for instance, [**32**, p. 257]). Sometimes, abusing the terminology a little bit, we use the same name for analogous statements involving several subsets of ∂D.

Given a C^1-smooth function f in a domain D, we denote by $H_{f,D}$ the function

$$H_{f,D}(z) = f(z) - \frac{1}{\pi}\int_D \frac{\overline{\partial} f(\zeta)}{z-\zeta}\, dm_2(\zeta). \tag{3.3}$$

Proof of Theorem 3.1: If $f * g = 0$, then as a consequence of (3.1),

$$(f * g_+)(x) = -(f * g_-)(x) = O(|x|^{-7}e^{-l_p|x|}), \qquad |x| \to \infty.$$

Now, applying Theorem 2.4 (a), we deduce that for every $u \in C_0^\infty(\mathbb{R})$ the function $\widetilde{\mathcal{F}} f \cdot \mathcal{F} g_+ \cdot \mathcal{F} u$ extends continuously across the strip $\Omega = \{z : |\operatorname{Im} z| \le l_p\}$ and analytically in its interior to the function $-\widetilde{\mathcal{F}} f \cdot \mathcal{F} g_- \cdot \mathcal{F} u$ in the half-plane $\mathbb{C}_- - il_p$. This continuation is bounded in $\Omega_1 = \{z : |\operatorname{Im} z| < l_p + 1\}$. In addition, the restriction of this continuation to $\overline{\Omega}$ coincides with the following products of Fourier transforms:

$$\mathcal{F}(f * g_+) \cdot \mathcal{F} u = -\mathcal{F}(f * g_-) \cdot \mathcal{F} u.$$

These facts imply the meromorphic continuability of $\mathcal{F} g_+$ to $\mathcal{F} g_-$, the restriction on the divisor of poles of this continuation $\mathcal{F} g$, and the equality

$$\mathcal{F}(f * g_+) = -\mathcal{F}(f * g_-) = \mathcal{F} f \cdot \mathcal{F} g$$

in the strip $\overline{\Omega}$. Here we use quasianalyticity of the function $\mathcal{F} f$. The only thing we need to verify is the restriction on the poles of $\mathcal{F} g$. Our argument is as follows. The zeros of $\mathcal{F} f$ on $\overline{\Omega}$ are of at most finite order. Let $\mathcal{F} f$ have a zero of order n at z_0, $\operatorname{Im} z_0 = l_p$. We know that $\mathcal{F} g$ is analytic in a deleted neighborhood of z_0,

$$B = \{z : 0 < |z - z_0| < \varepsilon\}.$$

If ε is small enough, then for some c_1, c_2

$$0 < c_1 \le \frac{|\mathcal{F} f(z)|}{|z - z_0|^n} \le c_2, \qquad z \in B \cap \Omega. \tag{3.4}$$

Let us suppose that the function $\mathcal{F} g(z)(z - z_0)^n$ is unbounded in B. Then it should be large on some connected set γ in $B \setminus \Omega$, whose radial projection on the ray $z_0 + \mathbb{R}_+$ contains a neighborhood of z_0: for some $c > 0$

$$\log |\mathcal{F} g(z)| > \frac{c}{|z - z_0|}, \qquad z \in \gamma.$$

Indeed, the function $z \mapsto \mathcal{F} g(z_0 + 1/z) z^{-n}$ is analytic and unbounded on $\mathbb{C}_+$ and bounded on $\partial(\mathbb{C}_+ \setminus R\mathbb{D})$ for big R, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Now our statement follows from a standard Phragmen–Lindelöf type argument.

Since $\widetilde{\mathcal{F}}f \cdot \mathcal{F}g$ is bounded near z_0, for some c_1,

$$\log|\widetilde{\mathcal{F}}f(z)| < c_1 - \frac{c}{|z-z_0|}, \qquad z \in \gamma.$$

Put $D = z_0 + \delta\,\mathbb{D}$, $0 < \delta < \varepsilon$, and consider the function $H = H_{\widetilde{\mathcal{F}}f,D}$, defined by (3.3), analytic in D. We obtain that H is small on γ. Namely, by (2.4) and (2.8) we have for small $\delta > 0$ and some $c > 0$,

$$\log|H(z)| < -c\delta_* := -c\min\left(\frac{1}{\delta}, \log\frac{w(\delta+l_p)}{(r(\delta+l_p))^2}\right), \qquad z \in \gamma.$$

Now, by the theorem on two constants applied to H in the connected component of $D\setminus\gamma$ containing $D\cap\Omega$, we get for some $c > 0$,

$$\log|H(z)| < -c\delta_*, \qquad |z-z_0| < \delta/2.$$

Therefore, for some $c_1 > 0$,

$$\log|\widetilde{\mathcal{F}}f(z)| < -c_1\delta_*, \qquad |z-z_0| < \delta/2,$$

that contradicts to (3.4) for small δ, because by (3.2) for every $n > 0$ we have $w(x)(x-l_p)^n(r(x))^{-2} \to \infty$, $x \to l_p + 0$. Thus, $\mathcal{F}g(z)(z-z_0)^n$ is bounded in B, and $\mathcal{F}g$ has at z_0 a pole of degree at most n.

Note that the function F, $F = \widetilde{\mathcal{F}}f \cdot \mathcal{F}g$, is bounded in $\mathbb{C}\setminus\Omega_1$ because $\widetilde{\mathcal{F}}f$ and $\mathcal{F}g_\pm$ are bounded there. We already know that $F\cdot\mathcal{F}u$ is bounded in Ω_1.

Lemma 3.2. *If Ω is a strip parallel to the real line, F is an unbounded C^1-function in Ω, bounded on $\partial\Omega$ and with bounded $\overline{\partial}$ derivative, then for some $c > 0$,*

$$\max_{|\operatorname{Re} z|=x} |F(z)| > c\exp\exp cx.$$

Proof: Applying the theorem on two constants to $D = \{z \in \Omega : a - x < \operatorname{Re} z < a + x\}$ and $H = H_{F,D}$ we get that for big fixed a, for sufficiently big $\max_{\operatorname{Re} z = x}|F(z)|$ and for some $c > 1$,

$$\max_{|\operatorname{Re} z - x| = a} \log|F(z)| > c \max_{\operatorname{Re} z = x} \log|F(z)|.$$

Iterating this estimate we obtain our assertion. ∎

Let us return to the proof of the theorem. If F is not bounded in Ω_1, then by Lemma 3.2, the growth of F is sufficiently large. This growth cannot be compensated by the decay of $\mathcal{F}u$. Thus, F is bounded on Ω_1, $F \in L^\infty(\mathbb{C})$.

Let us turn now to the opposite implication. For every $u \in C_0^\infty(\mathbb{R})$,

$$(f * g_+ * u)(x) = \frac{1}{2\pi}\int_{\mathbb{R}+il_p} (\mathcal{F}f\cdot\mathcal{F}g_+\cdot\mathcal{F}u)(z)e^{-ixz}\,dz =$$
$$-\frac{1}{2\pi}\int_{\mathbb{R}-il_p} (\mathcal{F}f\cdot\mathcal{F}g_-\cdot\mathcal{F}u)(z)e^{-ixz}\,dz = -(f * g_- * u)(x).$$

The middle equality follows from the inclusion

$$\mathcal{F}f \cdot \mathcal{F}g \in C(\Omega) \cap H^\infty(\operatorname{int}\Omega),$$

the other ones follow from Theorem 2.4 (b). Therefore, $(f*g)*u = 0$ for every $u \in C_0^\infty(\mathbb{R})$, $f*g=0$. ∎

The implication

$$\bigcap_{f:f*g=0} \{z : \mathcal{F}f(z) = 0\} = \emptyset \Longrightarrow \mathcal{F}g \in A(\mathbb{C})$$

can be found in many papers, as was mentioned above. Usually it is proved via certain localization of the functions $\mathcal{F}f$ and $\mathcal{F}g_\pm$. In the quasianalytic case this method cannot be applied, and the corresponding result was obtained by Y. Domar in 1975 [**12**] using the technique of commutative Banach algebras. The equality

$$\mathcal{F}(f*g_+)(z) = (\mathcal{F}f \cdot \mathcal{F}g)(z) = -\mathcal{F}(f*g_-)(z), \qquad z \in \mathbb{R} + [-l_p, l_p]i,$$

could be demonstrated (at least in the analytic-non-quasianalytic case) with the use of an argument similar to that in [**28**]. However, necessary and sufficient conditions for the equality $f*g=0$ in the quasianalytic case have not been obtained until now. The reason is that the information on the behavior of $\mathcal{F}f$ and $\mathcal{F}g$ only in Ω is insufficient. We show it by using the following construction.

Example 3.3. *Let p be a sufficiently regular (analitically-) quasianalytic weight. There exist $f \in \mathcal{L}_p$, $g \in \mathcal{M}_p^*$ such that the functions $\mathcal{F}g_+$ and $-\mathcal{F}g_-$ extend to an entire function $\mathcal{F}g$,*

$$(\mathcal{F}g \cdot \mathcal{F}f)(z) = O(|z|^{-n}), \qquad |z| \to \infty, \quad \operatorname{Im} z \in [-l_p, l_p], \quad n \geq 0,$$

but $f*g \neq 0$.

(A similar example is sketched in [**25**], see also [**37**, Sections 17, 18]).

Proof: Let α be a sufficiently regular positive function, rapidly decreasing to 0 for $x \to \infty$,

$$\Omega = \{z : \operatorname{Re} z > 0,\, l_p + \alpha(\operatorname{Re} z)/2 < \operatorname{Im} z < l_p + \alpha(\operatorname{Re} z)\},$$
$$\Omega_0 = \{z : \operatorname{Re} z > 0,\, |\operatorname{Im} z| < \pi\},$$

and let ψ be a conformal map from Ω onto Ω_0 such that $\psi(\infty) = \infty$,

$$\varphi(z) = \frac{1}{(z+1)^2} \exp\exp\psi(z), \qquad \varphi \in C(\overline{\Omega}) \cap A(\Omega),$$
$$G(z) = \int_{\partial\Omega} \frac{\varphi(\xi)}{z-\xi}\, d\xi, \qquad G \in A(\mathbb{C} \setminus \overline{\Omega}).$$

We denote by the same symbol G the analytic continuation of G into Ω. Then G is an entire function. For $z \in \Omega$ we have

$$G(z) = -2\pi i \varphi(z) + \int_{\partial\Omega} \frac{\varphi(\xi)}{z-\xi}\, d\xi.$$

Furthermore, by Warschawski's theorem (see [**40**]), if α is sufficiently regular we have

$$\sup_{\operatorname{Im} z > l_p + t} |\varphi(z)| < \exp\exp\Big[4\pi \int^{\alpha^{-1}(t)} \frac{dx}{\alpha(x)}\Big] =: k(\alpha^{-1}(t)).$$

Put

$$g(t) = i \int_{\partial\Omega} e^{-izt} \varphi(z)\, dz. \tag{3.5}$$

Then $\mathcal{F} g_{\pm} = \pm G$ in $\mathbb{C}_{\pm} \pm l_p i$. To verify that $g \in \mathcal{L}_p^*$ we deform the integration contour in (3.5) to $\Gamma = \partial(\Omega \cap (\mathbb{C}_- + (l_p + t)i))$ and obtain

$$|g(x)| \le c \inf_{t>0} \Big\{ \sup_{\operatorname{Im} z = l_p + t} |\varphi(z)| \exp[(l_p + t)x] \Big\}, \qquad x \ge 0,$$

and an analogous inequality for $x \le 0$. To get that

$$\inf_{t>0} \Big\{ \sup_{\operatorname{Im} z = l_p + t} |\varphi(z)| \exp[(l_p + t)x] \Big\} \le cp(x)$$

it is enough to know that

$$\sup_{\operatorname{Im} z = l_p + t} |\varphi(z)| \le cw(l_p + t)$$

which follows if α satisfies the inequality

$$4\pi \int^{\alpha^{-1}(t)} \frac{dx}{\alpha(x)} \le N(l_p + t), \qquad N(t) = \log\log w(t).$$

In particular, it holds if $4\pi(\alpha^{-1})'(t) \ge tN'(l_p + t)$. This inequality shows that such a function exists if the integral $\int_{l_p} N(x)\, dx$ diverges, which is equivalent (see Lemma 5.6, [**3**]) to the quasianalyticity of the weight p (for sufficiently regular p). To find $g \in \mathcal{M}_p^*$ we repeat the process described above for sufficiently regular p_1 such that $\mathcal{L}_{p_1}^* \subset \mathcal{M}_p^*$ (see also Lemma 3.5).

Since $|G|$ is estimated from above on $\mathbb{R} + l_p i$ by $\max_{s>0}\{c/(\alpha(s)k(s))\}$, for sufficiently regular α we obtain that G is bounded on $\overline{\mathbb{C}}_- + l_p i$. Now, if we put $f(x) = \exp(-x^2)$, then the functions f and g satisfy the conditions of Example, but by Theorem 3.1, $f * g \ne 0$. Indeed, $\overline{\partial}(\widetilde{\mathcal{F}} f)$ and $\widetilde{\mathcal{F}} f$ are bounded. Since $\widetilde{\mathcal{F}} f(x) = \mathcal{F} f(x) = \sqrt{\pi} e^{-x^2/4}$, $x \in \mathbb{R}$, for fixed p and sufficiently small α, using the theorem on two constants we obtain that $|\widetilde{\mathcal{F}} f(z)| \ge \exp(-c(\operatorname{Re} z)^2)$ on Ω. Therefore, for sufficiently small α the product $\widetilde{\mathcal{F}} f \cdot \mathcal{F} g$ is unbounded in Ω. ∎

To be in a position to treat convolution equations on the half-line, we need the following result.

Theorem 3.4. *Let the weight p satisfy conditions (2.1) and (3.1), $f \in (\mathcal{K}_p)_+ \setminus \{0\}$, $g \in (\mathcal{M}_p^*)_-$. Then $(f * g)_- = 0$ if and only if the function $\mathcal{F}g$ extends meromorphically to $\mathbb{C}$, the divisor of its poles is subordinated to the divisor of zeros of $\mathcal{F}f$, and*

$$\mathcal{F}g \cdot \widetilde{\mathcal{F}}f \in L^\infty(\mathbb{C}).$$

In this situation

$$\mathcal{F}(f * g)(z) = (\mathcal{F}f \cdot \mathcal{F}g)(z), \qquad z \in \overline{\mathbb{C}}_+ - il_p.$$

Proof: As a consequence of (3.1),

$$(f * g)(x) = O(|x|^{-7} e^{-l_p x}), \qquad x \to +\infty.$$

By Theorem 2.4 (a), modified a little bit, for every $u \in C_0^\infty(\mathbb{R})$ the function $\widetilde{\mathcal{F}}f \cdot \mathcal{F}g \cdot \mathcal{F}u$ extends continuously across the line $\mathbb{R} - il_p$ to the function $F = \mathcal{F}(f * g) \cdot \mathcal{F}u$ which is analytic in the half-plane $\mathbb{C}_+ - l_p i$. Now the function $F/(\mathcal{F}u \cdot \mathcal{F}f)$ is the meromorphic continuation of the function $\mathcal{F}g$. (Again we use the fact that $\mathcal{F}f$ is quasianalytic). Thus, we get the equality

$$\mathcal{F}(f * g)(z) = (\mathcal{F}f \cdot \mathcal{F}g)(z), \qquad z \in \overline{\mathbb{C}}_+ - il_p,$$

and the property of the poles' divisor of $\mathcal{F}g$ we are interested in. Finally, the boundedness of the function $\widetilde{\mathcal{F}}f \cdot \mathcal{F}g$ is proved in just the same way as in Theorem 3.1.

