Invariant subspaces of given index in Banach spaces of analytic functions

Alexander Borichev

0. Abstract

For a wide class of Banach spaces of analytic functions in the unit disc including all weighted Bergman spaces with radial weights and for weighted ℓ^p_A spaces we construct z-invariant subspaces of index n, $2 \le n \le +\infty$, without common zeros in the unit disc.

1. Introduction

Consider the operator M_z of multiplication by z acting (continuously) on a Banach space B of analytic functions on the unit disc $\mathbb{D}$. We assume that

(i) B is a linear subspace of the space $\mathrm{Hol}(\mathbb{D})$ of all analytic functions on $\mathbb{D}$ equipped with the topology of uniform convergence on compact subsets, and the imbedding $B \hookrightarrow \mathrm{Hol}(\mathbb{D})$ is continuous.

A (closed) subspace E of B is called (z-)invariant if $zE = M_z E \subset E$. The index of E is the dimension of the quotient space $E/\operatorname{Clos}_B(zE)$. It is denoted by $\operatorname{ind} E$, and takes values in $\{0, 1, 2, \dots, +\infty\}$. Clearly, $\operatorname{ind} E = 0 \iff E = \{0\}$.

Given $f \in B$, denote

$$[f] = [f]_B = \operatorname{Clos}_B \mathcal{L}\{z^n f, n \ge 0\},$$

where $\mathcal{L}$ stands for the linear hull. Then $\operatorname{ind}[f] = 1$ for every $f \ne 0$. In particular, Beurling's theorem implies that every non-zero invariant subspace in the Hardy space H^2 has index 1.

If we assume that the operator of multiplication by $z - \lambda$ is bounded from below on B for every $\lambda \in \mathbb{D}$, then for every invariant subspace E, $\lambda \in \mathbb{D}$, $(z-\lambda)E$ is closed, and $\dim(E/(z-\lambda)E) = \operatorname{ind} E$ does not depend on $\lambda \in \mathbb{D}$. Furthermore, under this assumption, $\operatorname{ind} E = 1$ if and only if E satisfies *the division property*: for every (some) $\lambda \in \mathbb{D}$ which is not a common zero of the functions in E, and for every $f \in E$ such that $f(\lambda) = 0$, there exists $g \in E$ such that $(z - \lambda)g = f$ (for this and other information on index see [**16**]).

Results of Apostol, Bercovici, Foiaş and Pearcy [**2**], [**4**, Corollary 5.5, Theorem 10.5] imply the existence of invariant subspaces of arbitrary index in the class $\mathcal{H}$ of weighted Hilbert spaces

$$H = \ell^2_A(\{\gamma_n\}_{n\ge 0}) = \Big\{ f = \sum_{n \ge 0} a_n z^n : \|f\|^2_{\ell^2_A(\{\gamma_n\}_{n\ge0})} = \sum_{n\ge 0} \gamma_n^2 |a_n|^2 < \infty \Big\},$$

that satisfy the conditions $\|M_z\| = 1$, $M_z \in C_{00}$, $\sigma(M_z) = \operatorname{Clos}\mathbb{D}$. These conditions mean that

$$\lim_{n\to\infty} \|z^n f\| = 0, \quad f \in H, \qquad\qquad \|M_z^n\| = 1, \quad n \ge 1;$$

they can be rewritten in terms of the sequence $\{\gamma_n\}_{n\geq 0}$ as follows:

$$0<\gamma_{n+1}\leq\gamma_n,\quad n\geq 0,\qquad \lim_{n\to\infty}\gamma_n=0,\qquad \sup_k\frac{\gamma_{k+n}}{\gamma_k}=1,\quad n\geq 1.$$

In particular, given a finite positive measure μ on $[0,1]$ with $1\in\operatorname{supp}\mu$, $\mu(\{1\})=0$, the spaces

$$B^2(\mu)=\Big\{f\in\operatorname{Hol}(\mathbb{D})\ :\|f\|^2=\int_{re^{i\theta}\in\mathbb{D}}|f(re^{i\theta})|^2d\theta d\mu(r)<\infty\Big\}$$

belong to the class $\mathcal{H}$.

