DENSITY OF POLYNOMIALS AND THE HAMBURGER MOMENT PROBLEM

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Abstract. We solve the weighted approximation problem on discrete subsets of the real line in the scale of L^p spaces. This result is applied to the classical Hamburger moment problem. In particular, we give a correct description of the canonical measures.

Densité des polynômes et la problème des moments de Hamburger

Résumé. Nous résolvons la problème de l'approximation pondérée par des polynômes sur des sous-ensembles discrets de l'axe réel dans l'échelle des espaces L^p. Ce résultat est applique au problème classique des moments de Hamburger. En particulier, nous donnons une description correcte des mesures canoniques.

Version française abrégée

La classe de Hamburger $\mathfrak{H}$ est constitué par toutes les fonctions B entières transcendantes réelles de type exponentiel nul dont les zéros λ sont réels (et simples) et telles que

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_B}} \frac{|\lambda|^n}{|B'(\lambda)|} = 0, \qquad n \ge 0,$$

où Λ_B est l'ensemble des zéros de B.

Pour une fonction B de la classe de Hamburger et un poids w tels que $w(\lambda) > 0$, $\lambda \in \Lambda_B$, et

$$\lim_{\substack{\lambda\to\infty \\ \lambda\in\Lambda_B}} |\lambda|^n w(\lambda) = 0, \qquad n \ge 0$$

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nous considerons les espaces de Banach $\ell^p(w)=\ell^p(w,\Lambda_B)$, $1\le p<\infty$, des fonctions a sur Λ_B, avec la norme

$$||a||^p_{\ell^p(w)}=\sum_{\lambda\in\Lambda_B}[w(\lambda)\,|a(\lambda)|]^p\,.$$

Comme B est de type exponentiel nul, notre condition sur w implique que les polynômes appartiennent à tous les espaces $\ell^p(w)$, $1\le p<\infty$.

Théorème 1. *Pour que les polynômes soient denses dans $\ell^p(w)$, il faut et il suffit que, pour chaque $F\in\mathfrak{H}$ qui est un diviseur de B (c'est-à-dire $\Lambda_F\subset\Lambda_B$),*

$$\sum_{\lambda\in\Lambda_F}\Big|\frac{1}{w(\lambda)F'(\lambda)}\Big|^{p/(p-1)}=+\infty,\qquad 1<p<\infty,$$

$$\liminf_{\substack{|\lambda|\to\infty\\ \lambda\in\Lambda_F}} w(\lambda)|F'(\lambda)|=0,\qquad p=1\,.$$

Nous déduisons notre résultat de son analogue pour l'espace $c_0(w)$ et $p=\infty$; cet analogue est le cas particulier du théorème de de Branges [4].

L'application principale de Théorème 1 concerne le problème des moments de Hamburger [1]. Considérons une mesure positive μ dans $\mathbb{R}$ qui a des moments de tous ordres $n\ge 0$. Une mesure μ est appelée *indéterminée* s'il existe une autre mesure positive ν, $\nu\ne\mu$ également supportée par $\mathbb{R}$ avec les mêmes moments: $\int_{\mathbb{R}}t^n d\nu=\int_{\mathbb{R}}t^n d\mu$. Une mesure indéterminée μ est appelée *canonique* si les polynômes sont denses dans $L^2(\mu)$. Un théorème de Marcel Riesz [1, Theorem 2.4.3] affirme que le support de chaque mesure canonique est l'ensemble des zéros d'une fonction entière de type exponentiel nul.

En 1944 H. Hamburger [5] déclara que la proposition suivante est correcte.

Proposition. *Une mesure positive μ est canonique si et seulement si pour chaque fonction $B\in\mathfrak{H}$ nous avons*

$$\text{(i)}\quad \mu=\sum_{\lambda\in\Lambda_B}\mu_\lambda\delta_\lambda;\qquad \sum_{\lambda\in\Lambda_B}|\lambda|^n\mu_\lambda<\infty,\quad n\ge 0,$$

$$\text{(ii)}\quad \sum_{\lambda\in\Lambda_B}\frac{1}{\mu_\lambda[B'(\lambda)]^2(1+\lambda^2)}<\infty,$$

$$\text{(iii)}\quad \sum_{\lambda\in\Lambda_B}\frac{1}{\mu_\lambda[B'(\lambda)]^2}=+\infty.$$

En particulier, pour les masses $\mu_\lambda=[B'(\lambda)]^{-2}$, $\lambda\in\Lambda_B$, les conditions (i)–(iii) sont satisfaisantes, et en conséquence, l'ensemble des zéros d'une fonction arbitraire dans $\mathfrak{H}$ serait le support d'une mesure canonique.