In the converse direction, for every $u \in C_0^\infty(\mathbb{R})$, if $F = \mathcal{F}f \cdot \mathcal{F}g \cdot \mathcal{F}u$, then by Theorem 2.4 (b)

$$(f * g * u)(x) = \frac{1}{2\pi} \int_{\mathbb{R} - l_p i} F(z) e^{-ixz}\, dz.$$

Now for each u such that $\operatorname{supp} u \subset \mathbb{R}_+$ we have

$$F(z) e^{-ixz} \in H^\infty(\mathbb{C}_+ - il_p) \cap L^1(\mathbb{R} - il_p), \qquad x \le 0,$$

therefore, $\operatorname{supp}(f * g * u) \subset \mathbb{R}_+ \cup \{0\}$. As a result, $(f * g)_- = 0$. ∎

Lemma 3.5. *If $x = o(\varphi(x))$, $\zeta(x) \to \infty$ as $x \to \infty$, then there exists a C^2-smooth convex function ψ such that $\psi'(x) \le \zeta(x)$, $x = o(\psi(x))$, $\psi(x) = o(\varphi(x))$, $\psi''(x) = O(1)$, $x \to \infty$.*

Proof: This statement is rather standard. Put $\psi_0(0) = \psi_0'(0) = 0$,

$$\psi_0''(x) = \begin{cases} 1, & \text{if} \quad \psi_0'(x) < \min\{\sqrt{\inf_{y \ge x} \varphi(y)/y}, \inf_{y \ge x} \zeta(y)\}, \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$0 \le \psi_0''(x) \le 1, \qquad \min\left\{\sqrt{\inf_{y \ge x} \varphi(y)/y}, \inf_{y \ge x} \zeta(y)\right\} \ge \psi_0'(x) \to \infty, \quad x \to \infty,$$

$$\psi_0(x) = O\left(x \sqrt{\inf_{y \ge x} \varphi(y)/y}\right), \qquad x = o(\psi_0(x)), \quad x \to \infty.$$

The function $\psi(x) = \int_{x-1}^x \psi_0(x)\, dx$ satisfies all the conditions of the lemma. ∎

This lemma helps us to prove a variant of Theorem 3.1 for the pairs of spaces $\left(\mathcal{L}_p, \mathcal{L}_p^*\right)$ and $\left(\mathcal{M}_p, \mathcal{M}_p^*\right)$.

THEOREM 3.6. *Let $\mathfrak{A}$ be one of the spaces $\mathcal{L}_p$ and $\mathcal{M}_p$, the weight p satisfy conditions (2.1) and (3.1) and $f \in \mathfrak{A} \setminus \{0\}$, $g \in \mathfrak{A}^*$. Pick a weight $\widehat{p}$ satisfying (2.1) and (3.1) and such that $f \in \mathcal{L}_{\widehat{p}}$, $g \in M^*_{\widehat{p}}$. (The existence of such a weight follows from Lemma 3.5). Then $f * g = 0$ if and only if the function $\mathcal{F}g$ which coincides with $\mathcal{F}g_+$ and $-\mathcal{F}g_-$ on $\mathbb{C}_+ + il_p$ and $\mathbb{C}_- - il_p$ respectively, meromorphically extend to $\mathbb{C}$, the divisor of its poles is subordinated to the divisor of zeros of $\mathcal{F}f$, and for every $u \in C_0^\infty(\mathbb{R})$,*

$$\mathcal{F}g \cdot \widetilde{\mathcal{F}}(f * u) \in L^\infty(\mathbb{C}),$$

where the generalized Fourier transform $\widetilde{\mathcal{F}}(f * u)$ is taken with respect to the weight $\widehat{p}$. In this situation

$$\mathcal{F}(f * g_+)(z) = -\mathcal{F}(f * g_-)(z) = \mathcal{F}f \cdot \mathcal{F}g(z), \qquad z \in \mathbb{R} + [-l_p, l_p]i.$$

REMARK 3.7: Theorem 3.4 extends analogously.

REMARK 3.8: Let us sketch how to find $\widehat{p}$ mentioned in Theorem 3.6, for example, in the case $\mathfrak{A} = \mathcal{M}_p$. Put

$$\varphi(x) = \log \min\Big(\frac{p(\exp x)}{|g(\exp x)|}, \frac{p(-\exp x)}{|g(-\exp x)|}\Big), \qquad \zeta(x) = -(1/2)(\log p)''(\exp x) \exp 2x.$$

Using Lemma 3.5 we find the function ψ and put

$$\widehat{p}\,(x) = p(x) \exp(-\psi(\log^+ |x|)).$$

The weight $\widehat{p}$ satisfies (2.1) and (3.1). For an arbitrary $u \in C_0^\infty(\mathbb{R})$ if $f_1 = f * u$, then $f \in \mathcal{L}_{\widehat{p}}$, $f_1 \in \mathcal{K}_{\widehat{p}}$, $g \in \mathcal{M}^*_{\widehat{p}}$, and we are in the conditions of Theorem 3.1.

4. ASYMPTOTICS OF QUASIANALYTIC AND ENTIRE FUNCTIONS. DISCUSSION.

Theorems 3.1 and 3.6 make it clear that it is important to have a description of the asymptotics of quasianalytically smooth functions on the line or on the strip and entire functions which are the Carleman transforms, to be able to investigate primary ideals at ∞ in $\mathcal{L}_p$ and $\mathcal{M}_p$.

The Fourier transform turns elements of $L^1_p(\mathbb{R})$ into functions in $C\{M_n\}(\mathbb{R})$,

$$C\{M_n\}(\mathbb{R}) := \Big\{f \in C^\infty(\mathbb{R}) : \sup_{n,x} \frac{|f^{(n)}(x)|}{M_n} < \infty\Big\}, \qquad M_n = \sup_{t>0} \frac{t^n}{p(t)}.$$

The first results on the asymptotics of these functions in the quasianalytic case were obtained in [**29,30**]. The limit rate of the decay of quasianalytically smooth functions on the real line was written down in [**34**]. A proof is presented in [**19**].

Furthermore, the limit rate of the decay for functions smooth (not necessarily quasianalytically smooth) on a closed strip and analytic inside it was obtained independently in [**39**]. Let us cite some results of this work (Theorems I and III):

Suppose that the weight $p(x) = p(|x|)$ increases for $x \ge 0$, $p(0) = 1$, the function $x \mapsto (\log p(x))/x$ decreases for $x \ge 0$, the function $\sup_{y\in\mathbb{R}} p(x+y)/p(y)$ is locally bounded, and

$$\int_{\mathbb{R}} \frac{\log p(x)}{1+x^2}\, dx = \infty.$$

THEOREM A. (a) *If $f \in L^1_p(\mathbb{R})$, $f \ne 0$, then*

$$\liminf_{x\to\infty} \big(h(-\log|\mathcal{F}f(x)|) - x\big) < \infty,$$

where

$$h(y) = \frac{2}{\pi}\int_0^y \frac{\log p(x)}{1+x^2}\, dx.$$

(b) *There exists $f \in L^1_p(\mathbb{R})$ such that*

$$\liminf_{x\to\infty} \big(h(-\log|\mathcal{F}f(x)|) - x\big) = 0.$$

If the function $x \mapsto g(x)(x^2+1)$ belongs to $L^\infty_p(\mathbb{R})$, and the Carleman transform of g, $\mathcal{F}g$, is an entire function, then for some $c > 0$,

$$\log^+\log^+|\mathcal{F}g(z)| \le c + \log^+ \sup_{x\ge 0}\big[\log p(x) - x|\operatorname{Im} z|\big] =: c + N(|\operatorname{Im} z|).$$

Two well-known results about the asymtotical behavior of entire functions G satisfying the estimate

$$\log^+\log^+|G(z)| < N(|\operatorname{Im} z|), \qquad z \in \mathbb{C},$$

are, in a sense, limite cases of the relation interesting for us.

THEOREM B. (The Phragmen-Lindelöf theorem for the strip) *If a function G, $G \in A(\mathbb{C}) \cap A_M$, $M = \exp\exp N$, is unbounded in the right half-plane, and*

$$N(t) = \begin{cases} 1, & t \ge \pi/2, \\ \infty, & t < \pi/2, \end{cases}$$

then

$$\lim_{x\to\infty} \sup_y \big[\log^+\log^+|G(x+iy)| - x\big] > -\infty.$$

THEOREM C. (The log-log theorem of Levinson) *If N decreases,*

$$\int_0 N(t)\, dt < \infty,$$

and $G \in A(\mathbb{C}) \cap A_M$, $M = \exp\exp N$, then G is a constant function.

We need to produce results sharpening Theorem A and filling the gap between Theorems B and C. First we construct some auxiliary functions with special asymptotical behavior at ∞. We do it in Section 6 using the conformal mappings technique. B. Nyman was the first to apply the so called "Nyman's bottle" construction in [**37**] (for the term see [**39**, p. 120]). Here, we need estimates essentially sharper than that in [**37**]. Therefore, our argument is necessarily more involved. Related notations are introduced in Section 5. Also, we discuss there regularity questions. In Sections 5 and 6 we deal with the case $l_p = 0$. The case $l_p > 0$ is considered in Section 7. Finally, results extending Theorems A – C are proved in Sections 8 and 9.

5. Auxiliary statements related to the Legendre transform.

Let $\rho(x)$ be a positive C^2-smooth strictly concave function on $\mathbb{R}_+$, $\rho(0)=0$, $\rho(x)=o(x)$, $x\to\infty$,

$$\int^\infty \frac{\rho(x)}{x^2}\,dx=\infty. \tag{5.1}$$

We introduce several auxiliary functions determined by ρ. Put

$$\begin{aligned}
h(x) &= \frac{2}{\pi}\int_1^x \frac{\rho(s)}{s^2}\,ds, \qquad x\ge 1,\\
F(x) &= \log h^{-1}(x), \qquad x\ge 0,\\
\gamma(x) &= \frac{\pi}{2}\frac{1}{F'(x)}, \qquad x\ge 0,\\
M(x) &= F(\gamma^{-1}(x))+\log|x\gamma'(\gamma^{-1}(x))|, \qquad 0<x\le\gamma(0),\\
Q(x) &= \sup_{y\ge 0}(\rho(y)-xy), \qquad x>0,\\
N(x) &= \log^+ Q(x), \qquad x>0.
\end{aligned}$$

Assuming some regularity of ρ, we are going to prove a number of statements concerning the regularity of these functions and relations between them. In particular, we show that the function N (which is the logarithm of the Legendre transform of ρ) is very close to M.

More specifically, we use the following assumptions on ρ:

$$\text{the function } q: x\mapsto \log\rho(\exp x) \text{ is strictly convex for big } x, \tag{5.2}$$

$$\text{for some } c>0, \text{ the function } q(x)+c\log x \text{ is concave for big } x, \tag{5.3}$$

$$\text{for every } A, \text{ the function } q(x)-A/x \text{ is convex for big } x. \tag{5.4}$$

Note, that condition (5.4) implies (5.2).

Lemma 5.1. (a) $\rho(x)\ge x\rho'(x)$, $x\ge 0$.

(b) *Under condition* (5.2),

$$\lim_{x\to\infty} q'(x)=\lim_{x\to\infty}\frac{\log\rho(x)}{\log x}=\lim_{x\to\infty}\frac{x\rho'(x)}{\rho(x)}=1.$$

Proof: (a) follows from the concavity of ρ, (b) follows from (5.1), the relation $\rho(x)=o(x)$, $x\to\infty$, and the convexity of q. ∎

LEMMA 5.2.

(a) $$h'(x) = \frac{2}{\pi}\frac{\rho(x)}{x^2}, \qquad h''(x) = \frac{2}{\pi}\frac{x\rho'(x) - 2\rho(x)}{x^3}, \qquad x \geq 1.$$

(b) $$F'(h(x)) = \frac{1}{xh'(x)},$$

$$F''(h(x)) = -\frac{h'(x) + xh''(x)}{x^2 h'^3(x)} = \frac{\pi^2}{4}x^2\frac{\rho(x) - x\rho'(x)}{\rho^3(x)}, \qquad x \geq 1.$$

(c) $$\gamma(h(x)) = \frac{\pi}{2}xh'(x) = \frac{\rho(x)}{x},$$

$$\gamma'(h(x)) = \frac{\pi}{2}\Big(1 + \frac{xh''(x)}{h'(x)}\Big) = \frac{\pi}{2}\frac{x\rho'(x) - \rho(x)}{\rho(x)},$$

$$\gamma(h(x))\gamma'(h(x)) = \frac{\pi^2}{4}\big(xh'(x) + x^2h''(x)\big) = \frac{\pi}{2}\frac{x\rho'(x) - \rho(x)}{x},$$

$$\gamma''(h(x))h'(x) = \frac{\pi}{2}\Big(\frac{x\rho'(x)}{\rho(x)}\Big)' = \frac{\pi}{2}\frac{q''(\log x)}{x}, \qquad x \geq 1.$$

THE PROOF is immediate. ▮

LEMMA 5.3. (a) *Under condition* (5.2), $q'(x) < 1$, $q''(x) < q'(x)(1 - q'(x)) < 1 - q'(x)$.
(b) *Under conditions* (5.2) *and* (5.3), $q'(x) > 1 - c/x$ *for some* $c > 0$ *and big* x.

PROOF: (a) By convexity of q, q' increases, and by Lemma 5.1 (b), $q'(x) \to 1$ as $x \to \infty$. Hence, $q'(x) < 1$ for big x. Expressing derivatives of ρ in terms of q we obtain

$$\rho'(\exp x) = q'(x)\exp(q(x) - x), \tag{5.5}$$

$$\rho''(\exp x) = \big(q''(x) + q'^2(x) - q'(x)\big)\exp(q(x) - 2x). \tag{5.6}$$

Since $\rho''(x) < 0$, we get $q''(x) + q'^2(x) < q'(x)$.

(b) The fact that $q''(x) < c/x^2$ for large x and the properties of q' mentioned in the proof of (a) imply the assertion. ▮

LEMMA 5.4. (a) *The function* F *is convex,* $F'(x) \to \infty$, $F''(x) = o(F'^2(x))$, $x \to \infty$.
(b) *The function* γ *decreases,* $\gamma'(x) \to 0$, *as* $x \to \infty$.
(c) *Under conditions* (5.2) *and* (5.3) *the function* $\gamma(x)$ *is convex for big* x,

$$\int^\infty \frac{\gamma'^2(x)}{\gamma(x)}\,dx < \infty, \qquad \int^\infty \frac{(\gamma\gamma')'^2(x)}{\gamma(x)}\,dx < \infty.$$

PROOF: (a) and (b) follow from Lemmas 5.1 and 5.2. The first part of (c) follows from Lemma 5.2 (c). To prove the second part of (c) note that

$$\int^\infty \frac{\gamma'^2(x)}{\gamma(x)}\,dx = \int^\infty \frac{\gamma'^2(h(x))}{\gamma(h(x))}h'(x)\,dx = \frac{\pi}{2}\int^\infty \Big(1 - \frac{x\rho'(x)}{\rho(x)}\Big)^2\frac{dx}{x} =$$
$$\frac{\pi}{2}\int^\infty (1 - q'(\log x))^2\,d\log x = \frac{\pi}{2}\int^\infty (1 - q'(x))^2\,dx \leq c\int^\infty \frac{dx}{x^2} < \infty$$

by Lemma 5.3 (b). To complete the proof of (c) we use the following estimates:

$$\int^{\infty} \frac{\gamma'^4(x)}{\gamma(x)}\,dx \le \sup|\gamma'^2(x)| \int^{\infty} \frac{\gamma'^2(x)}{\gamma(x)}\,dx < \infty,$$
$$\int^{\infty} \frac{\gamma(x)\gamma'^2(x)\gamma''(x)}{\gamma(x)}\,dx = \int^{\infty} \gamma'^2(x)\,d\gamma'(x) = \int^{\infty} \frac{d\gamma'^3(x)}{3} < \infty,$$
$$\int^{\infty} \frac{\gamma^2(x)\gamma''^2(x)}{\gamma(x)}\,dx = \int^{\infty} \gamma(h(x))\gamma''^2(h(x))h'(x)\,dx =$$

by Lemma 5.2 (c)

$$= \frac{\pi^3}{8}\int^{\infty} xh'(x)\frac{q''^2(\log x)}{x^2h'^2(x)}h'(x)\,dx = \frac{\pi^3}{8}\int^{\infty} q''^2(x)\,dx \le c\int^{\infty}\frac{dx}{x^4} < \infty. \ \blacksquare$$

LEMMA 5.5. *Under condition* (5.2),

(a) $M'(x) < 0$ *for small positive* x.
(b) $\gamma(x)\gamma'(x)M'(\gamma(x)) < \pi/2$ *for big* x.
(c) $\gamma(x)\gamma''(x) < (\pi/2)|\gamma'(x)|$ *for big* x.

