ESTIMATES FROM BELOW AND CYCLICITY IN BERGMAN-TYPE SPACES

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1. Introduction.

Let X be a linear topological space of analytic functions on the unit disc $\mathbb{D}$, closed under multiplication by the independent variable z. An element F in X is called cyclic (or weakly invertible) in X if

$$\operatorname*{clos}_X \mathcal{L}(\{z^n F, n \geq 0\}) = X,$$

where $\mathcal{L}(E)$ is the linear hull of E.

Usually, to describe cyclic vectors is a major step in investigating the lattice of all (closed) subspaces of X invariant under multiplication by z. Two important examples here are the Hardy spaces H^p and the Korenblum space $A^{-\infty}$ (see [1,2]), the space of the functions f holomorphic on $\mathbb{D}$ and such that for some c,

$$|f(z)| \leq c(1-|z|)^{-c}, \qquad z \in \mathbb{D}.$$

If the point evaluation functionals,

$$f \to f(z), \qquad z \in \mathbb{D},$$

are bounded, then an immediate necessary condition for f to be cyclic is that $f(z) \neq 0$, $z \in \mathbb{D}$. This is the case for the spaces H^p, $A^{-\infty}$, and the Bergman spaces B^p,

$$B^p = \Big\{ F \in \operatorname{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |F(z)|^p \, dm_2(z) < \infty \Big\}.$$

Since $A^{-\infty}$ is a topological algebra, the invertibility condition,

$$|f(z)| \geq c_1(1-|z|)^c, \qquad z \in \mathbb{D}, \tag{1}$$

for some positive c, c_1, is sufficient for cyclicity.

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For a long time, it was an open question (posed by H. S. Shapiro in [3,4]) whether condition (1) implies cyclicity in the space B^2. Recently, a counterexample was constructed in [5,6].

The next natural question (see, for example, Question 6 in [7]) is whether a stronger condition,

$$|f(z)| \ge \varphi(|z|), \qquad z \in \mathbb{D},$$

is sufficient for cyclicity, where φ has $\lim_{t\to 1} \varphi(t) = 0$. Note that the corresponding question for Hardy spaces has a negative answer (see, for instance, [7, p.327]). Here we prove that the answer is positive for the Bergman spaces with sufficiently slowly decreasing functions φ like

$$\varphi(t) = \exp\Big[-\Big(\log \frac{1}{1-t}\Big)^{1/(2+\varepsilon)}\Big], \qquad \varepsilon > 0.$$

The proof uses estimates on the sets of big values of functions harmonic on the unit disc. Our main results are formulated in Section 2. Lemmas on harmonic functions are given in Section 3. Main results are proved in Section 4.

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2. Main results.

We say that a positive nonincreasing continuous function ψ on $(0,1]$, $0 < \lim_{t\to 0} \psi(t) \le +\infty$, is a weight function if it does not decrease too rapidly near the point 0, namely if for some positive c,

$$\psi(t^2) \le c\psi(t), \qquad 0 < t \le 1.$$

For weight functions u and w consider the Banach spaces $A(u), A_0(u)$,

$$A(u) = \big\{F \in \operatorname{Hol}(\mathbb{D}) : \sup_{z\in\mathbb{D}} \big[|F(z)| \exp(-u(1-|z|))\big] < \infty\big\},$$
$$A_0(u) = \big\{F \in \operatorname{Hol}(\mathbb{D}) : \lim_{|z|\to 1} \big[|F(z)| \exp(-u(1-|z|))\big] = 0\big\},$$

and the Bergman spaces $B^p(w)$, $0 < p < \infty$,

$$B^p(w) = \Big\{F \in \operatorname{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |F(z)|^p \exp(-w(1-|z|))\, dm_2(z) < \infty\Big\},$$

which are Banach spaces for $1 \le p < \infty$, and complete metric spaces for $0 < p < 1$. Here dm_2 is Lebesgue area measure. Clearly, $A_0(u) = \{0\}$ if $\lim_{t\to 0} u(t) > 0$.

Note that for the spaces $X = A_0(u)$, $B^p(w)$,

$$\operatorname*{clos}_X \mathcal{L}(\{z^n F, n \ge 0\}) = \operatorname*{clos}_X H^\infty \cdot F, \qquad F \in X.$$

Theorem 1. *Suppose that weight functions u and v satisfy the condition*

$$\int^{\infty} \frac{v^2(\exp(-t))}{u(\exp(-t))}\frac{dt}{t} < \infty. \tag{2}$$

If a function $f \in A_0(u)$ satisfies the condition

$$|f(z)| \ge \exp(-v(1-|z|)), \qquad z \in \mathbb{D}, \tag{3}$$

then f is cyclic in $A_0(u)$.