Furthermore, Eschmeier [**8**] proved the existence of invariant subspaces of arbitrary index in the spaces B^p, $1\leq p<\infty$,

$$B^p=\Big\{f\in\operatorname{Hol}(\mathbb{D})\ :\|f\|^p=\int_{\mathbb{D}}|f(z)|^p dm_2(z)<\infty\Big\},$$

where dm_2 is Lebesgue planar measure.

The arguments given in [**2,8**] are pure existence proofs. The first explicit construction of a subspace in the Bergman space B^2 having index 2 was given by Hedenmalm [**11**]. This construction includes two ingredients. First, a result by Richter [**16**] implies that given two invariant subspaces E_1 and E_2 of index 1, the smallest invariant subspace containing their sum, $E_1\vee E_2$, has index 2 if the following inverse triangle inequality holds for some $\varepsilon>0$:

$$\|f_1+f_2\|\geq\varepsilon(\|f_1\|+\|f_2\|),\qquad f_1\in E_1, f_2\in E_2. \tag{1.1}$$

Note that for $\operatorname{ind}E_1=\operatorname{ind}E_2=1$,

$$\operatorname{ind}(E_1\vee E_2)=2\implies E_1\cap E_2=\emptyset. \tag{1.2}$$

Second, Seip's results on sampling and interpolation in Bergman spaces [**17**] are used to produce in an explicit way E_1 and E_2 satisfying property (1.1).

A construction of invariant subspaces of arbitrary index in the spaces

$$B^p(\varphi)=\Big\{f\in\operatorname{Hol}(\mathbb{D})\ :\|f\|^p=\int_{\mathbb{D}}|f(z)|^p\varphi(|z|)dm_2(z)<\infty\Big\},$$

with $\varphi(r)=(1-r^2)^a$, $0<p<\infty$, $-1<a<\infty$, is given in [**12**]. Let $\mathcal{N}$ be a finite or a countably infinite set. Once again, to verify that $\operatorname{ind}\big(\vee_{n\in\mathcal{N}}E_n\big)=\operatorname{card}\mathcal{N}$, with $\operatorname{ind}E_n=1$, $n\in\mathcal{N}$, the authors of [**12**] prove an inequality similar to (1.1). Furthermore, to produce E_n satisfying this inequality, they use results of Seip [**17,18**]. So, this construction depends heavily on specific properties of the weight φ.

We want to extend these results to more general spaces B. Let us return to the inequality (1.1). One possibility for this inequality to be fulfilled (in a weighted space of functions) is that

$$\operatorname{supp}f_1\cap\operatorname{supp}f_2=\emptyset,\qquad f_1\in E_1, f_2\in E_2. \tag{1.3}$$

Of course, (1.3) is impossible for analytic functions.

Assume that B is a subspace of an ideal normed space B_0. It means that $\|f\|_B = \|f\|_{B_0}$, $f \in B$, and

$$\text{if } f \in B_0,\ g \text{ is a measurable function on } \mathbb{D},\ |g(z)| \le |f(z)|,\ |z|<1,$$
$$\text{then } g \in B_0 \text{ and } \|g\|_{B_0} \le \|f\|_{B_0}.$$

Now, to get (1.1), we can try to find such invariant subspaces E_1 and E_2 and disjoint subsets Ω_1 and Ω_2 of $\mathbb{D}$ that for some $\varepsilon > 0$

$$\|f_1+f_2\|_B \ge \varepsilon \max\bigl\{\|\chi_{\Omega_1} f_1\|_{B_0}, \|\chi_{\Omega_2} f_2\|_{B_0}\bigr\} \ge \varepsilon^2 \max\bigl\{\|f_1\|_B, \|f_2\|_B\bigr\}, \quad f_1 \in E_1, f_2 \in E_2.$$

