

# The polynomial approximation property in Fock-type spaces

A. A. BORICHEV\*

## 1. INTRODUCTION.

Let  $\lambda(r)$  be a positive function on  $[0, \infty)$ ,  $\log r = o(\lambda(r))$ ,  $r \rightarrow \infty$ . Then the Fock space  $F_\lambda$ , associated with  $\lambda$ , consists of all entire functions  $f$  such that

$$\|f\|_{F_\lambda}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-\lambda(|z|)} dm_2(z) < \infty,$$

where  $dm_2$  is Lebesgue area measure. We deal with the following question, posed and discussed in [9] (and earlier in [8]).

*Do the polynomial multiples of an entire function  $\varphi$  span (in  $F_\lambda$ ,  $\lambda(r) = r^2$ ) all functions which vanish wherever  $\varphi$  vanishes?* We take into account the multiplicities of zeros, and require that all the polynomial (or even exponential polynomial) multiples of  $\varphi$  belong to  $F_\lambda$ . In other words, the question is whether the polynomials are dense in the Hilbert space  $F_{\lambda, \varphi}$  of analytic (entire) functions square summable with respect to the measure  $|\varphi(z)|^2 e^{-\lambda(|z|)} dm_2(z)$ . Let us remark here that the general problem of weighted polynomial approximation for weights vanishing at some inner points of the domain of analyticity is not yet sufficiently investigated.

For some classes of  $\lambda$  we construct  $\varphi$  for which the answer to our question is negative.

The method used is close to that in [2,3]. The idea is to produce an element  $\varphi$  of  $F_\lambda$  having extremal growth along a sufficiently massive subset of the plane. Then a Phragmén–Lindelöf type argument shows that if polynomial multiples  $p_n \varphi$  converge to an element in  $F_\lambda$ , then the polynomials  $p_n$  cannot be large enough to approximate all the elements in  $F_{\lambda, \varphi}$ . To use this argument we need first to consider (in Section 5) a problem on the harmonic measure in the unit disc. In Section 2 we formulate the main results. An element of extremal growth is constructed in Section 4. Our construction uses an auxiliary lemma describing general weight functions  $\lambda$  as restrictions on the positive axis of entire functions with positive coefficients, which is proved in Section 3. Finally, the Phragmén–Lindelöf type argument is given in Section 6.

The author is thankful to H. S. Shapiro for calling his attention to this problem.

## 2. MAIN RESULTS.

We are going to deal with the following two conditions on a strictly increasing  $C^2$ -smooth function  $\lambda$ .

- (i)  $\lambda'(2t) \leq c\lambda'(t)$  for some  $c$  and sufficiently big  $t$ .
- (ii)<sub>k</sub>  $\liminf_{t \rightarrow \infty} \exp(-kt)(\lambda(\exp t))'' > 0$ .

---

\*The research was supported in part by the Swedish Natural Science Research Council.

Note that condition (ii<sub>1</sub>) always holds for convex  $\lambda$ , and condition (ii<sub>k</sub>) holds for  $\lambda(x) = x^s$ ,  $s \geq k$ .

For  $0 < p < \infty$  consider the family of Fock-type spaces

$$F_\lambda^p = \text{the set of all entire functions } f \text{ such that } \int_{\mathbb{C}} |f(z)|^p e^{-\lambda(|z|)} dm_2(z) < \infty.$$

They are Banach spaces for  $1 \leq p < \infty$ , and complete metric spaces for  $0 < p < 1$ .

**THEOREM.** *Let  $\lambda$  satisfy conditions (i) and (ii<sub>k</sub>) for some integer number  $k$ ,  $k \geq 1$ ,  $0 < p < \infty$ , and let  $s$  be a positive number such that  $s < k/2$ . There exist an entire function  $f$  and a positive number  $a$  such that  $e^{-az^k} f \in F_\lambda^p$ ,*

$$\int_{\mathbb{C}} |f(z)|^p e^{|z|^s} e^{-\lambda(|z|)} dm_2(z) < \infty,$$

but

$$e^{-az^k} f \notin \text{clos}_{F_\lambda^p} \mathcal{L}(\{z^n f, n \geq 0\}).$$

Here  $\mathcal{L}(E)$  is the linear hull of  $E$ .