Theorem 2. *Suppose that weight functions w and v satisfy the condition*

$$\int^{\infty} \frac{v^2(\exp(-t))}{t + w(\exp(-t))}\frac{dt}{t} < \infty. \tag{4}$$

If a function $f \in B^p(w)$, $0 < p < \infty$, satisfies the condition

$$|f(z)| \ge \exp(-v(1-|z|)), \qquad z \in \mathbb{D},$$

then f is cyclic in $B^p(w)$.

Remarks. (1) If the weight function u satisfies the condition

$$\int^{\infty} \frac{dt}{tu(\exp(-t))} < \infty,$$

then there exists a weight function v such that (2) holds for the pair u, v, and $\lim_{t\to 0} v(t) = +\infty$.

(2) For every weight function w there exists a weight function v such that (4) holds for the pair w, v, and $\lim_{t\to 0} v(t) = +\infty$.

(3) In a similar way one can deal with smaller Bergman spaces like

$$\Bigl\{F \in \mathrm{Hol}\,(\mathbb{D}) : \int_{\mathbb{D}} |F(z)|^p (1-|z|)^{-s}\, dm_2(z) < \infty\Bigr\}, \qquad 0 < s < 1.$$

The proofs of Theorems 1 and 2 are presented in Section 4. They are based on results about harmonic functions given in Section 3. Under the conditions of Theorems 1 and 2 on u, v, and w we obtain that every function f in $X = A_0(u)$ or $B^p(w)$ such that $1/f \in A(v)$ can be multiplied by a bounded outer function in such a way that the product is a (bounded) multiplier from $A(v)$ to X.

3. Lemmas on harmonic functions.

Let h be a harmonic function on the unit disc such that

$$\left.\begin{aligned} h(0)=0, \\ -v(1-|z|)\le h(z)\le u(1-|z|), \end{aligned}\right\} \tag{5}$$

where u and v are two weight functions.

For a fixed α, $0<\alpha<1$, denote

$$\begin{aligned} E_h^\alpha &= \{z\in\mathbb{D}:\ h(z)\ge \alpha u(1-|z|)\}, \\ E_n &= E_h^\alpha \cap \{z\in\mathbb{D}:\ 2^{-2n}<1-|z|\le 2^{-n}\}, \qquad n\ge 1. \end{aligned}$$

Given a subset F of $\mathbb{D}$, consider its "shadow" F^* on the unit circle $\mathbb{T}$,

$$F^*=\{z\in\mathbb{T}:\ \text{for some } w\in F,\ |\arg w-\arg z|<1-|w|\}.$$

Lemma 1. *For some $c=c(\alpha,u,v)$,*

$$m(E_n^*)\le c\frac{v(2^{-n})}{u(2^{-n})},\qquad n\ge 1.$$

By m we denote normalized Lebesgue measure on the unit circle or on the circles $r\mathbb{T}$, $0<r<1$.

To prove Lemma 1, we need first to verify an auxiliary statement. Let K, $K>2$, be a number to be fixed later. Given $z\in\mathbb{D}$, $2^{-2n}<1-|z|\le 2^{-n}$, consider the set

$$F_z=\{w\in\mathbb{D}:\ |w|=1-2^{-3n},\ |\arg w-\arg z|<K(1-|z|)\}.$$

Lemma 2. *For sufficiently big K, $K=K(\alpha,u)$, there exists $c=c(\alpha,u)>0$, such that if $z\in E_n$, then*

$$\frac{1}{m(F_z)}\int_{F_z} h^+(\zeta)\,dm(\zeta)\ge cu(1-|z|),$$

where $h^+(\zeta)=\max(h(\zeta),0)$.

Proof. Consider a subdomain Ω_z of the unit disc,

$$\Omega_z=\{w\in\mathbb{D}:\ 2^{-3n}<1-|w|<4K\cdot(1-|z|),\ |\arg w-\arg z|<2K\cdot(1-|z|)\}.$$

When K is fixed and $|z|\to 1$, the domain Ω_z looks more and more like a square; the distance from z to the middle point of the side of $\partial\Omega_z$ closest to $\mathbb{T}$ is approximately $\frac{1}{4K}$ of the "side length" of Ω_z. Now a geometric argument shows that for z sufficiently close to $\mathbb{T}$, $r(K)\le|z|<1$,

$$\omega(\Omega_z,\partial\Omega_z\setminus F_z,z)\le c(K), \tag{6}$$

where $c(K)$ depends only on K and tends to 0 when K tends to $+\infty$, and

$$\frac{c_2(K)}{1-|z|} \le \frac{\omega(\Omega_z, d\zeta, z)}{dm(\zeta)} \le \frac{c_1}{1-|z|}, \qquad \zeta \in F_z, \tag{7}$$

where $c_1 > 0$ is an absolute constant, and $c_2(K) > 0$ depends only on K. Here $\omega(\cdot,\cdot,\cdot)$ is harmonic measure.