Furthermore, these inequalities hold if f_i are very big on Ω_i, and sufficiently small on Ω_j, $j \ne i$. Finally, what we need is to find invariant subspaces E_1 and E_2 such that all their elements have this specific behavior along Ω_i. Such subspaces can be generated by lacunary power series with prescribed asymptotics. This idea turns out to work under some natural assumptions on the space B_0. Moreover, the resulting construction looks quite elementary. We just use repeatedly the maximum principle and its (integral) analogs. Furthermore, this construction permits us to produce invariant subspaces of index bigger than 1 without common zeros in the unit disc. Previously, similar method was used by Krasichkov-Ternovskiĭ [**14**, Section 7]. The problem of constructing invariant subspaces without common zeros in different spaces of analytic functions in the unit disc was considered in recent papers by Atzmon [**3**] and Hedenmalm–Volberg [**13**]. See also earlier results by Beurling [**5**] and Nikolskiĭ [**15**, Section 2.8].

Weighted Hilbert spaces of sequences of class $\mathcal{H}$ are, generally speaking, not covered by our scheme: they are subspaces of (ideal) weighted L^2 spaces (against planar Lebesgue measure) only for logarithmically convex weight sequences $\{\gamma_n\}_{n\ge 0}$. However, our construction can be adapted for such spaces, and, more generally, for spaces $\ell^p_A(\{\gamma_n\}_{n\ge 0})$, $1 \le p \le \infty$.

The formulation of the main result is given in Section 2. We list general conditions on B implying the existence of invariant subspaces of arbitrary index and produce several examples of such spaces. An analog of condition (1.1) necessary to deal with the case $\operatorname{ind}(\vee E_n) = +\infty$ is given in Section 3. The proof of the main theorem is contained in Section 4. In Section 5 we deal with the spaces $\ell^p_A(\{\gamma_n\}_{n\ge 0})$. Final remarks are given in Section 6.

2. Main result

We deal with a Banach space B of analytic functions satisfying condition (i) and assume that

(ii) $1 \in B$.

This condition implies that B contains all the polynomials.

We suppose that

(iii) B is a subspace of a (radial) ideal normed space B_0: if $f \in B_0$, $0 \le a < b \le 1$, I is an interval (closed, open or semi-open) with endpoints a and b, $g(z) = f(z)\chi_I(|z|)$, $|z|<1$, then $g \in B_0$ and $\|g\|_{B_0} \le \|f\|_{B_0}$.

(iv) Two natural assumptions on the norm in B_0 are that

$$\lim_{a\to 1} \|\chi_{(a,1)}(|z|)\|_{B_0} = 0, \qquad \lim_{n\to\infty} \frac{\|z^n \chi_{[0,a)}(|z|)\|_{B_0}}{\|z^n\|_{B_0}} = 0, \quad 0 < a < 1.$$

The final of condition could be interpreted as a possibility to extract the circular part of the norm in B_0.

(v) Given a function h analytic in a neighborhood of $\operatorname{Clos}\mathbb{D}$, there exists a bounded function $\lambda(\cdot, h) : (0,1] \to [0,\infty)$ satisfying the inequality

$$|h(0)| \le \lambda(a,h) \le \frac{\|hz^n \chi_{[a,b]}(|z|)\|_{B_0}}{\|z^n \chi_{[a,b]}(|z|)\|_{B_0}} \le \lambda(b,h),$$

for every a, b, $0 < a < b \le 1$, and for every $n \ge 0$.

A weaker form of this condition is

(vi) For some $c = c(B_0) > 0$, and every function h analytic in a neighborhood of $\operatorname{Clos}\mathbb{D}$, there exists a bounded function $\lambda(\cdot, h) : (0,1] \to [0,\infty)$ satisfying the inequality

$$c|h(0)| \le c\lambda(d,h) \le \frac{\|hz^n \chi_{[a,b]}(|z|)\|_{B_0}}{\|z^n \chi_{[a,b]}(|z|)\|_{B_0}} \le \frac{1}{c}\lambda(b,h),$$

for every a, b, d, $1/2 < d < a^2 < a < b \le 1$, and for every $n \ge 0$.