**COROLLARY 1.** *Let  $\lambda$  satisfy conditions (i) and (ii<sub>1</sub>),  $0 < p < \infty$ . There exist an entire function  $f$  and a positive number  $a$  such that  $e^{-az} f \in F_\lambda^p$ , and for every  $n \geq 0$ ,  $z^n f \in F_\lambda^p$ , but*

$$e^{-az} f \notin \text{clos}_{F_\lambda^p} \mathcal{L}(\{z^n f, n \geq 0\}).$$

**COROLLARY 2.** *Let  $\lambda$  satisfy conditions (i) and (ii<sub>3</sub>),  $0 < p < \infty$ . There exist an entire function  $f$  and a positive number  $a$  such that  $e^{-az^3} f \in F_\lambda^p$ , and for every  $b \in \mathbb{C}$ ,  $e^{bz} f \in F_\lambda^p$ , but*

$$e^{-az^3} f \notin \text{clos}_{F_\lambda^p} \mathcal{L}(\{z^n f, n \geq 0\}).$$

**REMARK:** Actually, the proof of Theorem shows that for  $k \geq 3$  there exist such  $f$  that the function  $e^{-az^k} f$  cannot be approximated in  $F_\lambda^p$  even by exponential polynomial multiples of  $f$  or, in general, by products  $h^v f$  where  $h^v$  are entire functions of order less than or equal to  $v$ ,  $v < k/2$ .

**REMARK:** The (unbounded) operator  $T_f$  of multiplication by  $f$  and more general Toeplitz operators on the Fock space,  $p = 2$ ,  $\lambda(t) = t^2$ , were studied by J. Janas in a series of recent papers. J. Janas and J. Stochel use in [6] the notion of a core for an unbounded operator  $T$ . This is just a linear subspace  $L$  of the domain  $D(T)$  of  $T$  such that the graph of  $T$  is the closure of the graph of  $T|_L$ .

H. S. Shapiro pointed out that entire functions  $f$  constructed in Theorem (for  $p = 2$ ,  $0 < s < 1$ ,  $\lambda(t) = t^2$ ) satisfy the following property:

*Polynomials belong to  $D(T_f)$ , but the set of polynomials is not a core for  $T_f$ .*

### 3. AUXILIARY LEMMA.

To prove Theorem we use an auxiliary statement. It is similar to results on Legendre-type transforms in [10], [4], [1, Appendix]. In fact, to prove Theorem itself it would be enough to use only the result from [4]. However, our lemma which gives a criterion for a weight function to be representable up to a multiplicative constant by an entire function with positive coefficients seems to be of independent interest. The proof is close to that in [4].

LEMMA 1. *If  $\sigma$  is  $C^2$ -smooth function on  $\mathbb{R}_+$  such that  $\sigma'(0) = 1$ ,*

$$\sigma''(x) \geq \alpha > 0, \quad x \geq 0, \quad (1)$$

*then there exists an entire function  $g$ ,*

$$g(z) = \sum_{n \geq 0} x_n z^n, \quad x_n \geq 0,$$

*such that*

$$0 < c_1 \leq g(x)e^{-\sigma(\log x)} \leq c_2, \quad x \geq 1,$$

*where  $c_1$  and  $c_2$  depend only on  $\alpha$ .*

PROOF: Let points  $a_n \in \mathbb{R}_+$  be defined by the equalities

$$\sigma'(a_n) = n,$$

and put

$$b_n = \sigma(a_n).$$

For every  $m, n \in \mathbb{N}$ ,  $m < n$ , denote

$$\begin{aligned} I_{m,n} &= [a_m, a_n], \\ \delta_{m,n} &= \max_{t \in I_{m,n}} \min \{ \sigma(t) - b_m - m(t - a_m), \sigma(t) - b_n - n(t - a_n) \}, \\ \gamma_{m,n} &= \min \{ b_n - b_m - m(a_n - a_m), b_m - b_n - n(a_m - a_n) \}. \end{aligned}$$

It is clear that  $\gamma_{m,n} \geq \delta_{m,n}$ . Condition (1) implies that

$$0 < a_{n+1} - a_n < c$$

for some  $c$  depending only on  $\alpha$ . Choose a sequence  $\{n_k\}_{k \geq 1}$  in the following way. Put  $n_1 = 1$ . If  $n_k$  is already chosen, take as  $n_{k+1}$  the minimal integer number  $s$ ,  $s \geq n_k + 1$ , such that  $\delta_{n_k, s} \geq 1$ . Then

$$\gamma_{n_k, n_{k+1}} \geq \delta_{n_k, n_{k+1}} \geq 1. \quad (2)$$

It is clear that  $a_{n_{k+1}} - a_{n_k} < c$  for some  $c$  depending only on  $\alpha$ .