Since h is harmonic, we have

$$\alpha u(1-|z|) \le h(z) = \int_{\partial\Omega_z} h(\zeta)\,\omega(\Omega_z, d\zeta, z).$$

Furthermore, we know that

$$h(\zeta) \le u(1-|\zeta|).$$

Since u is a weight function, we obtain that

$$\max_{\zeta\in\partial\Omega_z} u(1-|\zeta|) \le cu(1-|z|)$$

for some positive c which depends only on u, and by (6) we have

$$\int_{\partial\Omega_z\setminus F_z} h(\zeta)\,\omega(\Omega_z, d\zeta, z) \le \alpha u(1-|z|)/2$$

for sufficiently big $K = K(\alpha, u)$. Now,

$$\int_{F_z} h(\zeta)\,\omega(\Omega_z, d\zeta, z) \ge \alpha u(1-|z|)/2,$$

and (7) implies that

$$\frac{1}{m(F_z)}\int_{F_z} h^+(\zeta)\,dm(\zeta) \ge cu(1-|z|)$$

for $K = K(\alpha, u)$ and some positive $c = c(\alpha, u)$ independent of z, z is sufficiently close to $\mathbb{T}$. $\square$

Proof of Lemma 1. Lemma 2 claims that the average value of h^+ on F_z is sufficiently big. We are going to use this information to estimate from above the size of the union of F_z, $z \in E_n$.

Our argument runs as follows. Fix K mentioned in Lemma 2 and apply the Vitali lemma (see, for instance, [8, Sections 1.1.6, 1.1.7]; the argument there works also in the circle case) to the family of the arcs F_z, $z \in E_n$, on the circle $T = \{|w| = 1 - 2^{-3n}\}$. By the lemma, there are finitely many disjoint arcs of the type F_z, where $z \in E_n$, enumerate them $F_{z_1}, \dots, F_{z_m}$, such that

$$\bigcup_{z\in E_n} F_z \subset \bigcup_{1\le k\le m} F^5_{z_k},$$

where $F^5_{z_k}$ is the arc on T with the same center as F_{z_k}, and 5 times longer.

Now, by the harmonicity of h,

$$0 = \int_T h(z)\,dm(z) \ge \sum_{k=1}^m \int_{F_{z_k}} h^+(z)\,dm(z) - v(2^{-3n}) \ge \text{ (by Lemma 2) } \ge$$
$$cu(2^{-n}) \sum_{k=1}^m m(F_{z_k}) - v(2^{-3n}).$$

Since v is a weight function, we get

$$v(2^{-n}) \ge c_1 u(2^{-n}) \sum_{k=1}^m m(F_{z_k}).$$

Finally, note that for $z \in E_n$ and big n, $n > n(K)$, we have $\{z\}^* \subset (F_z)^*$, $m\big((F^5_{z_k})^*\big) \le 2m\big(F^5_{z_k}\big)$. Therefore,

$$m(E_n^*) = m\big(\cup_{z\in E_n}\{z\}^*\big) \le m\big(\big(\cup_{z\in E_n} F_z\big)^*\big) \le m\big(\cup_{1\le k\le m}\big(F^5_{z_k}\big)^*\big) \le$$
$$2\sum_{k=1}^m m\big(F^5_{z_k}\big) \le c_2 \frac{v(2^{-n})}{u(2^{-n})}. \quad \square$$

For a summable function g on the unit circle, its Poisson integral Pg is defined as

$$Pg(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} g(e^{i\theta})\,d\theta.$$

Lemma 3. *Suppose that a harmonic function h and two weight functions u and v satisfy condition* (5), *and*

$$\sum_{n\ge1} \frac{v^2(2^{-2^n})}{u(2^{-2^n})} < \infty. \tag{8}$$

Then for every α, $0<\alpha<1$, there exists a non-positive summable function g on the unit circle such that

$$Pg(z) < -v(1-|z|), \qquad z \in E_h^\alpha. \tag{9}$$

Proof. For a subset S of $\mathbb{T}$ denote by χ_S its charactersitic function. We construct g just as the sum of the functions

$$-Cv\big(2^{-2^n}\big)\chi_{(E_{2^n})^*}$$

for a suitable C. Lemma 1 and condition (8) garantee that the sum converges in $L^1(\mathbb{T})$. Since u is a weight function, we can choose C in such a way that for $z \in E_{2^n}$, $n \ge 0$,

$$Pg(z) \le -Cv\big(2^{-2^n}\big)\,\omega\big(\mathbb{D},(E_{2^n})^*,z\big) \le -Cv\big(2^{-2^n}\big)\,\omega\big(\mathbb{D},\{z\}^*,z\big) < -v(1-|z|).$$