En 1989 une lacune dans le preuve de la Proposition de Hamburger a été trouvée par C. Berg et H. Pedersen. Bientôt P. Koosis [8] a construit un contre-exemple à la Proposition de Hamburger. La source de l'erreur de Hamburger réside dans le fait que la condition (iii) est nécessaire mais pas suffisante pour la densité des polynômes [3]. Dans [8] une fonction entière $B\in\mathfrak{H}$ est construite telle que pour la mesure $\mu=\sum_{\lambda\in\Lambda_B}[B'(\lambda)]^{-2}\delta_\lambda$, les polynômes ne sont pas denses dans $L^2(\mu)$, et alors, μ n'est pas canonique. Nous utilisons Théorème 1 pour deduire la version suivante correcte de la Proposition de Hamburger.

Théorème 2. *Pour rendre la Proposition de Hamburger correcte, il faut remplacer la condition* (iii) *par la condition suivante:*

(iii′) *pour chaque* $F \in \mathfrak{H}$ *telle que* $\Lambda_F \subset \Lambda_B$, *nous avons*

$$\sum_{\lambda\in\Lambda_F} \frac{1}{\mu_\lambda [F'(\lambda)]^2} = +\infty.$$

Corollaire. *Si* ν *est une mesure canonique et* $B \in \mathfrak{H}$, $\operatorname{supp}\nu = \Lambda_B$, *alors la mesure* $\mu = \sum_{\lambda\in\Lambda_B} [B'(\lambda)]^{-2}\delta_\lambda$ *est aussi une mesure canonique.*

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1. Introduction. Motivated by the classical problem of description of canonical solutions of the Hamburger moment problem in the indeterminate case (see [1, 2, 3]), we study a special case of the weighted polynomial approximation problem on the real line. In spite of significant efforts (see i.e. [7] and references therein) the general problem is still far from being completely solved. It is not even clear in what (explicit!) terms, if in any, such a solution might be formulated. We are interested here in a special case of the problem of density of the polynomials in $L^p(\mu)$ when the measure μ is supported by the zero set of an entire function of zero exponential type. Making use of an approach to the weighted approximation problem suggested by de Branges [4] in the general setting of the Bernstein approximation problem, we give a new necessary and sufficient condition for the density of the polynomials, which yields a complete description of canonical measures.

2. Main result. The Hamburger class $\mathfrak{H}$ consists of all transcendental real entire functions B of zero exponential type with only real (and simple) zeros λ such that

$$\lim_{\substack{|\lambda|\to\infty\\ \lambda\in\Lambda_B}} \frac{|\lambda|^n}{|B'(\lambda)|} = 0, \qquad n \ge 0,$$

where Λ_B is the zero set of B.

For a Hamburger class function B and a weight function w such that $w(\lambda) > 0$, $\lambda \in \Lambda_B$, and

$$\lim_{\substack{\lambda\to\infty\\ \lambda\in\Lambda_B}} |\lambda|^n w(\lambda) = 0, \qquad n \ge 0\,, \tag{2.1}$$

we consider the Banach spaces $\ell^p(w) = \ell^p(w, \Lambda_B)$, $1 \le p < \infty$, of functions a on Λ_B, with norm

$$\|a\|^p_{\ell^p(w)} = \sum_{\lambda\in\Lambda_B} [w(\lambda)\,|a(\lambda)|]^p\,.$$

As the limit case for $p = +\infty$, we consider the space $c_0(w) = c_0(w, \Lambda_B)$ of functions a on Λ_B such that

$$\lim_{\substack{|\lambda|\to\infty\\ \lambda\in\Lambda_B}} w(\lambda)\,|a(\lambda)| = 0,$$

with norm $\|a\|_{c_0(w)} = \sup_{\lambda\in\Lambda_B} w(\lambda)|a(\lambda)|$. For the sake of convenience, we set $\ell^p_*(w) = \ell^p(w)$, $1 \le p < \infty$, and $\ell^\infty_*(w) = c_0(w)$.

The dual spaces to our spaces $\ell^p_*(w)$ are $\ell^q(1/w) = \ell^q(1/w, \Lambda_B)$, $1/p + 1/q = 1$, with the pairing $(a, b) = \sum_{\lambda\in\Lambda_B} a(\lambda)\overline{b(\lambda)}$. Since B has zero exponential type, condition (2.1) implies that the polynomials belong to all the spaces $\ell^p_*(w)$, $1 \le p \le \infty$.