THEOREM 2.1. *Let B be a Banach space satisfying conditions* (i)–(v) (*or* (i)–(iv) *and* (vi)). *Then B contains invariant subspaces of index n, $2 \le n \le +\infty$, without common zeros in the unit disc.*

For a function f defined on $\mathbb{D}$, $0 < r < 1$ put $f_r(z) = f(rz)$, $z \in \operatorname{Clos}\mathbb{D}$; for $f(z) = \sum_{n\ge 0} a_n z^n$, $z \in \mathbb{T} = \partial\mathbb{D}$, define $\|\mathcal{F}f\|_p^p = \sum_{n\ge 0} |a_n|^p$, $\|\mathcal{F}f\|_\infty = \sup_{n\ge 0} |a_n|$.

THEOREM 2.2. (a) *If φ is a positive decreasing function on $[0,1)$, $\lim_{t\to 1}\varphi(t) = 0$, then the spaces*

$$A^p(\varphi) = \Big\{ f \in \operatorname{Hol}(\mathbb{D}) \,:\, \|f\|^p_{A^p(\varphi)} = \sup_{0\le r<1} \Big[\varphi(r) \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \Big] < \infty \Big\},$$
$$A^p_0(\varphi) = \Big\{ f \in \operatorname{Hol}(\mathbb{D}) \,:\, \varphi(r) \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta = o(1), r \to 1,\ \|f\|_{A^p_0(\varphi)} = \|f\|_{A^p(\varphi)} \Big\},$$

where $1 \le p < \infty$, *and*

$$A^\infty(\varphi) = \Big\{ f \in \operatorname{Hol}(\mathbb{D}) \,:\, \|f\|_{A^\infty(\varphi)} = \sup_{z\in\mathbb{D}} \big[|f(z)|\varphi(|z|) \big] < \infty \Big\},$$
$$A^\infty_0(\varphi) = \Big\{ f \in \operatorname{Hol}(\mathbb{D}) \,:\, |f(z)|\varphi(|z|) = o(1),\ |z| \to 1,\ \|f\|_{A^\infty_0(\varphi)} = \|f\|_{A^\infty(\varphi)} \Big\}$$

contain invariant subspaces of index n, $2 \le n \le +\infty$, without common zeros in the unit disc.

(b) *If μ is a finite positive Borelian measure on $[0,1]$ with $1 \in \operatorname{supp}\mu$, $\mu(\{1\}) = 0$, $1 \le p < \infty$, then the spaces*

$$B^{p,q}(\mu) = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) \ : \|f\|^p_{B^{p,q}(\mu)} = \int_0^1 \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^q d\theta\Big)^{p/q} d\mu(r) < \infty \Big\},$$

$$1 \le q < \infty,$$

$$B^{p,\infty}(\mu) = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) \ : \|f\|^p_{B^{p,\infty}(\mu)} = \int_0^1 \max_{-\pi<\theta\le\pi} |f(re^{i\theta})|^p d\mu(r) < \infty \Big\}$$

contain invariant subspaces of index n, $2 \le n \le +\infty$, without common zeros in the unit disc.

(c) *If μ is a finite positive Borelian measure on $[0,1]$ with $1 \in \operatorname{supp}\mu$, $\mu(\{1\}) = 0$, $1 \le p < \infty$, then the spaces*

$$C^{p,q}(\mu) = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) \ : \|f\|^q_{C^{p,q}(\mu)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(\int_0^1 |f(re^{i\theta})|^p d\mu(r)\Big)^{q/p} d\theta < \infty \Big\},$$

$$1 \le q < \infty,$$

$$C^{p,\infty}(\mu) = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) \ : \|f\|^p_{C^{p,\infty}(\mu)} = \sup_{-\pi<\theta\le\pi} \int_0^1 |f(re^{i\theta})|^p d\mu(r) < \infty \Big\}$$

contain invariant subspaces of index n, $2 \le n \le +\infty$, without common zeros in the unit disc.