PROOF: First of all, for big x,

$$\gamma(x)\gamma'(x)M'(\gamma(x)) = \gamma(x)\Big[F'(x) + \frac{\gamma''(x)}{\gamma'(x)} + \frac{\gamma'(x)}{\gamma(x)}\Big] = \frac{\pi}{2} + \frac{\gamma(x)\gamma''(x)}{\gamma'(x)} + \gamma'(x) < \frac{\pi}{2}. \quad (5.7)$$

Furthermore, by Lemmas 5.2 (c) and 5.3 (a),

$$\frac{\gamma(h(x))\gamma''(h(x))}{|\gamma'(h(x))|} = \frac{\pi}{2}\frac{xh'(x)q''(\log x)}{xh'(x)(1-q'(\log x))} = \frac{\pi}{2}\frac{q''(\log x)}{(1-q'(\log x))} < \frac{\pi}{2}, \quad (5.8)$$
$$\frac{\pi}{2} + \gamma'(h(x)) - \frac{\gamma(h(x))\gamma''(h(x))}{|\gamma'(h(x))|} = \frac{\pi}{2}\Big(q'(\log x) - \frac{q''(\log x)}{(1-q'(\log x))}\Big) > 0. \ \blacksquare$$

LEMMA 5.6. *The function* $Q(x)$ *is strictly convex for small* $x > 0$, $\int_0 N(x)\,dx = \infty$,

$$\rho(y) = \inf_{x>0}\big(xy + Q(x)\big), \qquad y \ge 0. \quad (5.9)$$

PROOF: The assertion is just a consequence of standard properties of the Legendre transform (see, for instance, [**3,32,36**]). ∎

EXAMPLE 5.7: Put $R(x) = -Q'(x) = -N'(x)\exp N(x)$. Then

$$F^{-1}(x) = \frac{2}{\pi}\int^{x} R^{-1}(e^t)\,dt + o(1), \qquad x \to \infty.$$

For example, if $N(x) = 1/x$, then for some constant c,

$$F(x) = \exp\Big(\frac{\pi}{2}x + c + o(1)\Big), \qquad x \to \infty.$$

LEMMA 5.8. (a) *Under condition* (5.4), $x^2(1-q'(x)) \to \infty$, $x \to \infty$.
(b) *Under conditions* (5.3) *and* (5.4), $\gamma(x)\gamma''(x) = o(|\gamma'(x)|)$, $x \to \infty$,

$$\lim_{x\to\infty} \gamma(x)\gamma'(x)M'(\gamma(x)) = \frac{\pi}{2}.$$

PROOF: According to (5.7) and (5.8), to prove (b) we need only to verify that

$$q''(x) = o(1-q'(x)), \qquad x \to \infty.$$

This relation is proved as follows. By (5.3), $q''(x) < c/x^2$, and by (5.4), for any fixed A the function $x \mapsto q(x) - x - A/x$ is strictly convex, $\lim_{x\to\infty}(q'(x) - 1 + A/x^2) = 0$, $q'(x) - 1 + A/x^2 < 0$, $1 - q'(x) > A/x^2$ for big x, $x > x(A)$. These estimates prove both (a) and (b). ∎

LEMMA 5.9. *Under condition* (5.4), $\log x = o(\rho(x) - x\rho'(x))$, $x \to \infty$.

PROOF: By Lemma 5.8 (a), for every A and big x,

$$\frac{A}{x^2} < 1 - q'(x) = \frac{\rho(\exp x) - \rho'(\exp x)\exp x}{\rho(\exp x)}, \tag{5.10}$$

and by Lemma 5.1 (b), $x = O(\rho(\exp x)/x^2)$, $x \to \infty$. ∎

LEMMA 5.10. *Under condition* (5.4), *for every* $c, c_1 > 0$, *and for sufficiently big* x, $x > x(c, c_1)$, *we have*

$$\begin{aligned} &(a) \quad \inf_{y>0}\left(xy + e^{N(y)-c_1}\right) \le \rho(x) - c\log x, \\ &(b) \quad \inf_{y>0}\left(xy + e^{N(y)+c_1}\right) \ge \rho(x) + c\log x. \end{aligned}$$

PROOF: (a) By Lemma 5.6, for small $y > 0$,

$$\rho\big(-N'(y)e^{N(y)}\big) = -yN'(y)e^{N(y)} + e^{N(y)}.$$

Therefore, we need only to verify that for small $y > 0$,

$$e^{N(y)} - e^{N(y)-c_1} \ge c\log\big(-N'(y)e^{N(y)}\big).$$

This inequality follows from relations

$$\begin{gathered} \exp(N(\rho'(x))) = \rho(x) - x\rho'(x), \\ -N'(\rho'(x))\exp(N(\rho'(x))) = x \end{gathered}$$

and Lemma 5.9. The proof of part (b) is analogous. ∎

Lemma 5.11. *Under conditions* (5.3) *and* (5.4) *we have for big x and y such that $|x-y| \le \gamma(x)$,*

$$\begin{aligned} |\gamma'(y)| &< 2|\gamma'(x)|, \\ |\gamma(x)-\gamma(y)| &< 2\gamma(x)|\gamma'(x)|, \\ |\gamma(y)\gamma'(y)| &< 3\gamma(x)|\gamma'(x)|. \end{aligned}$$

Proof: By Lemma 5.4 (b) and (c), γ is convex, and the assertions hold for $y \ge x$. For $y < x$ we use the following argument. Let s be such that $|\gamma'(x-s)| = 2|\gamma'(x)|$. Then, by Lemma 5.8 (b), we have

$$\gamma''(t) \le |\gamma'(x)|/\gamma(x), \qquad x-s \le t \le x, \quad x \to \infty,$$
$$|\gamma'(x)| = |\gamma'(x-s)| - |\gamma'(x)| = \int_{x-s}^{x} \gamma''(t)\,dt \le s|\gamma'(x)|/\gamma(x),$$
$$s \ge \gamma(x),$$

and the assertion follows. ∎

Lemma 5.12. *Under conditions* (5.3) *and* (5.4), $0 \le M(x) - N(x) \le 4$ *for small $x > 0$.*

Proof: We begin with a simple observation (see Lemma 5.2):

$$\begin{aligned} N(\rho'(x)) &= \log(\rho(x) - x\rho'(x)), \\ M\Big(\frac{\rho(x)}{x}\Big) &= M(\gamma(h(x))) = F(h(x)) + \log|\gamma(h(x))\gamma'(h(x))| \\ &= \log(\rho(x) - x\rho'(x)) + \log\frac{\pi}{2}. \end{aligned} \tag{5.11}$$

It remains only to verify that for big x,

$$0 \le M(\rho'(x)) - M\big(\rho(x)/x\big) \le 4 - \log\frac{\pi}{2}.$$

The left inequality follows from Lemmas 5.1 (a) and 5.5 (a). To verify the right inequality we use that by Lemma 5.5 (b) for big x,

$$|M'(x)| < \frac{\pi}{2}\frac{1}{x|\gamma'(\gamma^{-1}(x))|}.$$

Therefore, it is enough to estimate the expression

$$\frac{1}{\rho'(x)}\int_{\rho'(x)}^{\rho(x)/x} \frac{dx}{|\gamma'(\gamma^{-1}(x))|} = \frac{1}{\rho'(x)}\Big(\gamma^{-1}(\rho'(x)) - \gamma^{-1}\Big(\frac{\rho(x)}{x}\Big)\Big).$$

Put $u = \gamma^{-1}(\rho(x)/x)$, $v = u + 2\rho(x)/x$. If $\gamma^{-1}(\rho'(x)) > v$, then

$$\gamma(u)/2 < \rho'(x) < \gamma(v).$$

By Lemma 5.8 (b) for large x and $u \le y \le v$,

$$\gamma''(y) \le \frac{1}{4}\frac{|\gamma'(u)|}{\gamma(\gamma^{-1}(\rho(x)/x))} = \frac{x}{4\rho(x)}|\gamma'(u)|.$$

Therefore, $|\gamma'(y)| \ge |\gamma'(u)|/2$, $u \le y \le v$,

$$\frac{\rho(x)}{x} - \rho'(x) > \gamma(\gamma^{-1}(\rho(x)/x)) - \gamma(\gamma^{-1}(\rho(x)/x) + 2\rho(x)/x) = \int_u^v |\gamma'(t)|\,dt \ge$$
$$\frac{v-u}{2}|\gamma'(h(x))| = \frac{\pi}{2}\frac{\rho(x)}{x}\frac{\rho(x) - x\rho'(x)}{\rho(x)},$$

which is impossible. Hence, $\gamma^{-1}(\rho'(x)) - \gamma^{-1}(\rho(x)/x) \le 2\rho(x)/x$, and Lemma 5.1 (b) completes the proof of our assertion. ∎

LEMMA 5.13. *Under conditions* (5.3) *and* (5.4), *for every* $k \ge 0$ *and big* x, $x > x(k)$,

$$M(\gamma(x) + k\gamma(x)\gamma'(x)) - M(\gamma(x)) < 4k.$$

THE PROOF is analogous to that of Lemma 5.12. ∎

LEMMA 5.14. *Under conditions* (5.3) *and* (5.4), *if* ψ *is a* C^2*-smooth function,* $\psi''(x) = O(1)$, $x \to \infty$, *then the functions* $\widetilde{\rho}_{\pm}(x) = \rho(x) \pm \psi(\log x)$ *are concave for big* x, *and the functions* $\widetilde{q}_{\pm}(x) = \log \widetilde{\rho}_{\pm}(\exp x)$ *satisfy conditions* (5.3) *and* (5.4).

PROOF: First, note that by (5.3), for big x and some c,

$$\frac{c}{x\log^2 x} > \frac{q''(\log x)}{x} = \frac{x\rho''(x)}{\rho(x)} + \frac{\rho'(x)(\rho(x) - x\rho'(x))}{\rho^2(x)}.$$

By (5.10) and Lemma 5.1 (b), for every $A > 0$, and for big x, $x > x(A)$,

$$\frac{\rho'(x)(\rho(x) - x\rho'(x))}{\rho^2(x)} > \frac{A}{x\log^2 x},$$

and as a result we obtain,

$$\frac{x\rho''(x)}{\rho(x)} < -\frac{A}{x\log^2 x},$$
$$\rho''(x) < -\frac{A\rho(x)}{x^2\log^2 x},$$
$$\widetilde{\rho}''_{\pm} = \rho''(x) \pm \Big(\frac{\psi''(\log x)}{x^2} - \frac{\psi'(\log x)}{x^2}\Big) < 0.$$

Now let us turn to q and $\widetilde{q}_{\pm}$.

$$q''(\log x) = \frac{x\rho'(x) + x^2\rho''(x)}{\rho(x)} - \frac{x^2\rho'^2(x)}{\rho^2(x)} =: a - b.$$

Since $q''(x) > 0$, a is positive. Since ρ is concave, by Lemma 5.1 (a) we have $0 < b < a < 1$. Furthermore,

$$\widetilde{q}''_{\pm}(\log x) = \frac{x\rho'(x) + x^2\rho''(x) \pm \psi''(\log x)}{\rho(x) \pm \psi(\log x)} - \frac{(x\rho'(x) \pm \psi'(\log x))^2}{(\rho(x) \pm \psi(\log x))^2} =$$
$$a\left(\frac{1}{1 \pm \psi(\log x)/\rho(x)}\right) \pm \frac{\psi''(\log x)}{\rho(x) \pm \psi(\log x)} - b\left(1 \pm \frac{\psi'(\log x)}{x\rho'(x)}\right)^2 \left(\frac{1}{1 \pm \psi(\log x)/\rho(x)}\right)^2,$$

and for big x and some c,

$$\left|\widetilde{q}''_{\pm}(\log x) - q''(\log x)\right| \le c\frac{|\psi''(\log x)| + |\psi'(\log x)| + |\psi(\log x)|}{\rho(x)} \le \frac{1}{\log^3 x}. \quad \blacksquare$$

Lemma 5.15. *Under condition* (5.2) *for small* $x > 0$ *we have*

$$Q''(x) + Q'(x) \ge 0.$$

Proof: The assertion follows from the equalities

$$\begin{aligned} Q'(\rho'(x)) &= -x, \\ Q''(\rho'(x))\rho''(x) &= -1, \end{aligned} \tag{5.12}$$

based on (5.9). We need only to apply relation (5.6) and Lemma 5.3 (a). $\blacksquare$

Lemma 5.16. *For big* x,

$$|Q'(\gamma(x))| \le \frac{\exp M(\gamma(x))}{\gamma(x)|\gamma'(x)|} = \exp F(x).$$

Proof: First, $\exp F(h(t)) = t$. Since Q is convex, we obtain from Lemma 5.1 (a) and equality (5.12) that

$$|Q'(\gamma(h(t)))| = \left|Q'\left(\frac{\rho(t)}{t}\right)\right| \le |Q'(\rho'(t))| = t. \quad \blacksquare$$

6. Analytic functions generated by conformal mappings.

Let $\rho(x)$ be a positive C^2-smooth strictly concave function on $\mathbb{R}_+$, $\rho(0) = 0$, $\rho(x) = o(x)$, $x \to \infty$,

$$\int^{\infty} \frac{\rho(x)}{x^2}\, dx = \infty.$$

We use the notations of the preceding section and extend the function γ to the whole real line C^1-smoothly, and possibly, redefine it on a finite interval in such a way that $1/2 < \gamma(t) < 1$, $t \in \mathbb{R}_-$, $-1/100 < \gamma'(t) < 0$, $t \in \mathbb{R}$.