Let us verify that for some positive constants  $c_1$  and  $c_2$  depending only on  $\alpha$ ,

$$c_1 \leq \exp(-\sigma(t)) \sum_{k \geq 1} \exp[b_{n_k} + n_k(t - a_{n_k})] \leq c_2, \quad t \geq 0.$$

Indeed, the left estimate follows from the inequality

$$\delta_{n_k, n_{k+1}} < c,$$

where  $c$  depends only on  $\alpha$ , which is an immediate consequence of convexity of  $\sigma$  and the inequalities

$$\begin{aligned} \delta_{n_k, n_{k+1}-1} &< 1, \\ a_{n_{k+1}} - a_{n_k} &< c. \end{aligned}$$

The right estimate is a consequence of the inequality

$$\gamma_{n_k, n_m} \geq |k - m|,$$

which follows from (2) by induction.

Thus, we can put

$$g(z) = \sum_{k \geq 1} e^{(b_{n_k} - n_k a_{n_k})} z^{n_k},$$

and the lemma is proved. ■

#### 4. THE PROOF OF THEOREM. CONSTRUCTION OF AN EXTREMAL FUNCTION.

For the sake of simplicity we consider only the case  $p = 1$ . For other  $p$  the argument is identical. In this Section we construct  $f$  and verify its growth properties. In the next two sections we prove that  $e^{az^k}f$  cannot be approximated by polynomial multiples of  $f$ .

Pick  $\theta, b$  such that  $s < b < \theta < k/2$ . Condition (ii<sub>k</sub>) on  $\lambda$  implies that for some  $a > 0$ , we can redefine the function  $\sigma$ ,

$$\sigma(t) = \lambda(e^{t/k}) - ae^t - e^{bt/k},$$

on a finite interval of  $\mathbb{R}_+$  in such a way that it satisfies condition (1). Lemma 1 gives us an entire function  $g$  with positive Taylor coefficients such that

$$0 < c_1 \leq g(x) \exp(-\lambda(x^{1/k}) + ax + x^{b/k}) \leq c_2 < \infty, \quad x \geq 0. \quad (3)$$

Put  $f(z) = g(z^k)e^{az^k}$ . Then

$$\int_{\mathbb{C}} |f(z)| e^{|z|^s} e^{-\lambda(|z|)} dm_2(z) \leq C \int_0^\infty \exp(\lambda(r) - ar^k - r^b + ar^k + r^s - \lambda(r)) r dr < \infty.$$

Analogously,  $g = e^{-az^k}f \in F_\lambda^1$ .

## 5. HARMONIC MEASURE IN THE UNIT DISC.

To complete the proof we need first to consider the following problem. Let  $0 < \delta < 1/2$ ,  $k \geq 1$ ,

$$\Omega_k = \mathbb{D} \setminus \bigcup_{n=0}^{k-1} [\delta, 1) e^{2\pi i n/k},$$

$$a_k = \max_{|z|=\delta} \omega(\Omega_k, \mathbb{T}, z),$$

where  $\omega(\cdot, \cdot, \cdot)$  is the harmonic measure. What is the asymptotics of  $a_k$  for  $k \rightarrow \infty$ ?

LEMMA 2. *For an absolute constant  $\beta$ ,*

$$a_k < \beta \delta^{k/2}.$$

PROOF: We just apply two conformal mappings:  $z \rightarrow z^k$  and  $z \rightarrow (\delta^k - z)^{1/2}$  where the branch of the square root function is taken with cut along  $(-\infty, 0)$ . If  $\Omega'_k$  is the set of the points  $z$  in the right half-plane such that  $\delta^k - z^2 \in \mathbb{D}$ , and  $\Gamma_k$  is the subset of  $\Omega'_k$  consisting of the points  $z$  such that  $\delta^k - z^2 \in \delta^k \mathbb{T}$ , then

$$\max_{|z|=\delta} \omega(\Omega_k, \mathbb{T}, z) = \max_{z \in \Gamma_k} \omega(\Omega'_k, \Omega'_k \cap i\mathbb{R}, z).$$

The last maximum is easily estimated as  $\text{const} \cdot \delta^{k/2}$ . ■

Now fix  $k$ . Let  $\varepsilon$  be a small positive number,  $D \geq 2$ , and let  $E_n$ ,  $0 \leq n \leq k-1$ , be open subsets of angles

$$\left\{ z = r e^{i\alpha} : \left| \alpha - \frac{2\pi n}{k} \right| < \varepsilon \right\}$$

such that their radial projections  $E'_n$  satisfy the condition

$$m([1/D, 1) \setminus E'_n) < \varepsilon,$$

where  $m$  is Lebesgue measure on  $\mathbb{R}$ .

Hall's lemma (see [5, p.367]) and the theorem on two constants (see [7, p.257]) permit us to draw the following corollary of Lemma 2.