(d) *If μ is a finite positive Borelian measure on $[0,1]$ with $1 \in \operatorname{supp}\mu$, $\mu(\{1\}) = 0$, $1 \le p < \infty$, then the spaces*

$$F^p(\mu) = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) \ : \|f\|^p_{F^p(\mu)} = \int_0^1 \|\mathcal{F}(f_r)\|_p^p d\mu(r) < \infty \Big\}$$

contain invariant subspaces of index n, $2 \le n \le +\infty$, without common zeros in the unit disc.

(e) *If φ is a positive decreasing function on $[0,1)$, $\lim_{t\to 1}\varphi(t) = 0$, then the spaces*

$$F(\varphi) = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) \ : \|f\|_{F(\varphi)} = \sup_{0<r<1} \big[\varphi(r)\cdot\|\mathcal{F}(f_r)\|_\infty\big] < \infty \Big\},$$

$$F_0(\varphi) = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) \ : \varphi(r)\cdot\|\mathcal{F}(f_r)\|_\infty = o(1), r \to 1, \ \|f\|_{F_0(\varphi)} = \|f\|_{F(\varphi)} \Big\}$$

contain invariant subspaces of index n, $2 \le n \le +\infty$, without common zeros in the unit disc.

Proof of Theorem 2.2: To apply Theorem 2.1 we need only to define B_0, $\lambda(r,h)$, and to verify condition (v) (or (vi)). For the spaces $B^{p,q}(\mu)$, $C^{p,q}(\mu)$, the function $\lambda(r,h)$ is just the norm of h_r in $L^q(\mathbb{T})$. For the spaces $A^p(\varphi)$, $A^p_0(\varphi)$, the function $\lambda(r,h)$ is the maximum of $|h(z)|$ on $r\mathbb{T}$. For the space $F^p(\mu)$, the function $\lambda(r,h)$ is $\|\mathcal{F}(h_r)\|_p$; for the spaces $F(\varphi)$, $F_0(\varphi)$, the function $\lambda(r,h)$ is $\|\mathcal{F}(h_r)\|_\infty$.

A radial ideal normed space B_0 for the space $B = F^p(\mu)$ is defined as follows: take the set S of the Borelian measurable functions f on $\mathbb{D}$ such that the functions f_r, $0 < r < 1$, are continuous on $\mathbb{T}$ and are of the form $f_r(z) = \sum_{n\geq 0} a_n^r z^n$. Then B_0 is the space of the elements f in S such that

$$\|f\|_{B_0}^p = \int_0^1 \|\mathcal{F}(f_r)\|_p^p d\mu(r) < \infty.$$

The spaces $F(\varphi)$, $F_0(\varphi)$ are treated in an analogous way. When B is one of the spaces $A^p(\varphi)$, $A_0^p(\varphi)$, $B^{p,q}(\mu)$, $C^{p,q}(\mu)$, the space B_0 is the space of the measurable functions on $\mathbb{D}$ satisfying the same inequalities as that in the definition of B.

Condition (v) evidently holds for the spaces $A^p(\varphi)$, $A_0^p(\varphi)$, $B^{p,q}(\mu)$, $F^p(\mu)$, $F(\varphi)$ and $F_0(\varphi)$. Let us prove that the spaces $C^{p,q}(\mu)$ satisfy condition (vi). Indeed, if h_r^+ is the radial maximal function for $|h_r|$ (see, for example, [**10**, Chapter 3, Section 3], then

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\int_a^b |h(re^{i\theta})|^p r^{np} d\mu(r)\Big)^{q/p} d\theta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\int_a^b h_b^+(e^{i\theta})^p r^{np} d\mu(r)\Big)^{q/p} d\theta = \\ = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\int_a^b r^{np} d\mu(r)\Big)^{q/p} h_b^+(e^{i\theta})^q d\theta = \\ = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_b^+(e^{i\theta})^q d\theta \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\int_a^b r^{np} d\mu(r)\Big)^{q/p} d\theta \leq \\ \leq c(q)\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} |h(be^{i\theta})|^q d\theta \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\int_a^b r^{np} d\mu(r)\Big)^{q/p} d\theta.\end{aligned}$$

The case $q = \infty$ is evident.