For $k \in \mathbb{R}$, $|k| < 50$, define the strips

$$\Omega_k^+ = \{z \in \mathbb{C} : |\operatorname{Im} z| < \gamma(\operatorname{Re} z) + k\gamma(\operatorname{Re} z)\gamma'(\operatorname{Re} z),\ \operatorname{Re} z > 0\},$$
$$\Omega_k = \{z \in \mathbb{C} : |\operatorname{Im} z| < \gamma(\operatorname{Re} z) + k\gamma(\operatorname{Re} z)\gamma'(\operatorname{Re} z)\},$$

and the standard strips

$$\Omega_*^+ = \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi/2,\ \operatorname{Re} z > 0\},$$
$$\Omega_* = \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi/2\}.$$

Let the conformal mappings ψ_k^+ of Ω_k^+ onto Ω_*^+ and ψ_k of Ω_k onto Ω_* be defined in such a way that

$$\psi_k^+(\mathbb{R}_+) = \mathbb{R}_+, \qquad \psi_k^+(\Omega_k^+ \cap \mathbb{C}_+) \subset \mathbb{C}_+, \qquad \psi_k^+(\infty) = \infty,$$
$$\psi_k(\mathbb{R}_+) = \mathbb{R}_+, \qquad \psi_k(\mathbb{R}) = \mathbb{R}, \qquad \psi_k(\Omega_k \cap \mathbb{C}_+) \subset \mathbb{C}_+, \qquad \psi_k(+\infty) = +\infty.$$

We are interested in getting precise estimates on the asymptotical behavior of ψ_k^+ and ψ_k. First, we use a theorem of S. E. Warschawski [**40**, p. 296]. Its conditions on γ are as follows:

$$\lim_{t\to\infty} (\gamma + k\gamma\gamma')'(t) = 0, \qquad \int^\infty \frac{(\gamma + k\gamma\gamma')'^2(t)}{\gamma(t) + k\gamma(t)\gamma'(t)}\, dt < \infty \tag{6.1}$$

If ρ satisfies properties (5.2) and (5.3), then conditions (6.1) hold for every k because of Lemma 5.4 (b) and (c), and Lemma 5.5 (c). By Warschawski's theorem we have

$$\operatorname{Re} \psi_k^+(x + iy) = \frac{\pi}{2} \int^x \frac{dt}{\gamma(t) + k\gamma(t)\gamma'(t)} + c_{\gamma,k} + o(1) =$$
$$F(x) - \frac{\pi}{2} k \log \gamma(x) + c'_{\gamma,k} + o(1), \qquad x \to \infty,\ x + iy \in \Omega_k^+. \tag{6.2}$$

Correspondingly, for ψ_k we have

$$\operatorname{Re} \psi_k(x + iy) = F(x) - \frac{\pi}{2} k \log \gamma(x) + c''_{\gamma,k} + o(1), \qquad x \to +\infty,\ x + iy \in \Omega_k. \tag{6.3}$$

Now we would like to estimate $|\operatorname{Im} \psi_k(x + iy)|$, $|\operatorname{Im} \psi_k^+(x + iy)|$. Assume that $y > 0$. Clearly, for big x,

$$1 < \frac{\pi/2 - \operatorname{Im} \psi_k(x + iy)}{\omega(\psi_k(x + iy), \mathbb{R}_+, \Omega_* \cap \mathbb{C}_+)} < 2.$$

Furthermore,

$$\omega(\psi_k(x + iy), \mathbb{R}_+, \Omega_* \cap \mathbb{C}_+) = \omega(x + iy, \mathbb{R}_+, \Omega_k \cap \mathbb{C}_+).$$

From now on let us suppose that ρ satisfies conditions (5.3) and (5.4).

A simple geometric argument using Lemma 5.11 shows that for some absolute constants c_1, c_2, c_3, and $|k| \le 20$, $x \ge 0$,

$$\left.\begin{aligned} 0 < c_1 \le \frac{\omega\big(x+iy, \mathbb{R}_+, \Omega_k \cap \mathbb{C}_+\big)}{1-(y/\gamma(x))} \le c_2, \quad 0 \le y \le \gamma(x) - (3k^+ + 3)\gamma(x)|\gamma'(x)|, \\ \omega\big(x+iy, \mathbb{R}_+, \Omega_k \cap \mathbb{C}_+\big) \le c_3|\gamma'(x)|, \quad k < \frac{\gamma(x)-y}{\gamma(x)|\gamma'(x)|} < 3k^+ + 3, \end{aligned}\right\} \tag{6.4}$$

where $k^+ = \max(k, 0)$. Analogous estimates hold for Ω_k^+.

We introduce two functions f_1 and g_1 analytic correspondingly in Ω_0 and Ω_{-2}^+,

$$f_1(z) = \exp(-\exp\psi_0(z))/(z^2+4),$$
$$g_1(z) = \exp(-z + \exp\psi_{-2}^+(z)).$$

Asymptotical estimates (6.2), (6.3) and Lemma 5.4 (a) imply that

$$\lim_{x\to+\infty}\left[h\Big(\log\frac{1}{|f_1(x)|}\Big) - x\right] = 0,$$
$$\lim_{x\to+\infty}\big(h(\log|g_1(x)|) - x\big) = 0.$$

Lemma 6.1. *For every $c > 0$ there exists t_c such that*

$$\inf_{y>0}\sup_{x: x+iy\in\Omega_{-2}^+}\big[|g_1(x+iy)|\exp(ty)\big] \le \exp(\rho(t) - c\log t), \qquad t \ge t_c.$$

Proof: By Lemmas 5.10 (a) and 5.12 it is sufficient to verify that for some c and big x,

$$\log|g_1(x+iy)| \le y\exp(M(y)+c), \qquad 0 \le y < \gamma(x) + 2\gamma(x)|\gamma'(x)|. \tag{6.5}$$

Note that by (6.2) and (6.4), for some c,

$$\log\log|g_1(x+iy)| \le F(x) + \pi\log\gamma(x) + c + \log\cos\big(\operatorname{Im}\psi_{-2}^+(x+iy)\big) <$$
$$F(x) + 2\log\gamma(x) + \begin{cases} \log\Big[1 - \dfrac{y}{\gamma(x)}\Big], & 0 \le y \le \gamma(x) - 3\gamma(x)|\gamma'(x)| \\ \log|\gamma'(x)|, & y \in Y(x) = \gamma(x) + (-3\gamma(x)|\gamma'(x)|, 2\gamma(x)|\gamma'(x)|). \end{cases}$$

Furthermore,

$$\frac{d}{dx}\left[F(x) + 2\log\gamma(x) + \log\Big(1 - \frac{y}{\gamma(x)}\Big)\right] = 0 \Longrightarrow \frac{\pi/2 + 2\gamma'(x)}{\gamma(x)} + \frac{y\gamma'(x)}{\gamma(x)(\gamma(x)-y)} = 0 \Longrightarrow$$
$$\frac{\pi}{2}\gamma(x) - \frac{\pi}{2}y + 2\gamma(x)\gamma'(x) - y\gamma'(x) = 0 \Longrightarrow y \ge \gamma(x) + \gamma(x)\gamma'(x).$$

Therefore, for a fixed $y > 0$,

$$\sup_{x+iy\in\Omega_{-2}^+}\log\log|g_1(x+iy)| \le \sup_{x: y\in Y(x)}\big[F(x) + \log|\gamma'(x)\gamma^2(x)|\big] = \sup_{x: y\in Y(x)}\big[M(\gamma(x))\big] + \log y.$$

Finally, by Lemmas 5.4 (b) and 5.5 (b),

$$M(\gamma(x)) - M(\gamma(x) + 2\gamma(x)|\gamma'(x)|) \le \frac{\pi}{2}\frac{2\gamma(x)|\gamma'(x)|}{\gamma(x)|\gamma'(x)|} = \pi. \quad \blacksquare$$

LEMMA 6.2. *For every $c>0$ there exists t_c such that*

$$\left|\int_{\mathbb{R}} f_1(z)\exp(itz)\,dz\right| \le \exp(-\rho(|t|)-c\log|t|), \qquad |t|\ge t_c.$$

PROOF: The function f_1 is bounded and analytic in Ω_0. Therefore, for some c independent of $t\ge 0$,

$$\left|\int_{\mathbb{R}} f_1(z)\exp(itz)\,dz\right| \le c\sup_{x\ge 0}\Big[\big|f_1(x+i(\gamma(x)+\gamma(x)\gamma'(x)))\big|\exp(-t(\gamma(x)+\gamma(x)\gamma'(x)))\Big].$$

By Lemmas 5.10 (b) and 5.12, to prove our assertion it is sufficient to verify that for some c and big x,

$$\log\frac{1}{|f_1(x+i(\gamma(x)+\gamma(x)\gamma'(x)))|} \ge \frac{\exp(M(\gamma(x)+\gamma(x)\gamma'(x))-c)}{\gamma(x)+\gamma(x)\gamma'(x)}. \tag{6.6}$$

It follows from (6.3), (6.4) and Lemma 5.13, that for some constant $c=c(\gamma)$,

$$\log\log\frac{1}{|f_1(x+i(\gamma(x)+\gamma(x)\gamma'(x)))|} \ge F(x)+\log|\gamma'(x)|-c \ge$$
$$M(\gamma(x)+\gamma(x)\gamma'(x))-c-\log(\gamma(x)+\gamma(x)\gamma'(x)).$$

This proves (6.6) and the lemma for positive t. Similar argument works for negative t. ∎

Let $m(z)=\gamma(\operatorname{Re} z)|\gamma'(\operatorname{Re} z)|$. Define

$$f_2(z)=\frac{1}{\pi m^2(z)}\int_{\substack{|z-w|<m(z)\\ w\in\Omega_2^+}} f_1(w)\,dm_2(w).$$

By Lemma 5.11 for big x,

$$f_2(x+iy)=f_1(x+iy), \qquad x+iy\in\Omega_{10}^+.$$
$$f_2(x+iy)=0, \qquad x+iy\notin\Omega_{-10}^+.$$

LEMMA 6.3. *For big x and some constant c,*

(a) $\log|f_2(x+iy)|\le -\exp(M(y)-c)/y, \qquad x+iy\notin\Omega_{10}^+.$

(b) $\log|\bar\partial f_2(x+iy)|\le -\exp(M(y)-c)/y.$

PROOF: As in the proof of Lemma 6.2, for some c,

$$\log|f_2(x+iy)|\le -\exp(F(x)+\log|\gamma'(x)|-c)-2\log|\gamma(x)\gamma'(x)|, \quad |\gamma(x)-y|\le 10\gamma(x)|\gamma'(x)|.$$

Therefore, we need only to verify that for every c and for big x, $x>x(c)$,

$$-c\log|\gamma(x)\gamma'(x)|\le \exp(M(\gamma(x)))/\gamma(x).$$

This inequality follows from the relation

$$\log \frac{1}{|\gamma'(h(x))|} < \log \frac{\rho(x)}{\rho(x) - x\rho'(x)} \leq \frac{\pi}{2}\big(\rho(x) - x\rho'(x)\big) = \exp M(\gamma(h(x))),$$

which holds by Lemmas 5.2 (c) and 5.9 and equality (5.11). This proves assertion (a). Part (b) is proved analogously. We need only to apply additionally Lemma 5.8 (b). ∎

Define

$$g_2(z) = \int_{\partial\Omega^+_{-2}} \frac{g_1(w)}{z - w}\, dw, \qquad z \in \mathbb{C} \setminus \overline{\Omega^+_{-2}}.$$

The integral in the right hand side converges because g_1 is summable on $\partial\Omega^+_{-2}$. The function g_1 is analytic in Ω^+_{-2}. Hence, g_2 extends analytically to the whole complex plane. This extension is also denoted by g_2.

LEMMA 6.4. (a) *The function g_2 is the Carleman transform of an element g_0 in $\mathcal{M}^*_p$.*

(b) *For every $c > 0$ and small $\operatorname{Im} z > 0$,*

$$\log^+ |g_2(z)| < c \exp N(\operatorname{Im} z).$$

(c) $\big|\log\log|g_1(x)| - \log\log|g_2(x)|\big| = o(1)$, $x \to \infty$.

PROOF: (a) The proof is very close to the argument used in Example 3.3. Put

$$g_0(t) = i \int_{\partial\Omega^+_{-2}} g_1(z) e^{-itz}\, dz.$$

Then, $\mathcal{F}((g_0)_\pm) = \pm g_2$ on $\mathbb{C}_\pm$, and for $t \geq 0$,

$$|g_0(t)| = \Big|\int_{\partial\Omega^+_{-2}} g_1(z)e^{-itz}\, dz\Big| \leq \int_{\partial(\Omega^+_{-2} \cap (\mathbb{C}_- + iy))} \big|g_1(z)e^{-itz}\big|\, dz.$$

The last integral is estimated using Lemma 6.1. Analogous argument works for $t \leq 0$.

(b) If p_1 corresponds to $N + c$ as p corresponds to N, then the proof of (a) together with estimate (6.5) show that for some c_1,

$$|g_0(t)| \leq c_1 \frac{p_1(|t|)}{1 + t^2}.$$

By Lemma 2.1, we get our assertion.

(c) It is enough to verify that $|g_1(x) - g_2(x)| = o(|g_1(x)|)$, $x \to \infty$, that is,

$$\Big|\int_{\partial\Omega^+_{-2}} \frac{g_1(w)}{(x - w)}\, dw\Big| = o(|g_1(x)|), \qquad x \to \infty.$$

This last relation follows from the estimates

$$|g_1(x)| > \exp\exp(F(x) + 4\log\gamma(x)), \qquad \operatorname{dist}(x, \partial\Omega^+_{-2}) > \frac{\gamma(x)}{2}, \qquad \exp F(x) > \frac{1}{\gamma^c(x)},$$

which hold for every c and big x by (6.2) and Lemmas 5.11 and 5.4 (a). ∎

Suppose that p is a weight function as introduced in Section 2, $l_p = 0$, and let the function $\rho = \log p$ satisfy the conditions given in the beginning of this section and conditions (5.3), (5.4). Then we can summarize the results of this section as follows.

THEOREM 6.5. *There exist two functions $f_0 \in \mathcal{L}_p \cap \mathcal{K}_p$ and $g_0 \in \mathcal{M}_p^*$ such that $\mathcal{F} f_0 = f_1$, the Fourier transforms of the functions $(g_0)_+$ and $-(g_0)_-$ are the restrictions of the entire function $\mathcal{F} g_0 = g_2$ to $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively, and such that*

$$\lim_{x\to\infty} \Big[h\Big(\log\Big|\frac{1}{\mathcal{F} f_0(x)}\Big|\Big) - x \Big] = 0,$$
$$\lim_{x\to\infty} \big[h\big(\log|\mathcal{F} g_0(x)|\big) - x \big] = 0.$$

7. THE CASE $l_p > 0$.

To be able to deal with the case $l_p > 0$ we need to make slight changes in definitions and arguments in Sections 5 and 6. Put $\rho = \log p$. We suppose that $\rho(x)$ is a positive C^2-smooth strictly concave function on $\mathbb{R}_+$, $\rho(0) = 0$, $\rho(x) = l_p x + o(x)$, $x \to \infty$,

$$\int^{\infty} \frac{\rho(x) - l_p x}{x^2}\, dx = \infty.$$

The functions h, F and γ are defined as in Section 5,

$$\lim_{x\to\infty} \gamma(x) = l_p,$$

the functions M, Q and N are defined, correspondingly, for $l_p < x \le \gamma(0)$, $x > l_p$ and $x > l_p$, the function q is defined as

$$q(x) = \log\big[\rho(\exp x) - l_p \exp x\big].$$

Then Lemmas 5.1 – 5.3 hold in the case $l_p > 0$, with the exception of the last equality in Lemma 5.2 (c). Instead of it we have

$$\gamma'(h(\exp x)) = \frac{\pi}{2} \frac{\rho'(\exp x)\exp x - \rho(\exp x)}{\rho(\exp x) - l_p \exp x} \frac{\rho(\exp x) - l_p \exp x}{\rho(\exp x)} =$$
$$\frac{\pi}{2}(q'(x) - 1)\Big(1 - \frac{l_p \exp x}{\rho(\exp x)}\Big), \tag{7.1}$$
$$\gamma''(h(\exp x)) h'(\exp x) \exp x =$$
$$\frac{\pi}{2} q''(x)\Big(1 - \frac{l_p \exp x}{\rho(\exp x)}\Big) + \frac{\pi}{2}(1 - q'(x)) \frac{l_p \exp x}{\rho(\exp x)} \Big(1 - \frac{\rho'(\exp x)\exp x}{\rho(\exp x)}\Big). \tag{7.2}$$

In Lemma 5.1 (b) we have also

$$\lim_{x\to\infty} \frac{\log(\rho(x) - l_p x)}{\log x} = \lim_{x\to\infty} \frac{x\rho'(x) - l_p x}{\rho(x) - l_p x} = 1.$$

Equality (5.6) still holds, though (5.5) should be replaced by

$$\rho'(\exp x) - l_p = q'(x)\exp(q(x) - x).$$

In Lemma 5.4 we have now

$$\lim_{x\to\infty} F'(x) = \frac{\pi}{2l_p},$$

and the corresponding changes should be made in (b). To prove part (c) we use (7.1) and (7.2).