LEMMA 3. *For every  $\theta$ ,  $0 < \theta < k/2$ , there exist such small positive number  $\varepsilon$ ,  $\varepsilon = \varepsilon(\theta, k)$ , and big integer number  $D$ ,  $D = D(\theta, k)$ , that for every function  $h$  subharmonic in  $\mathbb{D}$  and such that*

$$\begin{aligned} h(z) &\leq c_1, & z &\in \mathbb{D}, \\ h(z) &\leq c_2, & z &\in E_n, \quad n = 0, \dots, k-1, \end{aligned}$$

where  $E_n$  are as above, we have

$$h(z) \leq c_1 D^{-\theta} + c_2 (1 - D^{-\theta}), \quad |z| = 1/D.$$

## 6. THE END OF THE PROOF. A PHRAGMÉN–LINDELÖF TYPE ARGUMENT.

Now, suppose that

$$p_k f \xrightarrow{F_\lambda^1} g$$

for a sequence of polynomials  $\{p_k\}_{k \geq 1}$ . Then for some  $c$  independent of  $k$  we have

$$\int_{\mathbb{C}} \left| \frac{p_k(z)}{c} \right| |f(z)| e^{-\lambda(|z|)} dm_2(z) \leq 1. \quad (4)$$

We are going to verify that (4) implies an estimate (uniform in  $k$ ) on the polynomials  $p_k$  which is incompatible with pointwise convergence of  $p_k$  to  $e^{-az^k}$ .

Take a polynomial  $p$  such that

$$\int_{\mathbb{C}} |p(z)| |f(z)| e^{-\lambda(|z|)} dm_2(z) \leq 1.$$

Fix  $\varepsilon = \varepsilon(\theta, k)$ ,  $D = D(\theta, k)$  from Lemma 3 ( $\theta$  is chosen in Section 4). Put  $\gamma(t) = \sigma(k \log t) + at^k = \lambda(t) - t^b$ . Furthermore, take  $m \geq 1$ , and let  $N$  be the minimal integer number bigger than  $\lambda'(D^m)$ .

Consider the rectangle  $A_m$ ,

$$A_m = \{z \in \mathbb{C} : D^m \leq \operatorname{Re} z \leq D^{m+1}, N|\operatorname{Im} z| \leq 1\},$$

and its subset  $E_m$ ,

$$E_m = \{z \in A_m : |p(z)| < N \exp(D^{b(m+2)})\}.$$

$A_m$  is the union of  $(D^{m+1} - D^m)N/2$  squares

$$A_{m,s} = \{z \in \mathbb{C} : D^m + 2(s-1)/N \leq \operatorname{Re} z \leq D^m + 2s/N, N|\operatorname{Im} z| \leq 1\},$$

$$1 \leq s \leq (D^{m+1} - D^m)N/2.$$

By conditions (i) on  $\lambda$  and (1) on  $\sigma$ ,

$$\max\{\gamma'(x) : D^{m-1} \leq x \leq D^{m+1}\} \leq C_1 N,$$

$$|\gamma(\log |z|) - \gamma(\log(D^m + (2s-1)/N))| \leq C_2, \quad z \in A_{m,s}, \quad 1 \leq s \leq (D^{m+1} - D^m)N/2,$$

where  $C_1, C_2$  do not depend on  $m$  and  $s$ . Furthermore, by (3),

$$\begin{aligned} \log |f(z)| &\leq \gamma(|z|) + C_3, & z \in \mathbb{C}, \\ \log |f(x)| &\geq \gamma(x) + C_4, & x \geq 0. \end{aligned}$$

Since the function  $\log |f|$  is subharmonic, we obtain that

$$\log |f(z)| \geq \gamma(|z|) + C_5$$

on a subset  $B_{m,s}$  of  $A_{m,s}$  whose Lebesgue measure  $m_2(B_{m,s})$  is at least  $(1-\varepsilon/2)m_2(A_{m,s})$ , for some constants  $C_3, C_4, C_5$  independent of  $m$  and  $s$ . Denote  $B_m = \cup_s B_{m,s}$ . We have

$$\begin{aligned} 1 &\geq \int_{B_m \setminus E_m} |p(z)| |f(z)| e^{-\lambda(|z|)} dm_2(z) \geq \\ &\geq C \int_{B_m \setminus E_m} N \exp(D^{b(m+2)}) e^{\gamma(|z|)} e^{-\lambda(|z|)} dm_2(z) = \\ &= CN \exp(D^{b(m+2)}) \int_{B_m \setminus E_m} e^{-|z|^b} dm_2(z). \end{aligned}$$

Therefore, for sufficiently big  $m$ ,  $m_2(B_m \setminus E_m) < \varepsilon m_2(B_m)/2$ , and as a result, the length of the radial projection of  $E_m$  on  $\mathbb{R}$  is bigger than  $(1-\varepsilon)(D^{m+1} - D^m)$ .