To get the estimate from below we deal with the cases $p \geq q$ and $p < q$ separately. If $p \geq q$, then by the Hölder inequality applied to the inner integral we get

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\int_a^b |h(re^{i\theta})|^p r^{np} d\mu(r)\Big)^{q/p} d\theta \geq \\ \geq \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\int_a^b |h(re^{i\theta})|^q r^{np} d\mu(r)\Big)\cdot\Big(\int_a^b r^{np} d\mu(r)\Big)^{(q-p)/p} d\theta = \\ = \Big(\int_a^b r^{np} d\mu(r)\Big)^{(q-p)/p}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_a^b |h(re^{i\theta})|^q r^{np} d\theta d\mu(r) \geq \\ \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} |h(ae^{i\theta})|^q d\theta \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\int_a^b r^{np} d\mu(r)\Big)^{q/p} d\theta.\end{aligned}$$

In the case $p < q$, we use subharmonicity of $|h|^p$ in the following way. Put

$$H(se^{i\theta}) = \int_a^b |h(rse^{i\theta})|^p r^{np} d\mu(r), \qquad se^{i\theta} \in \operatorname{Clos}\mathbb{D}.$$

Then H, and consequently, $H^{q/p}$ are continuous and subharmonic in the unit disc, and for $a \geq 1/2$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\int_a^b |h(re^{i\theta})|^p r^{np} d\mu(r)\Big)^{q/p} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\big(H(e^{i\theta})\big)^{q/p} d\theta \geq$$
$$\geq \frac{1}{2\pi\cdot 2(1-a)(1-a^2)}\int_{-\pi}^{\pi}\int_{-1+a}^{1-a}\int_{a^2}^{1}\big(H(se^{i(\theta+\delta)})\big)^{q/p} d\theta d\delta ds \geq$$
$$\geq \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{1}{2(1-a)(1-a^2)}\int_{-1+a}^{1-a}\int_{a^2}^{1} H(se^{i(\theta+\delta)}) d\delta ds\Big)^{q/p} d\theta =$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{1}{2(1-a)}\int_{-1+a}^{1-a}\Big(\frac{1}{1-a^2}\int_{a^2}^{1}\int_a^b |h(rse^{i(\theta+\delta)})|^p r^{np} d\mu(r) ds\Big) d\delta\Big)^{q/p} d\theta \geq$$
$$\geq \frac{(c_{\rm abs})^{q/p}}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{1}{2(1-a)}\int_{-1+a}^{1-a}\Big(\frac{1}{a-a^2}\int_{a^2}^{a} |h(se^{i(\theta+\delta)})|^p ds \int_a^b r^{np} d\mu(r)\Big) d\delta\Big)^{q/p} d\theta =$$
$$= \frac{(c_{\rm abs})^{q/p}}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{1}{2(1-a)(a-a^2)}\int_{-1+a}^{1-a}\int_{a^2}^{a} |h(se^{i(\theta+\delta)})|^p ds d\delta\Big)^{q/p} d\theta \times$$
$$\times \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\int_a^b r^{np} d\mu(r)\Big)^{q/p} d\theta \geq$$
$$\geq (c'_{\rm abs})^{q/p}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}|h(a^{3/2}e^{i\theta})|^q d\theta \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\int_a^b r^{np} d\mu(r)\Big)^{q/p} d\theta,$$

where $c_{\rm abs}$ and $c'_{\rm abs}$ are absolute constants. An easy modification of this argument works for $q = \infty$. ∎

3. Lemma on summation of indices

We construct invariant subspaces $\vee_{n\in\mathcal{N}} E_n$ of index $\operatorname{card}\mathcal{N}$ from invariant subspaces E_n, $\operatorname{ind} E_n = 1$, using a statement which is just Theorem 2.1 of [**12**]. We give the proof here to make our presentation self-contained.