Furthermore, Lemmas 5.5 and 5.6 hold with the only changes in Lemma 5.5 (a) being that $M'(x+l_p) < 0$ for small positive x, in Lemma 5.5 (c) being that $\gamma'(x)\gamma''(x) < \pi|\gamma'(x)|$ for big x, and in Lemma 5.6 being that $\int_0$ and $\inf_{x>0}$ are replaced, correspondingly, by $\int_{l_p}$ and $\inf_{x>l_p}$. In the proof of Lemma 5.5 we use instead of (5.8) that by (7.1) and (7.2),

$$\frac{\gamma(h(\exp x))\gamma''(h(\exp x))}{|\gamma'(h(\exp x))|} = \frac{\pi}{2}\left[\frac{q''(x)}{1-q'(x)} + (1-q'(x))\frac{l_p \exp x}{\rho(\exp x)}\right]. \tag{7.3}$$

As an analog of Example 5.7 we present the following statement.

EXAMPLE 7.1: If $N(x) = 1/(x-1)$, $x > l_p = 1$, then for some constant c,

$$F(x) = \frac{\pi}{2}x - \log x + c + o(1), \qquad x \to \infty.$$

Finally, Lemmas 5.8 – 5.16 hold for $l_p > 0$ with obvious changes: the infimums in Lemma 5.10 are taken by $y > l_p$, the inequalities in Lemmas 5.12 and 5.15 hold for small positive values of $x - l_p$, and the functions $\widetilde{q}_\pm$ are defined in Lemma 5.14 as $\widetilde{q}_\pm(x) = \log\bigl(\widetilde{\rho}_\pm(\exp x) - l_p \exp x\bigr)$. In the proof of Lemma 5.8 we use (7.3), in the proof of Lemma 5.9 we use that for every A and big x,

$$\frac{A}{x^2} < 1 - q'(x) = \frac{\rho(\exp x) - \rho'(\exp x)\exp x}{\rho(\exp x) - l_p \exp x} = \frac{\rho(\exp x) - \rho'(\exp x)\exp x}{\exp q(x)}$$

and the fact that $x = O\bigl((\exp q(x))/x^2\bigr)$, $x \to \infty$.

To extend the results of Section 6, we suppose that ρ satisfies conditions (5.3) and (5.4), extend the function γ to the whole real line C^1-smoothly, and possibly, redefine it on a finite interval in such a way that $l_p + 1/2 < \gamma(t) < l_p + 1$, $t \in \mathbb{R}_-$, $-1/100 < \gamma'(t) < 0$, $t \in \mathbb{R}$. The strips $\Omega_k^+, \Omega_k, \Omega_*^+, \Omega_*$ and conformal mappings ψ_k^+, ψ_k are defined as in Section 6. Warschawski's theorem gives us analogs of (6.2) – (6.4). Now, for sufficiently big C we put

$$f_1(z) = \exp(-\exp(\psi_0(z) + C))/(z^2 + 4),$$
$$g_1(z) = \exp(\exp(\psi_0^+(z) - C(z+1))),$$

introduce the function m, $m(z) = \gamma(\operatorname{Re} z)|\gamma'(\operatorname{Re} z)|$, define

$$f_2(z) = \frac{1}{\pi m^2(z)} \int_{\substack{|z-w|<m(z)\\ w\in\Omega_2^+}} f_1(w)\, dm_2(w),$$

$$g_2(z) = \int_{\partial\Omega_0^+} \frac{g_1(w)}{(z-w)}\, dw, \qquad z \in \mathbb{C} \setminus \overline{\Omega_0^+}, \tag{7.4}$$

and extend g_2 into Ω_0^+ similarly to how it was done in Section 6. Lemmas 6.1 – 6.4 can be extended for $l_p > 0$ and sufficiently big C with the only changes in Lemma 6.1 being that $\inf_{y>0}$ is replaced by $\inf_{y>l_p}$, the assertion of Lemma 6.3 hold for small $y - l_p > 0$. We can obtain arbitrary (fixed) constants c in (6.5) and Lemmas 6.3 and 6.4 (b) depending on the choice of C. Note, that deforming the integration contour in (7.4) we can verify that for some c,

$$|g_2(z)| < c, \qquad z \in \mathbb{C} \setminus \Omega_0^+. \tag{7.5}$$

Finally, we get an analog of Theorem 6.5.

THEOREM 7.2. *There exist two functions $f_0 \in \mathcal{L}_p \cap K_p$ and $g_0 \in \mathcal{M}_p^*$ such that $\mathcal{F} f_0 = f_1$, the Fourier transforms of the functions $(g_0)_+$ and $-(g_0)_-$ are the restrictions of the entire function $\mathcal{F} g_0 = g_2$ to $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively, and such that for some constant C,*

$$\lim_{x\to\infty} \Big[h\Big(\log\Big|\frac{1}{\mathcal{F} f_0(x)}\Big|\Big) - x\Big] = C,$$
$$\lim_{x\to\infty} \big[h\big(\log|\mathcal{F} g_0(x)|\big) - x\big] = -C.$$

Note, that here we are not able to make the limits to be equal to 0. The reason is that F' is bounded and $\gamma(x) > l_p > 0$ here.

8. Asymptotics of quasianalytic and entire functions. Continuation.

The constructions of the previous sections permit us here to improve Theorem A and to prove uniqueness theorems for entire functions analogous to Theorems B and C of Section 4.

The main method used in this section is inspired by a work of E. M. Dyn'kin [**16**], where Levinson's log-log theorem is proved with the help of the technique of asymptotically holomorphic function. Throughout this section we assume that $\rho = \log p$ satisfies the following conditions:

The function $\rho(x)$ is positive, C^2-smooth and strictly concave on $\mathbb{R}_+$, $\rho(0) = 0$, $\rho(x) = l_p x + o(x)$, $x \to \infty$,

$$\int^\infty \frac{\rho(x) - l_p x}{x^2}\, dx = \infty,$$

and the function $q(x) = \log\big[\rho(\exp x) - l_p \exp x\big]$ satisfies conditions (5.3) and (5.4). It is easy to see that in this case the function p satisfies condition (2.1) and (3.1). We need only to use relation (5.6), Lemma 5.1 (a) and the argument in the proof of Lemma 5.8.

PROPOSITION 8.1. *Suppose that $l_p = 0$ and $f \in \mathcal{L}_p \setminus \{0\}$.*

(a) *For some set E of (asymptotical) density 1 on $\mathbb{R}_+$,*

$$\lim_{x\to\infty} m(E \cap (x, x+1)) = 1,$$

we have

$$\lim_{\substack{x\to+\infty\\ x\in E}} \Big[h\Big(\log^+\Big|\frac{1}{\mathcal{F} f(x)}\Big|\Big) - x\Big] = \liminf_{x\to+\infty} \Big[h\Big(\log^+\Big|\frac{1}{\mathcal{F} f(x)}\Big|\Big) - x\Big] =: L(f).$$

(b) *For every* $s \le L(f)$ *and for every* $u \in C_0^\infty(\mathbb{R})$ *we have*

$$\mathcal{F}g_0(\cdot + s)\widetilde{\mathcal{F}}(f * u) \in L^\infty(\mathbb{C}),$$

where the function $g_0 \in \mathcal{M}_p^*$ *is constructed in Theorem* 6.5, *and the generalized Fourier transform* $\widetilde{\mathcal{F}}(f * u)$ *is constructed in Section* 2.

(c) $L(f) < \infty$.

PROOF: (a) Suppose that $L(f) < \infty$ and ε is sufficiently small. In the case $L(f) > -\infty$ put $\varphi(z) = (\widetilde{\mathcal{F}}f)(z - L(f) - \varepsilon)$, otherwise put $\varphi = \widetilde{\mathcal{F}}f$.

First, note that both limits in (a) do not change when we replace f by $f * u$, at least with

$$|\mathcal{F}u(x)| > \exp(-\exp x). \tag{8.1}$$

Indeed, by Lemmas 5.1 (b) and 5.2 (a), h is concave,

$$\lim_{x\to\infty}\big(h(\exp x) - x\big) = \lim_{x\to\infty}\big(x - F(x)\big) = -\infty,$$
$$c \le \lim_{x\to\infty}\big(h(h^{-1}(x+c) + \exp x) - x\big) \le \lim_{x\to\infty}\big(h'(h^{-1}(x+c))\cdot \exp x + c\big) =$$
$$\lim_{x\to\infty}\Big(\frac{\exp x}{F'(x+c)\exp F(x+c)} + c\Big) = c.$$

For every smooth u condition (8.1) holds on a set of density 1 on $\mathbb{R}_+$. If there is no E satisfying the conditions of the proposition, then taking a suitable $f * u_t$, $u_t(x) = u(tx)$, $0 < t < 1$, we can assume that $x \mapsto (|x|+1)^2 f(x) \in L^\infty_{1/p}(\mathbb{R})$, $L(f) \le -\varepsilon$, and that for some $\delta > 0$ and a sequence of points x_k, $x_k \to +\infty$,

$$F^{-1}\left(\log^+\log^+\left|\frac{1}{\varphi(x)}\right|\right) > x_k + \varepsilon, \qquad x \in E_k,$$
$$E_k \subset (x_k,\, x_k + \gamma(x_k)), \qquad mE_k > \delta\gamma(x_k).$$

Put $x_* = x_k + \gamma(x_k)/2$. Using (2.5) and (2.8) we obtain

$$|\bar\partial(\widetilde{\mathcal{F}}f)(z)| \le c\exp(-\exp N(|\operatorname{Im} z|)). \tag{8.2}$$

We are going to show that

for every k the function $\varphi \cdot \mathcal{F}g_0$ is bounded (uniformly in k) on the line $x_* + i\mathbb{R}$. (8.3)

To prove this statement, consider the rectangle R_k,

$$R_k = \big\{x + iy : x_k < x < x_k + \gamma(x_k),\ |y| < \gamma(x_k)/2\big\},$$

and the auxiliary function $\varphi_k = H_{\varphi,R_k}$, analytic in R_k. Then for some c, every c_1 and sufficiently big k,

$$\log^+\log^+\Big(\frac{1}{|\varphi_k(z)|}\Big) \ge \min\big\{M(\gamma(x_k)/2), F(x_k+\varepsilon)\big\} - c > F(x_k) + c_1, \qquad z \in E_k, \tag{8.4}$$
$$\|\varphi_k\|_{H^\infty(R_k)} < c.$$

The second inequality in (8.4) follows from Lemmas 5.4 (a) and 5.8 (b) and the relation

$$1/\gamma(t) = O(\exp(1/(2|\gamma'(t)|))), \qquad t \to \infty,$$

which is a consequence of Lemmas 5.1 (b) and 5.3 (b) and the equality

$$\frac{\exp(-1/(2|\gamma'(h(t))|))}{\gamma(h(t))} = \frac{x}{\rho(x)} \exp\Bigl(-\frac{\rho(t)}{\pi(\rho(t) - t\rho'(t))}\Bigr) = \frac{x}{\rho(x)} \exp\Bigl(-\frac{1}{\pi(1 - q'(\log t))}\Bigr).$$

By the theorem on two constants applied to φ_k in $R_k \setminus E_k$, we obtain for some $c > 0$,

$$\log^+ \log^+\Bigl(\frac{1}{|\varphi_k(x+iy)|}\Bigr) \ge F(x_k) + c_1 - c, \qquad |x - x_*| \le \gamma(x_k)/3,\ |y| \le \gamma(x_k)/3, \quad (8.5)$$

and the same estimate holds for φ. In order to avoid applying the theorem on two constants to an infinitely connected domain we can just apply it first to $R_k \cap \mathbb{C}_\pm$ and then to $R_k \cap (\mathbb{C}_\pm \pm 2\gamma(x_k)/5)$.

Introduce a harmonic function ψ on

$$B_k = \bigl\{z : 0 < \operatorname{Im} z < \gamma(x_k) + \gamma(x_k)|\gamma'(x_k)|,\ x_k < \operatorname{Re} z < x_k + \gamma(x_k)\bigr\}$$

as follows:

$$\psi(x+iy) = (\exp F(x_k))(\gamma(x_k) + \gamma(x_k)|\gamma'(x_k)| - y).$$

Estimate (8.2) and Lemma 5.12 imply that for sufficiently big k and some $c_0 > 0$,

$$\log \frac{1}{|\bar\partial\varphi(z)|} > c_0\psi(z), \qquad z \in B_k. \qquad (8.6)$$

Indeed, we need only to verify that

$$M(y) > F(x) + \log\bigl[\gamma(x) + \gamma(x)|\gamma'(x)| - y\bigr] - c, \qquad 0 < y < \gamma(x) + \gamma(x)|\gamma'(x)|,$$

which follows from Lemmas 5.8 (b) and 5.13.

Put $\Omega = \{z \in B_k : \log^+(1/|\varphi(z)|) < c_0\psi(z)\}$. Inequality (8.5) for φ shows that

$$\Omega \cap \bigl\{x+iy : |x - x_*| \le \gamma(x_k)/3,\ |y| \le \gamma(x_k)/3\bigr\} = \emptyset.$$

Dividing φ by $2\|\varphi\|_\infty$ we obtain that

$$\overline{\Omega} \cap \bigl\{x+iy : x_k < x < x_k + \gamma(x_k),\ y = \gamma(x_k) + \gamma(x_k)|\gamma'(x_k)|\bigr\} = \emptyset.$$

If O is a connected component of Ω relatively compact in B_k, then we consider an auxiliary function $\varphi(O, \cdot)$,

$$\varphi(O, z) = \varphi(z) \exp\Bigl\{-\frac{1}{\pi}\int_O \frac{\bar\partial\varphi(\zeta)}{\varphi(\zeta)}\frac{1}{z-\zeta}\,dm_2(\zeta)\Bigr\},$$

where the integral in the right-hand side converges and is bounded uniformly in z because of (8.6). This function is analytic in O, for some c which does not depend on k,

$$\log|\varphi(O,z)| \le -c_0\psi(z) - c, \qquad z \in \partial O,$$

and by the maximum principle,

$$\log|\varphi(O,z)| \le -c_0\psi(z) - c, \qquad z \in O.$$

If $z_0 = x_* + iy \in B_k$ belongs to a connected component O of Ω such that $\overline{O} \cap \partial B_k \ne \emptyset$, then we can again consider the function $\varphi(O,z)$. A simple geometric estimate

$$\int_{\partial O \setminus \partial B_k} \psi(\zeta)\omega(z_0, d\zeta, O) \ge c \int_{\partial O \cap \partial B_k} \psi(\zeta)\omega(z_0, d\zeta, O), \tag{8.7}$$

for some c independent of k, follows if we consider harmonic functions $x+iy \mapsto \psi(x+iy)(x-x_k-\varepsilon)$ and $x+iy \mapsto \psi(x+iy)(x-x_k-\gamma(x_k)+\varepsilon)$. Estimate (8.7) implies that for some c depending only on c_0,

$$\log|\varphi(z_0)| \le -c\psi(z_0) - c.$$

As a result,

$$\log|\varphi(x_*+iy)| \le -c\psi(x_*+iy) - c, \qquad 0 \le y < \gamma(x_k). \tag{8.8}$$

Analogous argument using the domains

$$B_{k,s} = \bigl\{z : 0 < \operatorname{Im} z < s,\ x_k < \operatorname{Re} z < x_k + \gamma(x_k)\bigr\},$$

with $\gamma(x_k) < s < 3\gamma(x_k)$ and the harmonic functions

$$\psi_s(x+iy) = (\exp M(s))\bigl[s|M'(s)| + 1 - y|M'(s)|\bigr]$$

shows that

$$\log|\varphi(x_*+iy)| \le -c\exp M(y) - c, \qquad \gamma(x_k) \le y \le 2\gamma(x_k). \tag{8.9}$$

Finally, since $\gamma(y)|M'(\gamma(y))| \to \infty$, $y \to \infty$, we have $\log(1/y) = o(M(y))$, $y \to 0$. Therefore, using a "spreading lemma" argument like that in [**5,6,7**], we get

$$\log|\varphi(x_*+iy)| \le -c/y^2, \qquad 2\gamma(x_k) \le y \le 1. \tag{8.10}$$

Now let us turn to the function $g_2 = \mathcal{F}g_0$. Estimates (6.2) and (6.4) and Lemma 5.11 imply that for every $c > 0$ and sufficiently big k,

$$\log^+|g_2(x_*+iy)| < \max\Bigl[c\psi(x_*+iy), \log\frac{1}{\gamma(x_k) + 2\gamma(x_k)|\gamma'(x_k)| - y}\Bigr], \qquad 0 \le y \le \gamma(x_k) + \gamma(x_k)|\gamma'(x_k)|,$$

$$\log^+|g_2(x_*+iy)| < \log\frac{1}{y - \gamma(x_k) - 2\gamma(x_k)|\gamma'(x_k)|}, \qquad \gamma(x_k) + 3\gamma(x_k)|\gamma'(x_k)| \le y \le 1.$$

Furthermore, Lemma 6.4 (b) gives us that for every $c > 0$ and small $\operatorname{Im} z > 0$,

$$\log^+ |g_2(z)| < c \exp M(\operatorname{Im} z).$$

Taking into account (8.8), (8.9), (8.10) and the estimate

$$\log \frac{1}{t - 2\gamma(x)|\gamma'(x)|} \le \exp M(t + \gamma(x)), \qquad 3\gamma(x)|\gamma'(x)| < t < \gamma(x),$$

we obtain that the function Φ, $\Phi = \varphi \cdot g_2$, is bounded (uniformly in k) on the lines $x_* + i\mathbb{R}_+$. The same argument applied in the lower half-plane completes the proof of (8.3).