The same argument works for the rectangles  $A_m^n$ ,

$$A_m^n = e^{2\pi i n/k} A_m, \quad 0 \leq n \leq k-1,$$

and their subsets  $E_m^n$ ,

$$E_m^n = \{z \in A_m^n : |p(z)| < N \exp(D^{b(m+2)})\}.$$

Now we use the following Phragmén–Lindelöf type reasoning. Since  $p$  is a polynomial, there exists  $C_p$  such that

$$\log |p(z)| \leq C_p \log(1 + |z|).$$

Fix sufficiently big  $m_0$ , and take some  $m$ ,  $m \geq m_0$ . Then the function  $h(z) = \log |p(z)|$  satisfies the following estimates:

$$\begin{aligned} h(z) &\leq cC_p m, \quad |z| = D^{m+1}, \\ h(z) &\leq cD^{bm} + \log(\lambda'(D^m)), \quad z \in E_m^n, \quad 0 \leq n \leq k-1, \end{aligned}$$

for some  $c$  which does not depend on  $p$ . Thus, Lemma 3 applies to the function  $h(D^{m+1}z)$  subharmonic in  $\mathbb{D}$  and shows that

$$h(z) \leq cC_p m D^{-\theta} + [cD^{bm} + \log(\lambda'(D^m))], \quad |z| = D^m.$$

Repeating this procedure  $(m - m_0)$  more times we obtain that

$$h(z) \leq cC_p m D^{-\theta(m-m_0+1)} + \sum_{j=0}^{m-m_0} D^{-\theta j} [cD^{b(m_0+j)} + \log(\lambda'(D^{m_0+j}))], \quad |z| = D^{m_0}.$$

When  $m$  tends to  $\infty$ , we get

$$h(z) \leq \sum_{j=0}^{\infty} D^{-\theta j} [cD^{b(m_0+j)} + \log(\lambda'(D^{m_0+j}))], \quad |z| = D^{m_0}.$$

Condition (i) on  $\lambda$  and the inequality  $b < \theta$  give us now that

$$h(z) \leq c_1 D^{bm_0}, \quad |z| = D^{m_0},$$

and by the maximum principle,

$$|p(z)| \leq e^{c_2(|z|^b+1)}, \quad z \in \mathbb{C},$$

where  $c_1$  and  $c_2$  do not depend on  $p$ .

Therefore, if polynomials  $p_k$  satisfy estimate (4), they cannot approximate the function  $\exp(-az^k)$ . Theorem is proved. ■

## REFERENCES

- [1] A. A. Borichev, H. Hedenmalm, *Completeness of translates in weighted spaces on the half-line*, Acta Mathematica **174** no.1 (1995), 1–84.
- [2] A. A. Borichev, H. Hedenmalm, *Cyclicity in Bergman-type spaces*, International Mathematics Research Notices no.5 (1995), 253–262.
- [3] A. A. Borichev, H. Hedenmalm, *Cyclicity in Bergman-type spaces and harmonic functions of maximal growth*, Uppsala University Department of Mathematics Preprints 1994:47, 25 pp.
- [4] Y. Domar, *Closed primary ideals in a class of Banach algebras*, Math. Scand. **7** (1959), 109–125.
- [5] J. Garnett, *Bounded analytic functions*, Academic Press, 1981.
- [6] J. Janas, J. Stochel, *Unbounded Toeplitz operators in the Segal–Bargmann space, II*, J. Funct. Anal. **126** (1994), 418–447.
- [7] P. Koosis, *The logarithmic integral, vol. I*, Cambridge University Press, Cambridge, 1988.
- [8] D. J. Newman, H. S. Shapiro, *Fisher spaces of entire functions*, Proc. Symp. Pure Math., Amer. Math. Soc. **11** (1968), 360–369.
- [9] H. S. Shapiro, *Comparative approximation in two topologies*, Approximation Theory, Banach Center Publ., Warsaw **4** (1979), 225–232.
- [10] V. Vidensky, *On normally increasing functions*, Uspekhi Mat. Nauk **9** no.2 (1954), 212–213 [in Russian].

*Keywords.* Polynomial approximation, Fock spaces

DEPARTMENT OF MATHEMATICS, UPPSALA UNIVERSITY, SWEDEN

Address after September 1, 1996:

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BORDEAUX I, 351, COURS DE LA LIBERATION, 33405 TALENCE, FRANCE

E-mail: borichev@math.u-bordeaux.fr