Lemma 3.1. *Let B be a Banach space of analytic functions satisfying conditions* (i), *and let $\mathcal{N}$ be a finite or countably infinite set. Suppose that E_n are invariant subspaces of B, $\operatorname{ind} E_n = 1$, $f_n \in E_n$, $f_n(0) \neq 0$, $n \in \mathcal{N}$. Furthermore, suppose that for every $n \in \mathcal{N}$ there exists $c_n > 0$ such that*

$$c_n|g(0)| \leq \|g+h\|_B, \qquad g \in E_n, h \in E_n^0 = \vee_{k\in\mathcal{N}\setminus\{n\}} E_k. \tag{3.1}$$

Then $\operatorname{ind} E = \operatorname{card}\mathcal{N}$, where $E = \vee_{n\in\mathcal{N}} E_n$.

Proof: First, $\operatorname{ind} E \leq \operatorname{card}\mathcal{N}$ (for a proof see [**16**, Proposition 2.16] with insignificant modifications). Furthermore, denote by Q the quotient map $Q : E \to E/\operatorname{Clos}(zE)$. If $\operatorname{ind} E < \operatorname{card}\mathcal{N}$, then the set $\{Qf_n\}_{n\in\mathcal{N}}$ is linearly dependent, and for some $n \in \mathcal{N}$, $f \in E_n^0$, we have $Qf_n = Qf$, that is $f_n - f \in \operatorname{Clos} zE$. Therefore, there exist $f_n^k \in E_n$, $f^k \in E_n^0$ such that $\|f_n - f - zf_n^k - zf^k\|_B \to 0$, $k \to \infty$. Since $f_n - zf_n^k \in E_n$, $(f_n - zf_n^k)(0) = f_n(0) \neq 0$, $f + zf^k \in E_n^0$, we get a contradiction to (3.1). ∎

This lemma is a generalization of the corresponding result for $n = 2$ given in [**16**]. A similar statement for a special case is contained in [**14**, Theorem 5.4].

In our construction the spaces E_n will be just $[f_n]$ for some $f_n \in B$, $f_n(0) \neq 0$. Then the condition (3.1) of Lemma 3.1 can be rewritten as follows:

There exists a sequence $\{c_n\}$ of positive numbers such that for every finite sequence of polynomials P_n (all P_n except a finite number of them are 0) and for every n we have

$$c_n|(P_n f_n)(0)| \leq \left\|\sum P_k f_k\right\|_B.$$

4. The proof of Theorem 2.1

We assume that conditions (i)–(v) hold. If condition (v) is replaced by condition (vi), then the argument does not change much.

A. We begin with an auxiliary construction. From now on we use the notation $\|\cdot\|$ for the norm in B_0.

Consider the function

$$\Phi(a, b, n) = \frac{\|z^n \chi_{[a,b]}(|z|)\|}{\|z^n\|}.$$

By (iii), $\Phi(a, b, n)$ depends monotonically on each of the first two variables. Since

$$\|z^n \chi_{[a,1]}(|z|)\| \geq \|z^n\| - \|z^n \chi_{[0,a)}(|z|)\|,$$

property (iv) implies that for every a, $0 < a < 1$,

$$\lim_{n\to\infty} \Phi(a, 1, n) = 1.$$

Choose a sequence $a_n \to 1$, $0 < a_n < 1$, in such a way that

$$\Phi(a_n, 1, n) > \frac{1}{2}, \qquad n \geq 0.$$

Furthermore, properties (iv) and (v) imply that

$$\|z^n \chi_{[b,1]}(|z|)\| \leq \lambda(1, z^n)\|\chi_{[b,1]}(|z|)\| \to 0, \qquad b \to 1, n \geq 0,$$
$$\lim_{b\to 1} \Phi(a_n, b, n) > \frac{1}{2}, \qquad n \geq 0.$$