To get (9) for z in $(1/2)\,\mathbb{D}$ we additionally increase C. $\square$

4. Proofs of Theorems.

Proof of Theorem 1. Without loss of generality assume that $f(0)=1$. Since $f\in A_0(u)$ and estimate (3) holds for f, the function $h=\log|f|$ satisfies the conditions of Lemma 3. Furthermore, (2) is equivalent to (8) for weight functions u and v. Therefore, applying Lemma 3 to $\alpha=1/2$ and h we get a summable function g satisfying (9), and a bounded outer function G, $\log|G|=Pg$. Denote $f_r(z)=f(rz)$, $0<r<1$, $E=E_h^{1/2}$. Now, for every r, $0<r<1$,

$$\frac{f}{f_r}G\in H^\infty f.$$

Furthermore,

$$\left\|\frac{f}{f_r}G\right\|_{A_0(u)}=\sup_{z\in\mathbb{D}}\left[\left|\frac{f(z)}{f_r(z)}\right||G(z)|e^{-u(1-|z|)}\right]\le$$

$$\sup_{z\in E}\left[\left|\frac{f(z)}{f_r(z)}\right|e^{Pg(z)-u(1-|z|)}\right]+\sup_{z\notin E}\left[\left|\frac{f(z)}{f_r(z)}\right|e^{-u(1-|z|)}\right]\le$$

$$\sup_{z\in E}\left[|f(z)|e^{-u(1-|z|)+v(1-r|z|)-v(1-|z|)}\right]+\sup_{z\notin E}\left[e^{v(1-r|z|)-u(1-|z|)/2}\right]\le\|f\|+c,$$

$$\frac{f}{f_r}G\overset{A_0(u)}{\longrightarrow}G.$$

Here we use that under condition (2),

$$v(x)=o(u(x)),\qquad x\to0. \tag{10}$$

Indeed, if v is bounded, then u should be unbounded. On the other hand, if v is unbounded, and $v(\exp(-t))>\varepsilon u(\exp(-t))$, then

$$\int_t^{2t}\frac{v^2(e^{-t})}{u(e^{-t})}\frac{dt}{t}>c\varepsilon v(e^{-t}),$$

where c depends only on u and v. For fixed ε, the expression in the right hand side tends to ∞ as $t\to\infty$, which contradicts to (2). Now, (10) follows.

As a result, we obtain that for the outer function G,

$$G\in\operatorname*{clos}_{A_0(u)}H^\infty f.$$

Hence,

$$1\in\operatorname*{clos}_{A_0(u)}H^\infty f,$$

and f is cyclic. $\square$

Proof of Theorem 2. We assume that $f(0) = 1$. By the mean value property for harmonic functions, since $f \in B^p(w)$, we have

$$\log|f(z)| < c \log\frac{1}{1-|z|} + \frac{1}{p}w(|z|), \qquad z \in \mathbb{D},$$

for some positive c, depending on f and p. Put

$$u(t) = c\log\frac{1}{t} + \frac{1}{p}w(t).$$

Then u is a weight function, u and v satisfy condition (2), and as a consequence, property (10).

We can fix a small positive ε such that

$$\int_{\mathbb{D}} e^{2\varepsilon pu(1-|z|)-w(1-|z|)}\,dm_2(z) < \infty. \tag{11}$$

Note that condition (4) on w and v is equivalent to condition (8) on u and v. Therefore, we can apply Lemma 3 to $\alpha = \varepsilon$ and $h = \log|f|$. As a result, there exist a summable function g and a bounded outer function G, $\log|G| = Pg$, such that

$$|G(z)| < \exp(-v(1-|z|)), \qquad z \in E_h^\varepsilon.$$

As in the proof of Theorem 1, we denote $f_r(z) = f(rz)$, $0 < r < 1$, $E = E_h^\varepsilon$, and obtain

$$\int_{\mathbb{D}}\left|\frac{f(z)}{f_r(z)}\right|^p |G(z)|^p e^{-w(1-|z|)}\,dm_2(z) \le$$

$$\int_E \left|\frac{f(z)}{f_r(z)}\right|^p e^{-pv(1-|z|)-w(1-|z|)}\,dm_2(z) + \int_{\mathbb{D}\setminus E}\left|\frac{f(z)}{f_r(z)}\right|^p e^{-w(1-|z|)}\,dm_2(z) \le$$

$$\int_E |f(z)|^p e^{pv(1-r|z|)-pv(1-|z|)-w(1-|z|)}\,dm_2(z) +$$

$$\int_{\mathbb{D}\setminus E} e^{\varepsilon pu(1-|z|)+pv(1-r|z|)-w(1-|z|)}\,dm_2(z) \le$$

$$\int_{\mathbb{D}} |f(z)|^p e^{-w(1-|z|)}\,dm_2(z) + c,$$

where c does not depend on r and f. The last integral over $\mathbb{D}\setminus E$ is estimated using (10) and (11).

Now, our argument is completed like in the previous proof. □

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