Theorem 1. *The polynomials are dense in $\ell^p_*(w)$ if and only if for every $F \in \mathfrak{H}$ which is a divisor of B (that is $\Lambda_F \subset \Lambda_B$),*

$$\sum_{\lambda\in\Lambda_F}\left|\frac{1}{w(\lambda)F'(\lambda)}\right|^{p/(p-1)} = +\infty, \qquad 1 < p \le \infty,$$

$$\liminf_{\substack{|\lambda|\to\infty\\ \lambda\in\Lambda_F}} w(\lambda)|F'(\lambda)| = 0, \qquad p = 1\,.$$

For $p=\infty$, this is a special case of de Branges' theorem [4]. We give a sketch of the proof of Theorem 1, in which we reduce the case of an arbitrary exponent $p \ge 1$ to the case $p=\infty$.

(a) The polynomials are not dense in $\ell^p_*(w)$, $1 \le p \le \infty$, if and only if there exists an entire function $f \ne 0$ of zero exponential type such that $f \in \ell^q\bigl(1/(w|B'|)\bigr)$, $1/p+1/q=1$, and

$$\lim_{|y|\to\infty}\left[|y|^n\left|\frac{f(iy)}{B(iy)}\right|\right] = 0, \qquad n \ge 0. \tag{2.2}$$

The proof basically repeats the argument of Koosis [8] which was given there in the special case $p=2$, $w(\lambda)=1/|B'(\lambda)|$. Suppose, for instance, that the polynomials are note dense in $\ell^p_*(w)$. Then there exists a non-zero element b of $\ell^q(1/w)$ with $\sum_{\lambda\in\Lambda_B}\lambda^n b(\lambda)=0$ for $n\ge 0$. With $c=bB'$, the formula $f(z)=B(z)\sum_{\lambda\in\Lambda_B}c(\lambda)/[(z-\lambda)B'(\lambda)]$ is readily seen to define a function f of zero exponential type, for which one also has $z^n f(z) = B(z)\sum_{\lambda\in\Lambda_B}\lambda^n c(\lambda)/[(z-\lambda)B'(\lambda)]$, $n\ge 0$. ¿From this, (2.2) is easily established. The argument is reversible.

(b) If $F\in\mathfrak{H}$ is a divisor of B such that $\mathbf{1}\in\ell^q(1/(w|F'|),\Lambda_F)$, then the polynomials are not dense in $\ell^p_*(w)$, $1/p+1/q=1$, $1\le p\le\infty$. Here $\mathbf{1}$ is the function identically equal to one.

To prove this we just apply (a) with $f(z)=B(z)/F(z)$.

The necessity in Theorem 1 is proved. The statement converse to (b) (i.e., the sufficiency in Theorem 1) is more delicate. The following statement is a special case of de Branges' theorem [4,7, Section VI F]:

(c) If the polynomials are not dense in $c_0(w,\Lambda_B)$, then for some $F\in\mathfrak{H}$ which is a divisor of B, $\mathbf{1}\in\ell^1\bigl(1/(w|F'|),\Lambda_F\bigr)$.

Our aim is to extend this statement to all $p\ge 1$. We call an exponent p, $1\le p\le\infty$, *normal* (for the pair B,w) if $\mathbf{1}\notin\ell^q\bigl(1/(w|B'|)\bigr)$, $1/p+1/q=1$. If p is a normal exponent, then all r, $p<r\le\infty$, are normal exponents as well. For normal exponents every entire function f in (a) is automaticaly transcendental (since, f being in $\ell^q(1/(w|B'|))$, there is an infinite subsequence $\{\lambda_n\}\subset\Lambda_B$ with $f(\lambda_n)\to 0$ for $n\to\infty$), and as a consequence has infinitely many zeros. Dividing it by an arbitrary polynomial divisor, we get another function satisfying the conditions of (a). This gives us the following:

(d) Let p be a normal exponent. The polynomials are not dense in the space $\ell^p_*(w)$ if and only if for every n there exists an entire function $f\ne 0$ of zero exponential type satisfying condition (2.2) and such that $|f(\lambda)|\le(1+|\lambda|)^{-n}w(\lambda)|B'(\lambda)|$, for $\lambda\in\Lambda_B$.

(e) If the polynomials are dense in $\ell^p_*(w)$ for some normal exponent p, then they are dense in $\ell^p_*(w)$ for *all* normal exponents p.

Combining this result with (b) and (c), we obtain the sufficiency in Theorem 1 for normal exponents p. Finally, note that for exponents p that are not normal the polynomials are not dense as a consequence of (b) (with $F = B$). This completes the proof of Theorem 1.

Remark. In the proof of Theorem 1 we used duality arguments extensively. It would be interesting to study the same problem in the spaces $\ell^p(w)$ for $p < 1$.

Note that quasianalyticity theorems given in Chapter I of [6] may be interpreted as statements on weighted polynomial approximation on zero sets of entire functions of zero exponential type (and of convergence class) when the weight w is log-concave. In the opposite direction, de Branges' theorem provides results on the kind of quasianalyticity problems considered in [6].