Note, that (8.2) and Lemmas 6.4 (b) imply that $\bar\partial \Phi(z) \in L^\infty(\mathbb{C})$. Dividing Φ by z^2 and smoothing it up near the point 0 we can assume that $\bar\partial \Phi \in L^1(\mathbb{C})$. Now, the function Φ is unbounded on any sequence of points of $\mathbb{R}$ where the lower limit in the definition of $L(f)$ is attained and where $\mathcal{F}u$ is not very small. Put $\Delta = \{z : |\operatorname{Im} z| < 1\}$ and define $\Phi_\Delta = H_{\Phi,\Delta}$.

Then Φ_Δ is analytic in Δ, $\Phi - \Phi_\Delta \in L^\infty(\mathbb{C})$,

$$\Phi_\Delta \in L^\infty\big(\{z : |\operatorname{Im} z| = 1\}\big), \qquad \Phi_\Delta \in L^\infty\Big(\bigcup_k \Big(x_k + \frac{\gamma(x_k)}{2} + i\mathbb{R}\Big)\Big), \qquad \Phi_\Delta \notin L^\infty(\mathbb{R}_+).$$

This is a contradiction to the maximum principle. Thus, assertion (a) is proved.

(b) In the case $s < L(f) \le \infty$ our statement follows from the proof of the previous assertion. Therefore, by Theorem 3.1, if $s < L(f)$, then $(g_0(x) \exp isx) * f(x) = 0$.

By the continuity property of convolution, we have

$$\big(g_0(x) \exp(iL(f)x)\big) * f(x) = 0,$$

again by Theorem 3.1

$$\mathcal{F}g_0(\cdot + L(f))\widetilde{\mathcal{F}}f(\cdot) \in L^\infty(\mathbb{C}).$$

(c) If $\big((g_0(u)\exp(itu)) * f(u)\big)(x) = 0$ for all x, t, then $\mathcal{F}(g_0(\cdot) f(x - \cdot)) = 0$, $g_0(y) f(x - y) = 0$ for a.e. x and y, so we have $f = 0$. ∎

THEOREM 8.2. *Let $l_p = 0$, and $f \in \mathcal{M}_p \setminus \{0\}$. Then for some set E of density 1 on $\mathbb{R}_+$ we have*

$$\lim_{\substack{x \to +\infty \\ x \in E}} \Big[h\Big(\log^+ \Big|\frac{1}{\mathcal{F}f(x)}\Big|\Big) - x \Big] = \liminf_{x \to +\infty} \Big[h\Big(\log^+ \Big|\frac{1}{\mathcal{F}f(x)}\Big|\Big) - x \Big] < \infty.$$

PROOF: By Lemmas 3.5 and 5.14, as in Remark 3.8, we can find $\widehat{p} = p \cdot \exp(-\psi(\log))$ satisfying the conditions listed before Proposition 8.1 and such that $f \in \mathcal{L}_{\widehat{p}}$, $g_0 \in \mathcal{M}^*_{\widehat{p}}$, where the function g_0 is constructed by $\widehat{p}$ in Theorem 6.5.

To prove the existence of the limit $\lim_{x\to\infty} \big[h(x) - \widehat{h}(x)\big] > 0$, where $\widehat{h}$ corresponds to $\widehat{\rho} = \log \widehat{p}$, we need only to recall that $\int \psi(\log x)/x^2\, dx < \infty$. Now our assertion easily follows from Proposition 8.1 (a), (c). ∎

REMARK 8.3: Note, that since $\gamma(h(x)) = \rho(x)/x$,

$$\widehat{\gamma}(h(x)) < \widehat{\gamma}(\widehat{h}(x)) = \frac{\rho(x) - \psi(\log x)}{x} < \frac{\rho(x)}{x} = \gamma(h(x)),$$
$$\widehat{\gamma}(x) < \gamma(x), \qquad x \to \infty.$$

Therefore, the proofs of Proposition 8.1 and Theorem 8.2 show that under the same conditions if for some $a \in \mathbb{R}$ and for some set E on $\mathbb{R}_+$ such that

$$\limsup_{x\to\infty} \frac{m(E \cap (x, x + \gamma(x+a)))}{\gamma(x+a)} > 0$$

we have

$$\liminf_{\substack{x\to+\infty\\ x\in E}} \Bigl[h\Bigl(\log^+\Bigl|\frac{1}{\mathcal{F}f(x)}\Bigr|\Bigr) - x\Bigr] > a,$$

then $L(f) \geq a$.

In the case $l_p > 0$ we can prove in an analogous way only the following weaker statement.

PROPOSITION 8.4. *Let $l_p > 0$, $f \in \mathcal{M}_p \setminus \{0\}$. Then for some set E of density 1 on $\mathbb{R}_+$ we have*

$$\limsup_{\substack{x\to+\infty\\ x\in E}} \Bigl[h\Bigl(\log^+\Bigl|\frac{1}{\mathcal{F}f(x)}\Bigr|\Bigr) - x\Bigr] < \infty.$$

If for every E of density 1 on $\mathbb{R}_+$ we have

$$\limsup_{\substack{x\to+\infty\\ x\in E}} \Bigl[h\Bigl(\log^+\Bigl|\frac{1}{\mathcal{F}f(x)}\Bigr|\Bigr) - x\Bigr] > -\infty,$$

then there is $s \in \mathbb{R}$ such that for every $u \in C_0^\infty(\mathbb{R})$,

$$\mathcal{F}g_0(\cdot + s)\widetilde{\mathcal{F}}(f * u) \in L^\infty(\mathbb{C}),$$

where the function $g_0 \in \mathcal{M}_p^*$ is constructed in Theorem 7.2.

THE PROOF OF PROPOSITION 8.4 runs as follows. By (7.5), the function

$$\Phi = H_{\mathcal{F}g_0(\cdot - s)\widetilde{\mathcal{F}}(f*u),\mathbb{C}}$$

is bounded on the boundary of the strip Ω_0^+. Therefore, if Φ is unbounded on Ω_0^+, then by the Ahlfors–Heins theorem (see [**4**, Chapter 7]), the function $H = \Phi \circ \psi_0^{-1}$ should tend to ∞ along half-lines $iy + \mathbb{R}_+$, $-\pi/2 < y < \pi/2$, for a.e. y (see also the proof of Lemma 9.3). Now, if the function $\mathcal{F}(f * u)$ is sufficiently small on big sets near points x_k of $\mathbb{R}$, $x_k \to +\infty$, then it is small on the discs $(l_p/2)\mathbb{D} + x_k$, $k \to \infty$. Estimating ψ_0 we get that H is small on the discs $(1/10)\mathbb{D} + \psi_0(x_k)$, $k \to \infty$, that leads to a contradiction. ∎

REMARK 8.5: We cannot get here a complete analog of Proposition 8.1 because the function F' is bounded, and the implication

$$\varphi \in H^\infty(\mathbb{D}), \quad \|\varphi\|_\infty \le 1, \quad m\Big\{x \in [0,1] : \log^+ \log^+ \frac{1}{|\varphi(x)|} > F(a+s)\Big\} > \varepsilon,\ \varepsilon \le 1 \Longrightarrow$$
$$\log^+ \log^+ \frac{1}{|\varphi(x)|} > F(a), \quad |x| < \frac{1}{2}$$

holds only for $s > (\log(c/\varepsilon))/F'(a)$.

Now we turn to the investigation of the asymptotics of entire functions which are the Carleman transforms.

PROPOSITION 8.6. *Suppose that $l_p = 0$, $g \in \mathcal{M}_p^*$, and the Carleman transform of the function g is an entire function $\mathcal{F}g$, $\mathcal{F}g \in L^\infty(\{z : \operatorname{Re} z < 0\})$. Then*

(a) $\lim_{x\to+\infty} \Big[h\big(\log \max_{\operatorname{Re} z = x} |\mathcal{F}g(z)|\big) - x\Big] = \limsup_{x\to+\infty} \Big[h\big(\log \max_{\operatorname{Re} z = x} |\mathcal{F}g(z)|\big) - x\Big] =: L^M(g).$

(b) *For every $s \ge L^M(g)$,*

$$\mathcal{F}g\widetilde{\mathcal{F}}f_0(\cdot + s) \in L^\infty(\mathbb{C}),$$

where the function $f_0 \in \mathcal{L}_p \cap \mathcal{K}_p$ is constructed in Theorem 6.5.

(c) *If $L^M(g) = -\infty$, then $g = 0$.*

Before turning to the proof, we state the following auxiliary statement, which permits us to use different asymptotically holomorphic extensions.

LEMMA 8.7. *Let a decreasing function $m : (0,1) \to (0,\infty)$ be such that $\lim_{x\to+0} m(x) = \infty$, the function $\log m$ be convex, and let a function f, $f \in (L^\infty \cap C^1)(\{z : 0 \le \operatorname{Im} z < 1\})$ satisfy the following conditions:*

$$f|\mathbb{R} \equiv 0, \qquad |\bar\partial f(z)| < \frac{1}{m(\operatorname{Im} z)(1 + |\operatorname{Re} z|^2)}.$$

Then for some $c = c(f)$ we have

$$|f(z)| < \frac{c}{m(\operatorname{Im} z)}, \qquad 0 < \operatorname{Im} z < 1.$$

PROOF: Put

$$\Omega = \Omega(c) = \Big\{z : 0 < \operatorname{Im} z < 1,\ |f(z)| > \frac{c}{m(\operatorname{Im} z)}\Big\}.$$

For sufficiently large c the strip $\big(\{z : 1/2 \le \operatorname{Im} z < 1\}\big)$ does not intersect with Ω. Put

$$\varphi(z) = f(z) \exp\Big(-\frac{1}{\pi}\int_\Omega \frac{\bar\partial f(w)}{f(w)} \frac{1}{z-w}\, dm_2(w)\Big).$$

Given $z \in \Omega$, let a and b be numbers such that

$$\log m(\operatorname{Im} z) = b - a \operatorname{Im} z, \qquad \log m(y) \ge b - ay, \qquad y > 0.$$

Then

$$\log|\varphi(x+iy)| \le \log c - b + ay, \qquad x+iy \in \partial\Omega,$$

and by the maximum principle,

$$\log|\varphi(z)| \le \log c - b + a \operatorname{Im} z = \log c - \log m(\operatorname{Im} z), \qquad z \in \Omega. \quad \blacksquare$$

PROOF OF PROPOSITION 8.6: Let us suppose that $L^M(g) = 0$ and there is a sequence of points $\{x_k\}$, $x_k \to \infty$, such that

$$\log\log \max_{\operatorname{Re} z = x_k} |\mathcal{F}g(z)| < F(x_k - \varepsilon).$$

Put

$$S = \{x+iy : y > 0,\, x_k - \gamma(x_k) < x < x_k\},$$
$$x_* = x_k - \gamma(x_k)/2.$$

The function Φ, $\Phi = f_2 \cdot \mathcal{F}g$, is analytic in Ω_{10}^+ and is equal to zero on the set $\{z \in \Omega_{-10}^+ : \operatorname{Re} z > c\}$ for sufficiently big c. By Lemmas 2.1 and 6.3 (b), the function $\bar\partial\Phi$ is bounded. Dividing Φ by $(z+1)^2$ we can assume that $\bar\partial\Phi$ is summable on $\mathbb{C}_+$. Furthermore, for big k,

$$\log|\Phi(z)| < -F(x_k - \varepsilon/2), \qquad z \in \partial S \cap \mathbb{R},$$
$$\log|\Phi(z)| < F(x_k - \varepsilon), \qquad z \in \partial(S \cap \Omega_{10}^+) \setminus (\mathbb{R} \cup \partial\Omega_{10}^+),$$

and by Lemma 6.3 (a),

$$\log|\Phi(z)| < 0, \qquad z \in S \cap \partial\Omega_{10}^+.$$

By the theorem on two constants applied to Φ in $S \cap \Omega_{10}^+$ we get

$$\log|\Phi(x_* + iy)| < 0, \qquad x_* + iy \in \Omega_{10}^+, \quad y > 0.$$

Moreover, by Lemma 6.3 (a),

$$|\Phi(z)| < 1, \qquad z \in S \setminus \Omega_{10}^+.$$

The same argument applied for the lower half-plane gives us that

$$|\Phi(x_* + iy)| \le 1, \qquad y \in \mathbb{R}.$$

As in the proof of Proposition 8.1, the maximum principle leads to contradiction.

Assertions (b) and (c) are verified as in Proposition 8.1. Here we apply Lemma 6.3 (b) and Lemma 8.7 which imply that

$$\widetilde{\mathcal{F}} f_0(\cdot + s)\mathcal{F}g \in L^\infty(\mathbb{C}) \iff f_2(\cdot + s)\mathcal{F}g \in L^\infty(\mathbb{C}). \quad \blacksquare$$

THEOREM 8.8. *Suppose that $l_p = 0$, and $g \in \mathcal{L}_p^* \setminus \{0\}$. If the Carleman transform of g is an entire function $\mathcal{F}g$, $\mathcal{F}g \in L^\infty(\{z : \operatorname{Re} z < 0\})$, then*

$$\lim_{x\to+\infty} \Big[h\big(\log \max_{\operatorname{Re} z = x} |\mathcal{F}g(z)|\big) - x\Big] > -\infty.$$

THE PROOF is analogous to that of Theorem 8.2. ∎

In the case $l_p > 0$ we can prove in an analogous way only a weaker statement.

PROPOSITION 8.9. *Let $l_p > 0$, $g \in \mathcal{L}_p^* \setminus \{0\}$, and let the Carleman transform of g be an entire function $\mathcal{F}g$, $\mathcal{F}g \in L^\infty(\{z : \operatorname{Re} z < 0\})$. Then*

$$\liminf_{x\to+\infty} \Big[h\big(\log \max_{\operatorname{Re} z = x} |\mathcal{F}g(z)|\big) - x\Big] > -\infty.$$

If

$$\liminf_{x\to+\infty} \Big[h\big(\log \max_{\operatorname{Re} z = x} |\mathcal{F}g(z)|\big) - x\Big] < \infty,$$

then for some $s \in \mathbb{R}$,

$$\mathcal{F}g(\cdot + s)\widetilde{\mathcal{F}} f_0 \in L^\infty(\mathbb{C}),$$

where the function $f_0 \in \mathcal{L}_p \cap \mathcal{K}_p$ is constructed in Theorem 7.2.