3. Hamburger moment problem. The main application of Theorem 1 pertains to the Hamburger moment problem [1]. Consider a positive measure μ on $\mathbb{R}$ which has moments of all orders $n \geq 0$. A measure μ is called *indeterminate* if there exists another positive measure ν, $\nu \neq \mu$ also supported by $\mathbb{R}$ with the same moments: $\int_{\mathbb{R}} t^n d\nu = \int_{\mathbb{R}} t^n d\mu$. An indeterminate measure μ is called *canonical* if the polynomials are dense in $L^2(\mu)$ (the classical definition sounds quite different [1, Section 3.4], however, M. Riesz' theorem [1, Theorem 2.3.2] asserts the equivalence of the classical definition and the definition we use). This class of measures plays an important rôle in function theory. For results in operator theory related to the Hamburger moment problem see [1, Chapters 3, 4]. Another theorem of M. Riesz [1, Theorem 2.4.3] implies that the support of every canonical measure is the zero set of an entire function of zero exponential type.

In 1944 H. Hamburger claimed the following statement to be valid.

Statement (Hamburger [5], [1, Addenda and Problems to Chapter 4])**.** *A positive measure μ is a canonical measure if and only for some function $B \in \mathfrak{H}$ we have*

$$\text{(i)} \quad \mu = \sum_{\lambda\in\Lambda_B} \mu_\lambda \delta_\lambda; \qquad \sum_{\lambda\in\Lambda_B} |\lambda|^n \mu_\lambda < \infty, \quad n \geq 0,$$

$$\text{(ii)} \quad \sum_{\lambda\in\Lambda_B} \frac{1}{\mu_\lambda [B'(\lambda)]^2 (1+\lambda^2)} < \infty,$$

$$\text{(iii)} \quad \sum_{\lambda\in\Lambda_B} \frac{1}{\mu_\lambda [B'(\lambda)]^2} = +\infty.$$

In particular, for the masses $\mu_\lambda = [B'(\lambda)]^{-2}$, $\lambda \in \Lambda_B$, conditions (i)–(iii) are fulfilled, and as a result, the zero set of an arbitrary entire functions in $\mathfrak{H}$ should be the support of a canonical measure.

In 1989 a gap in the proof of Hamburger's statement was found by C. Berg and H. Pedersen. Soon P. Koosis [8] constructed a counterexample to Hamburger's statement. The source of Hamburger's mistake was in the fact that condition (iii) is necessary but not sufficient for the density of the polynomials [3]. In [8] an entire function $B \in \mathfrak{H}$ is constructed such that for the measure $\mu = \sum_{\lambda\in\Lambda_B} [B'(\lambda)]^{-2} \delta_\lambda$, the polynomials are not dense in $L^2(\mu)$, and hence, μ is not canonical. Lemma 2 and Theorem 1 in [1, Addenda and Problems to Chapter 4]) show that the measure μ is canonical if and only if conditions (i) and (ii) from Hamburger's statement are fulfilled and the polynomials are dense in $L^2(\mu)$. Thus Theorem 1 permits us to derive a correct version of Hamburger's statement:

Theorem 2. *To make Hamburger's statement correct, condition* (iii) *should be replaced by the following:*

(iii′) *for every $F \in \mathfrak{H}$ such that $\Lambda_F \subset \Lambda_B$, we have*

$$\sum_{\lambda\in\Lambda_F} \frac{1}{\mu_\lambda[F'(\lambda)]^2} = +\infty.$$

Corollary. *If ν is a canonical measure and $B \in \mathfrak{H}$, $\operatorname{supp}\nu = \Lambda_B$, then the measure $\mu = \sum_{\lambda\in\Lambda_B}[B'(\lambda)]^{-2}\delta_\lambda$ is also a canonical measure (which may correspond to another indeterminate moment problem).*

Proof. If μ is not a canonical measure, then by Theorem 2, for some divisor $F \in \mathfrak{H}$ of B we have $\sum_{\lambda\in\Lambda_F} |B'(\lambda)|^2/|F'(\lambda)|^2 < \infty$, and in particular B/F is not a constant. Pick a zero w of B/F and consider $F_0(z) = F(z)(z-w)$. Then by condition (ii) of Hamburger's statement, $\nu_\lambda[B'(\lambda)]^2(1+\lambda^2) \geq c > 0$, so

$$\sum_{\lambda\in\Lambda_F} \frac{1}{\nu_\lambda[F_0'(\lambda)]^2} \leq C \sum_{\lambda\in\Lambda_F} \left|\frac{B'(\lambda)}{F'(\lambda)}\right|^2 \frac{1+\lambda^2}{(\lambda-w)^2} < \infty\,,$$

and it follows from Theorem 2 that ν cannot be a canonical measure.

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