The results proved in the beginning of this section imply that if $f \in \mathcal{M}_p \setminus \{0\}$, and $L(f) > -\infty$, then $\widetilde{\mathcal{F}} f$ is sufficiently small near $\mathbb{R}$. The following statement shows how small should be $\widetilde{\mathcal{F}} f$ outside Ω_0^+ in the case $l_p = 0$.

PROPOSITION 8.10. *Let $l_p = 0$, $f \in \mathcal{L}_p \setminus \{0\}$, and $L(f) > 0$. Then for some $c = c(f)$,*

$$|\widetilde{\mathcal{F}} f(z)| \le c \exp(-\exp N(|\operatorname{Im} z|)), \qquad \gamma(\operatorname{Re} z) \le |\operatorname{Im} z| \le 1,\ \operatorname{Re} z > 0.$$

PROOF: Pick ε, $0 < \varepsilon < \min(L(f), 1)$, and for h, $0 < h \le \varepsilon$, and a, $a \in \mathbb{R}_+$, put

$$O_{a,h} = \big\{x + iy : a - \varepsilon < x < a + \varepsilon,\ 0 < y < h\big\},$$

and introduce $\psi_h = \omega(\cdot, \mathbb{R} \cap O_{a,h}, O_{a,h})$.

We are interested in the function φ_h,

$$\varphi_h(y) = \sup_{a-\varepsilon<x<a+\varepsilon} \psi_h(x + iy) = \psi_h(a + iy), \qquad 0 < y < h.$$

Estimating ψ_h in $O_{a,h}$ we obtain for some c_1, c_2, c_3 depending only on ε,

$$\begin{aligned} 0 < \varphi_h(y) &< 1, && 0 < y < h, \\ 0 < c_1 \le -h\varphi_h'(y) &\le c_2, && 0 < y < h, \\ |\varphi_h''(y)| &\le c_3, && 0 < y < h. \end{aligned}$$

Recall that $Q = \exp N$ is a strictly convex function. We introduce $y(t)$, $0 < y(t) < t$, uniquely depending on $t > 0$, such that

$$(y(t) - t)Q'(y(t)) = Q(y(t)).$$

Put $B(t) = t|Q'(y(t))|/c_1$. Clearly, $B(t)$ depends on t continuously.

Let us prove that the equation

$$B(t)\varphi_t'(r) = Q'(r), \tag{8.11}$$

has a unique solution r in the interval $(0, t]$ for small t. Indeed,

$$|Q'(y(t))| \le -B(t)\varphi_t'(r) \le (c_2/c_1)|Q'(y(t))|, \qquad 0 < r \le t,$$
$$\lim_{r\to 0} Q'(r) = -\infty,$$

and both $B(t)\varphi_t'$ and Q' are continuous. Therefore, there exists at least one solution of (8.11) and all solutions are in $(0, y(t)]$. For sufficiently small t,

$$B(t)\varphi_t''(r) \le c_3 B(t) = (c_3 t/c_1)|Q'(y(t))| < Q''(r), \qquad 0 < r \le y(t),$$

because of Lemma 5.15 and the convexity of Q. This inequality proves the uniqueness of the solution of (8.11). Now, this solution $r_0 = r_0(t)$ depends on t continuously for small $t > 0$, because the family $B\varphi_h'$ is equicontinuous for $B_0 < B < 2B_0$, $h_0 < h < 2h_0$. If t_0 is such that $y(t_0) = \gamma(a)$, then $r_0(t_0) \le \gamma(a)$, and by Lemma 5.16, for big a,

$$B(t_0) \le \exp F(a).$$

Put

$$A(t) = \sup_{0<y\le t} \big(B(t)\varphi_t(y) - Q(y)\big) = B(t)\varphi_t(r_0(t)) - Q(r_0(t)).$$

Since

$$B(t)\varphi_t(y(t)) - Q(y(t)) \ge \frac{c_1 B(t)}{t}(t - y(t)) - Q(y(t)) = 0,$$

$A(t)$ is positive.

Thus, for big a and every r between $\gamma(a)$ and a fixed t_* independent of a, we have a number t, $0 < t \le \varepsilon$, such that $r = r_0(t)$, and a function $\psi_r^* = B(t)\psi_t - A(t)$ harmonic in $O_{a,t}$ such that

$$\psi_r^* \,\big|\, \mathbb{R} \cap \partial O_{a,t} \le \exp F(a),$$
$$\psi_r^* \,\big|\, \partial O_{a,t} \setminus \mathbb{R} \le 0,$$
$$\psi_r^*(x + is) \le Q(s), \qquad 0 < s < t,\ a - \varepsilon < x < a + \varepsilon,$$
$$\psi_r^*(a + ir) = Q(r).$$

Let $\widetilde{\psi}_r^*$ be the function harmonically conjugate to ψ_r^*, and $\Phi = \widetilde{\mathcal{F}} f \cdot \exp\bigl(\psi_r^* + i\widetilde{\psi}_r^*\bigr)$. Then by (8.2), for some c,

$$|\bar\partial \Phi(z)| \le c, \qquad z \in O_{a,t}.$$

Furthermore, since $L(f) > \varepsilon$, we get for big a,

$$|\Phi(z)| \le 1, \qquad z \in \partial O_{a,t}.$$

Therefore, for some c,

$$\begin{gathered} |\Phi(z)| \le c, \qquad z \in O_{a,t}, \\ |\widetilde{\mathcal{F}} f(a+ir)| \le c \exp(-Q(r)), \qquad \gamma(a) \le r \le t_*, \\ |\widetilde{\mathcal{F}} f(z)| \le c \exp(-Q(|\operatorname{Im} z|)), \qquad \gamma(\operatorname{Re} z) \le |\operatorname{Im} z| \le t_*. \ \blacksquare \end{gathered}$$

9. Primary ideals in Beurling-type algebras.

Let $\mathfrak{A} = \mathcal{L}_p$ or $\mathcal{M}_p$, and let $\rho = \log p$ satisfy the conditions in the beginning of Section 8. In this section we show how the numbers $L_\pm(f) := L(f(\pm\cdot))$ and $L^M_\pm(g) := L^M(g(\pm\cdot))$ determine whether the ideal generated by f in $\mathfrak{A}$ is orthogonal to a functional g with empty spectrum. (For the definitions of L and L^M see Propositions 8.1 and 8.6.)

After that, we describe primary ideals at infinity in $\mathcal{L}_p$ and in $\mathcal{M}_p$, and the asymptotics of elements of $\mathcal{F}L^1_p(\mathbb{R})$ and of the Carleman transforms of elements in $L^\infty_p(\mathbb{R})$ which are entire functions.

Theorem 9.1. *If $f \in \mathfrak{A} \setminus \{0\}$, $g \in \mathfrak{A}^* \setminus \{0\}$, $\mathcal{F}g \in A(\mathbb{C})$, then $f * g = 0$ is equivalent to the pair of inequalities*

$$L^M_+(g) \le L_+(f), \quad L^M_-(g) \le L_-(f).$$

Thus, when we solve convolution equations in the case of empty spectrum, $\mathcal{F}g \in A(\mathbb{C})$, it suffices to use the information contained in the usual Fourier transforms.

Proof of Theorem 9.1: Replacing p by $\widehat{p}$ and convoluting f with some $u \in C_0^\infty(\mathbb{R})$, as in the proof of Theorem 8.2, we can assume that $f \in \mathcal{L}_p \cap \mathcal{K}_p$, $g \in \mathcal{M}_p^*$. Taking into account Theorem 3.1, we see that it is sufficient to verify that

$$\widetilde{\mathcal{F}} f \cdot \mathcal{F} g \in L^\infty(\mathbb{C}) \iff L^M_+(g) \le L_+(f),\, L^M_-(g) \le L_-(f).$$

Moreover, Lemma 2.1 shows that for some $g_* \in \mathcal{M}_p^*$,

$$\mathcal{F} g_*(z) = (\mathcal{F} g(z) - \mathcal{F} g(0) - z(\mathcal{F} g)'(0))/z^2.$$

Now, an elementary argument involving Lemma 3.2 implies that

$$\widetilde{\mathcal{F}} f \cdot \mathcal{F} g \in L^\infty(\mathbb{C}) \iff \widetilde{\mathcal{F}} f \cdot \mathcal{F} g_* \in L^\infty(\mathbb{C}).$$

Therefore, the general case can be reduced to the case $\mathcal{F}g \in L^\infty\big(\{z : \operatorname{Re} z < 0\}\big)$. Continuity of the convolution operation implies that

$$(f(x)\exp i\varepsilon x) * g(x) = 0 \text{ for all } \varepsilon > 0 \implies f * g = 0.$$

Thus, again by Theorem 3.1, it is enough to verify that for $\mathcal{F}g \in L^\infty\big(\{z : \operatorname{Re} z < 0\}\big)$,

$$\widetilde{\mathcal{F}}f \cdot \mathcal{F}g \in L^\infty(\mathbb{C}) \Rightarrow L_+^M(g) \le L_+(f) \Rightarrow \text{for every } \varepsilon > 0,\ \widetilde{\mathcal{F}}f(\cdot + \varepsilon)\mathcal{F}g \in L^\infty(\mathbb{C}). \quad (9.1)$$

(a) The case $l_p = 0$. The first implication in (9.1),

$$\widetilde{\mathcal{F}}f \cdot \mathcal{F}g \in L^\infty(\mathbb{C}) \Rightarrow L_+^M(g) \le L_+(f),$$

is verified through the usual technique of asymptotically holomorphic functions. Let $\varepsilon > 0$ be fixed, x be sufficiently big, $N(t) = \exp Q(t) = F(x + L_+^M(g) - 2\varepsilon)$, $t < \varepsilon$, and

$$\Omega = \{z : x < \operatorname{Re} z < x + \varepsilon,\ |\operatorname{Im} z| < t\}.$$

We define also

$$J = \{z : x < \operatorname{Re} z < x + \varepsilon,\ |\mathcal{F}g(z)| > \exp\exp F(x + L_+^M(g) - \varepsilon)\}.$$

By Proposition 8.6,

$$\log\log \max_{\operatorname{Re} w = x} |\mathcal{F}g(w)| > F(x + L_+^M(g) - \varepsilon),$$

and by Lemma 2.1, $J \subset \Omega$ for large x. Since $\widetilde{\mathcal{F}}f \cdot \mathcal{F}g \in L^\infty(\Omega)$,

$$\log\log\left|\frac{1}{\widetilde{\mathcal{F}}f(z)}\right| > F(x + L_+^M(g) - 2\varepsilon), \quad z \in J.$$

If $\Phi = H_{\widetilde{\mathcal{F}}f,\Omega}$, then, by (2.5) and (2.8),

$$\log\log\left|\frac{1}{\Phi(z)}\right| > F(x + L_+^M(g) - 3\varepsilon), \quad z \in J.$$

By the theorem on two constants applied to Φ in the connected component O of $\Omega \setminus J$ containing the point $x + \varepsilon/2$ and a simple estimate of the harmonic measure,

$$\omega(x + \varepsilon/2, J, O) \ge c,$$

for some absolute constant $c > 0$, we get

$$\log\log\left|\frac{1}{\Phi(x + \varepsilon/2)}\right| > F(x + L_+^M(g) - 4\varepsilon),$$

$$\log\log\left|\frac{1}{\widetilde{\mathcal{F}}f(x + \varepsilon/2)}\right| > F(x + L_+^M(g) - 5\varepsilon),$$

$$L_+(f) > L_+^M(g) - 6\varepsilon.$$

Since ε is arbitrarily small, $L_+(f) \geq L_+^M(g)$.

On the other side, suppose that $L_+^M(g) < 0 < L_+(f)$, and put $\Phi = \widetilde{\mathcal{F}}f \cdot \mathcal{F}g$. By Lemma 2.1 and Proposition 8.10, for some c,

$$|\Phi(x+iy)| < c, \qquad |y| \geq \gamma(x).$$

Furthermore, we know that $|\bar\partial\Phi|$ is bounded and can assume that it is summable. If Φ is unbounded, then the function H, $H = H_{\Phi,\mathbb{C}}$ is also unbounded. Replacing, if necessary, H by $(H(z) - H(0) - zH'(0))/z^2$ we can guarantee that $H = \mathcal{F}h$ for some $h \in \mathcal{M}_p^*$ because of Lemma 2.1. Since H is bounded on the boundary of the domain Ω_0^+, Warschawski's theorem together with the Phragmen–Lindelöf theorem for the strip imply that $L_+^M(h) \geq 0$ that contradicts to the condition $L_+^M(g) < 0$. As a result, we conclude that

$$\Phi = \widetilde{\mathcal{F}}f \cdot \mathcal{F}g \in L^\infty(\mathbb{C}).$$

(b) In the case $l_p > 0$ our argument is somewhat different. The second implication in (9.1),

$$L_+^M(g) < 0 < L_+(f) \Longrightarrow \Phi := \widetilde{\mathcal{F}}f \cdot \mathcal{F}g \in L^\infty(\mathbb{C}),$$

is verified in the following way. The function H, $H = H_{\Phi,\mathbb{C}}$, is bounded on the lines $\mathbb{R}$, $\mathbb{R} \pm 3l_p i/2$. However, the growth of $\mathcal{F}g$ and, as a consequence, that of Φ is not sufficient for H to grow in a strip of width $3l_p/2$ and to be bounded on its boundary on account of the Phragmen–Lindelöf theorem for the strip.

Let us state two auxiliary statements.

LEMMA 9.2. *Let $l_p > 0$. Under the conditions of Theorem 9.1 if*

$$\widetilde{\mathcal{F}}f \cdot \mathcal{F}g \in L^\infty(\mathbb{C}),$$

then

$$L^M(g) = \limsup_{x\to\infty} \Big[h\big(\log \max_{\operatorname{Re} z = x} |\mathcal{F}g(z)|\big) - x\Big] < \infty. \tag{9.2}$$

THE PROOF is analogous to the reasoning in (a) and uses Proposition 8.4. ∎

LEMMA 9.3. *Let $l_p > 0$, and let $\rho = \log p$ satisfy the conditions in the beginning of Section 8. If a function $g \in \mathcal{M}_p^* \backslash \{0\}$, $\mathcal{F}g \in A(\mathbb{C})$, $\mathcal{F}g \in L^\infty(\{z : \operatorname{Re} z < 0\})$, satisfies condition (9.2), then there exists a curve S, $S = \{x + is(x), x \geq 0\}$, such that $\lim_{x\to\infty} s(x) = 0$, and*

$$L^M(g) = \lim_{\substack{z\in S \\ \operatorname{Re} z\to\infty}} \Big[h\big(\log^+ \big|\mathcal{F}g(z)\big|\big) - \operatorname{Re} z\Big].$$

PROOF: Suppose that $L^M(g) > 0$. By Lemma 5.12, $\exp N(\gamma(x)) = o(\exp F(x))$, $x \to \infty$, and by Lemma 2.1 we have

$$\limsup_{x\to\infty} \Big[h\big(\log \sup_{\substack{\operatorname{Re} z = x \\ z \in \Omega_0^+}} |\mathcal{F}g(z)|\big) - x\Big] > 0. \tag{9.3}$$

Let ψ be the conformal map of Ω_* onto Ω_0 which is inverse to ψ_0. By (6.3), replacing if necessary, ψ by $\psi(\cdot - c)$, we obtain

$$0 < x - \operatorname{Re}\psi(F(x) + iy) = O(1), \qquad x \to \infty,\ |y| \le \pi/2. \tag{9.4}$$

Put $G = \mathcal{F}g(\psi)$. Then for some c we get, using (9.4) and Lemmas 2.1 and 5.12 that

$$\begin{gathered} \log|G(F(x) + iy)| < c \exp M(\gamma(x)), \qquad y = \pm\pi/2, \\ \lim_{x\to\infty}\left(\log^+\log^+|G(x+iy)| - x\right) = -\infty, \qquad y = \pm\pi/2, \end{gathered} \tag{9.5}$$

and by (9.3),

$$\limsup_{x\to\infty}\left(\log\log \max_{\operatorname{Re} z = x}|G(z)| - x\right) > 0. \tag{9.6}$$

Furthermore,

$$\int^\infty \frac{\log^+|G(\log x \pm i\pi/2)|}{x^2}\,dx \le c\int^\infty \frac{\log^+|G(F(t) \pm i\pi/2)|}{\exp F(t)} F'(t)\,dt \le$$
$$c\int^\infty \frac{\exp M(\gamma(t))}{\exp F(t)} F'(t)\,dt = c\int^\infty \frac{\gamma(t)|\gamma'(t)|}{\gamma(t)}\,dt = c\int^\infty |\gamma'(t)|\,dt < \infty.$$

Therefore, there exists a function G_1, outer in the strip Ω_*, with the modulus of boundary values equal to $\max(|G|, 1)$ on $\partial\Omega_* \cap \mathbb{C}_+$. Standard estimates of outer functions and estimate (9.5) imply that

$$\lim_{x\to\infty}(\log\log \max_{\operatorname{Re} z = x}|G_1(z)| - x) = -\infty. \tag{9.7}$$

Put $G_2 = G/G_1$. Then $|G_2|$ is bounded on $\partial\Omega_*^+$. By (9.6) and (9.7) we obtain

$$\limsup_{x\to\infty}(\log\log \max_{\operatorname{Re} z = x}|G_2(z)| - x) > 0.$$

By condition (9.2) for some c we have

$$\log\log \max_{\operatorname{Re} z = x}|G_2(z)| < x + c.$$

Now the lemma is a consequence of the Ahlfors–Heins theorem (see [**4**, Chapter 7]), estimate (9.7), and property (9.4) of ψ. We find a curve S_0 in Ω_*^+ approaching to $\mathbb{R}$ at infinity and such that the maximal growth of G_2 (and, as a consequence, that of G) is almost attained along S_0:

$$\lim_{\substack{x\to\infty \\ x+iy\in S_0}}\left[\log^+\log^+|G(x+iy)| - x\right] = \lim_{x\to\infty}\left[\log\log \max_{\operatorname{Re} z = x}|G_2(z)| - x\right],$$

and map it into Ω_0^+ by ψ. ∎

COROLLARY 9.4. *Let $l_p > 0$, and let $\rho = \log p$ satisfy the conditions in the beginning of Section 8. If $g \in \mathcal{L}_p^* \setminus \{0\}$, $\mathcal{F}g \in A(\mathbb{C})$, $\mathcal{F}g \in L^\infty\big(\{z:\ \operatorname{Re} z < 0\}\big)$, then the limit*

$$\lim_{x\to\infty}\Big[h\big(\log \max_{\operatorname{Re} z = x} |\mathcal{F}g(z)|\big) - x\Big]$$

exists and is finite or equal to $+\infty$.

THE PROOF consists in applying Proposition 8.9 and Lemmas 9.2 and 9.3. ∎

Let us return to the proof of the first implication in (9.1) in the case $l_p > 0$. Lemma 9.3 claims that the limit $L^M(g)$ is attained on a curve approaching to $\mathbb{R}$. This fact and an elementary estimate using the theorem on two constants like that for the case $l_p = 0$ imply the required inequality. ∎

Now we pass to equations on the half-line:

$$(f * g)_- = 0.$$

THEOREM 9.5. *Let $\mathfrak{A} = \mathcal{L}_p$ or $\mathcal{M}_p$, and $\rho = \log p$ satisfy the conditions in the beginning of Section 8. If $f \in \mathfrak{A}_+$, $g \in (\mathfrak{A}^*)_-$, $\mathcal{F}f(z) \neq 0$ for $\operatorname{Im} z \geq -l_p$,*

$$\lim_{y\to\infty} \frac{\log|\mathcal{F}f(iy)|}{y} = 0,$$

and $(f * g)_- = 0$, then $g = 0$.

PROOF: Theorem 3.4 and Lemma 3.5 imply that $\widetilde{\mathcal{F}f}\cdot\mathcal{F}g \in L^\infty(\mathbb{C})$. Since $\mathcal{F}f \in H^\infty(\mathbb{C}_+ - il_p)$,

$$\int_{\mathbb{R}} \frac{\log \mathcal{F}f(x - il_p)}{1 + x^2}\, dx > -\infty. \tag{9.8}$$

An easy argument using the Phragmen–Lindelöf theorem for angles (see, for example, [**27**]) shows that unless $g = 0$, the entire function $\mathcal{F}g$, bounded in $\mathbb{C}_- - (l_p + 1)i$ and belonging to the Smirnov class in $\mathbb{C}_+ - (l_p - 1)i$ should grow along $\mathbb{R} - l_p i$ sufficiently rapidly: for some $\delta > 0$,

$$\max_{|\operatorname{Re} z| = x} \log\log|\mathcal{F}g(z)| > \delta x, \qquad x \to \infty.$$

Therefore, the function $\widetilde{\mathcal{F}f}$ is small enough on a massive set. Applying the theorem on two constants, we get a contradiction with (9.8). ∎

Let us formulate once again our conditions on p.

$$\left.\begin{array}{l}
\text{The function } \rho = \log p \text{ is positive, } C^2\text{-smooth and strictly concave on } \mathbb{R}_+,\\
\quad \rho(0) = 0, \qquad \rho(x) = l_p x + o(x), \quad x \to \infty,\\
\quad \displaystyle\int^\infty \frac{\rho(x) - l_p x}{x^2}\, dx = \infty,\\
\text{and the function } q(x) = \log\big[\rho(\exp x) - l_p \exp x\big] \text{ satisfies the following}\\
\text{conditions:}\\
\text{for some } c > 0, \text{ the function } q(x) + c\log x \text{ is concave for big } x,\\
\text{for every } A, \text{ the function } q(x) - A/x \text{ is convex for big } x.
\end{array}\right\} \tag{9.9}$$

The functions h and F are defined as

$$h(x) = \frac{2}{\pi}\int^{x} \frac{\rho(s)}{s^2}\,ds, \qquad x \geq 1,$$
$$F(x) = \log h^{-1}(x).$$

Let us recall that if $R(x) = -Q'(x)$, $Q(x) = \sup_{y\geq 0}(\log p(y) - xy)$, then

$$h(\exp x) = \frac{2}{\pi}\int^{x} R^{-1}(e^t)\,dt + o(1), \qquad x \to \infty.$$

THEOREM 9.6. *Under conditions* (9.9) *every primary ideal at infinity in algebras* $\mathfrak{A} = \mathcal{L}_p$ *or* $\mathcal{M}_p$ *is of the form*

$$J = J(j_+, j_-), \qquad -\infty \leq j_\pm < \infty,$$

where

$$J(j_+, j_-) = \{f \in \mathfrak{A} : L_\pm(f) \geq j_\pm\},$$
$$L_\pm(f) = \liminf_{x\to+\infty}\Big[h\Big(\log^+\Big|\frac{1}{\mathcal{F}f(\pm x)}\Big|\Big) - x\Big],$$
$$j_\pm = \inf_{f\in J} L_\pm(f).$$

In this situation $J(-\infty, -\infty) = \mathfrak{A}$, *and we have* $J(a, b) = J(c, d)$ *if and only if* $a = c$, $b = d$.

This theorem is just a corollary of Theorem 9.1 (and of Theorems 6.5 and 7.2 for the last statement). Analogously, as a consequence of Theorem 9.5, we get the following result on the ideals on the half-line.

THEOREM 9.7. *Under conditions* (9.9) *every (closed) ideal* J *in algebras* $\mathfrak{A}_+ = (\mathcal{L}_p)_+$ *or* $(\mathcal{M}_p)_+$ *without common zeros for the Fourier transforms in* $\overline{\mathbb{C}}_+ - l_p i$ *is of the form* $\tau_t\mathfrak{A}_+$, $t \geq 0$.

Furthermore, for entire functions, subordinate to a majorant depending on $|\operatorname{Im} z|$, we deduce from Theorem 8.8 and Corollary 9.4 the following result.

THEOREM 9.8. *Let* Q *be a monotone decreasing convex function on* (l_p, ∞), *such that the function* p, $p(x) = \exp\inf_{y>l_p}(Q(x) + xy)$, *satisfies conditions* (9.9). *If* G *is an entire function, which is unbounded in the right half-plane, and*

$$|G(z)| < \exp Q(|\operatorname{Im} z|), \tag{9.10}$$

then the limit

$$\lim_{x\to\infty}\Big[h\big(\log \max_{\operatorname{Re} z = x} |G(z)|\big) - x\Big]$$

exists and is finite or equal to $+\infty$.

It is useful to compare this statement with the log-log theorem of Levinson–Sjöberg (see [**11,15,24,35,38**]) and with the Phragmen–Lindelöf theorem for the strip.

Theorem 9.8 and Examples 5.7 and 7.1 show that in the cases $Q(x) = \exp(1/x)$, $Q(x) = \exp(1/\max(0, x-1))$, if an entire function G satisfies condition (9.10), and the function H, $H(x) = \max_{|\operatorname{Re} z|=x} |G(z)|$, is unbounded, then H should grow at least as, correspondingly, $\exp\exp\exp(\frac{\pi}{2}x + c)$ and $\exp\exp(\frac{\pi}{2}x - \log x + c)$.

Theorem 8.2 and an argument like that in the proof of Theorem 9.1 permit us to describe the asymptotics of quasianalytically smooth functions. Let us sketch here the argument for the case $l_p > 0$. If $f \in \mathcal{M}_p \setminus \{0\}$, then $\Phi = \widetilde{\mathcal{F}}f \cdot \mathcal{F}g_0(\cdot + L(f) + \varepsilon)$ is unbounded for g_0 constructed in Theorem 7.2 and every $\varepsilon > 0$. Put $g = \mathcal{F}^{-1}(H_{\Phi,\mathbb{C}}) \in \mathcal{L}_p^* \setminus \{0\}$. By Proposition 8.9, Lemmas 9.2 and 9.3, $\mathcal{F}g$ is big on a curve S approaching to $\mathbb{R}$, and $\widetilde{\mathcal{F}}f$ could not be very small on a set of positive lower density on $\mathbb{R}$. As a result we get the following statement.

Theorem 9.9. *If $f \in C\{M_n\}(\mathbb{R}) \setminus \{0\}$, where $M_n = \sup_{t>0} \dfrac{t^n}{p(t)}$, p satisfies (9.9), then there exists a set E of density 1 on $\mathbb{R}_+$, $\lim_{x\to\infty} m(E \cap (x, x+1)) = 1$, such that the limits*

$$\lim_{\substack{x\to+\infty \\ x\in E}} \Big[h\Big(\log^+\Big|\frac{1}{f(\pm x)}\Big|\Big) - x\Big]$$

exist and are finite or equal to $-\infty$.

This statement improves earlier results of [**10,29,34,39**]. See also [**1,18,19,20**]. In the description of the distribution of zeros of quasianalytically smooth functions we obtain only a partial statement improving an earlier result of [**30**].

Proposition 9.10. *Under the conditions of Theorem 9.9 if n_k is the number of zeros of the function f on the interval $[k, k+1]$, counted according to their multiplicities, then for some c,*

$$n_k < \exp F(|k| + c).$$

A sufficient condition,

$$\sum_{k\in\mathbb{Z}} n_k \exp(-F(|k| + c)) < \infty,$$

for some c, was obtained in [**31**]. A theorem on the asymptotical behavior of quasianalytically smooth functions formulated in [**20**] permits to show in [**31**] that this conditions is also necessary for $p(x) < x/\log\log x$.

Let us sketch the proof of Proposition 9.10 for $l_p = 0$. Without loss of generality we can deal with zeros of f on $\mathbb{R}_+$ and assume that $|f^{(n)}(x)| < M_n$, $x \in \mathbb{R}$, $n \geq 0$. If f has N zeros in $[k, k+\delta]$, $k \in \mathbb{R}$, then (see, for instance [**30**, p.402])

$$|f(x)| \leq M_N \delta^N / N!, \qquad k \leq x \leq k + \delta.$$

It follows from Lemmas 5.1 and 5.3 (b) that for some c

$$\begin{gathered} q(t) - t > q(q(t)) - q(t) - c, \\ h(\rho(s)) + c > h(10s). \end{gathered} \tag{9.11}$$

Therefore, if the estimate on n_k claimed in the proposition does not hold, we can find s and k such that $k = h(\rho(s))$ and f has at least $\rho(s)$ zeros in $I = [k, k + \rho(s)/(10s)]$. Then

$$|f(x)|^{1/\rho(s)} < \sup_{t\ge 0} \frac{t\rho(s)}{10s(p(t))^{1/\rho(s)}(\rho(s)!)^{1/\rho(s)}} < \sup_{t\ge 0} \frac{e\rho(s)}{10s}\frac{t}{\rho(s)} \exp\Bigl(-\frac{\rho(t)}{\rho(s)}\Bigr) < \frac{1}{e}, \quad x \in I,$$

because the last supremum is attained at t such that $\rho(s) = t\rho'(t) < \rho(t) < \rho(100s/99)$ (see Lemma 5.1). As a result,

$$|f(x)| < \exp(-h^{-1}(k))$$

on the interval I of length

$$\rho(s)/(10s) > c\rho(\rho(s))/\rho(s) = c\gamma(h(\rho(s))) = c\gamma(k).$$

Last inequality follows from (9.11). Now, Theorem 8.2 together with Remark 8.3 give us that $L_+(\mathcal{F}^{-1}f) \ge 0$. Since the same argument works for all $\tau_s f$, we have $L_+(\mathcal{F}^{-1}f) = \infty$, and Theorem 8.2 implies that $f = 0$.

10. Open questions.

Theorems 3.1 and 9.1 make the following statement plausible:

10.1. Let $f \in \mathcal{L}_p$, $g \in \mathcal{L}_p^*$. Then $f * g = 0$ if and only if the functions $\mathcal{F}g_+$ and $-\mathcal{F}g_-$ continue to a function $\mathcal{F}g$ meromorphic in the whole plane, the divisor of poles of $\mathcal{F}g$ is subordinated to the divisor of zeros of $\mathcal{F}f$ and

$$M_g \cdot \mathcal{F}f \in L^\infty(\mathbb{R}),$$

where $M_g(x) = \sup_{\operatorname{Re} z = x} |\mathcal{F}g(z)|$.

This statement would imply that in order to solve convolution equations in this situation it is enough to know just the behavior of usual Fourier transforms.

As potential objects of application of the methods used in this paper we would like to point out

10.2. The question on description of primary ideals at ∞ in $\mathcal{L}_p$ with strongly asymmetric weights p (a weak asymmetry is considered in [**21,22**]).

10.3. The problem of description of all (closed) ideals in $\mathcal{L}_p$ and all 1-invariant subspaces there. ($E \subset \mathcal{L}_p$ is called 1-invariant, if $\tau_t E \subset E$, $\tau_t E \ne E$, $t > 0$.)

It would be interesting to get any information on the multiplicative factorization of quasi-analytically smooth functions in the classes considered in this paper.

10.4. Is it true that for every function f satisfying the conditions of Theorem 9.9 there exist a number c_f and functions f_1, f_2, $f_1 \in H^\infty(\Omega_0^+ + c_f)$, $f_2, \dfrac{1}{f_2} \in L^\infty(\Omega_0^+ + c_f)$, such that

$$f = f_1 \cdot f_2\big|\mathbb{R}?$$

Positive answer would permit us to get a complete description of the zero sets for quasi-analytically smooth functions.

Theorem 9.8 extending the log-log Levinson theorem and belonging to the domain of Function Theory is proved here via the Functional Analysis technique (duality, continuity of the convolution operation).

10.5. Is it possible to prove Theorem 9.8 directly?

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