Thus, we can associate to every $n \geq 0$ an interval $I_n = [a_n, b_n]$, $a_n < b_n < 1$, in such a way that

$$\frac{1}{2}\|z^n\| \leq \|z^n \chi_{I_n}(|z|)\| \leq \|z^n\|, \tag{4.1}$$

and

$$\lim_{n\to\infty} a_n = 1, \tag{4.2}$$
$$\lim_{n\to\infty} \|\chi_{I_n}(|z|)\| = 0. \tag{4.3}$$

Denote

$$Z_n(z) = \frac{z^n}{\|z^n\|}, \qquad \|Z_n\| = 1, \qquad n \geq 0.$$

B. We are going to produce a sequence of functions $\{f_l\}$, $f_l \in B$, and a sequence of positive numbers $\{c_l\}$ such that for every finite sequence $\{h_l\}$ of polynomials and every m,

$$c_m|(f_m h_m)(0)| \leq \|\sum f_l h_l\|.$$

The functions f_l are constructed as the sums of lacunary series,

$$f_l(z) = 2^{-2l} + \sum_{k \geq l} Z_{s(k,l)} 2^{-k}, \qquad l \geq 1.$$

where the numbers $s(k,l) > 0$ will be fixed later. Clearly, $f_l(0) = 2^{-2l}$, $f_l \in B$, $\|f_l\| < 2^{-l+2}$. Put $\chi_{s(k,l)}(z) = \chi_{I_{s(k,l)}}(|z|)$.

The numbers $s(k,l)$ are to satisfy three properties:

(P_1) the intervals $I_{s(k,l)}$ are disjoint.

(P_2) for every k, l, p, n, $(k,l) \neq (p,n)$,

$$\|\chi_{s(k,l)} Z_{s(p,n)}\| 2^{2k+2|n-l|+5} \leq 1.$$

(P_3) for every k, l,

$$\|\chi_{s(k,l)}\| e^{2k+2l+3} \leq 1.$$

We choose the numbers $s(k,l)$ by an inductive process in such a way that

$$s(1,1) < s(2,1) < s(2,2) < s(3,1) < s(3,2) < s(3,3) < \dots$$

As a consequence of (4.2), condition (P_1) is fulfilled if each next $s(k,l)$ is taken sufficiently big. Therefore, on every step (k,l) we need only to verify that if $s(p,n) < s(k,l)$ (and, hence, $p \leq k$), then the following inequalities hold:

$$\|\chi_{s(k,l)} Z_{s(p,n)}\| \leq 2^{-2k-2|n-l|-5},$$
$$\|\chi_{s(p,n)} Z_{s(k,l)}\| \leq 2^{-2k-2|n-l|-5},$$
$$\|\chi_{s(k,l)}\| \leq 2^{-2k-2l-3}.$$

The first inequality holds for sufficiently big $s(k,l)$ because of (4.3) and (v), the second and the third inequalities follow from (iv).

Put $a_{k,l} = a_{s(k,l)}$, $b_{k,l} = b_{s(k,l)}$, $\chi_{k,l} = \chi_{s(k,l)}$, $Z_{k,l} = Z_{s(k,l)}$.

Properties (P_2)–(P_3) imply that for every k, l, $k \geq l \geq 1$,

$$\|\chi_{k,l}\| 2^{-2l} + \sum_{p \geq l,\, p \neq k} \|\chi_{k,l} Z_{p,l}\| 2^{-p} \leq 2^{-2k-4}, \tag{4.4}$$

$$\|\chi_{k,l}\| 2^{-2n} + \sum_{p \geq n} \|\chi_{k,l} Z_{p,n}\| 2^{-p} \leq 2^{-2k-2|n-l|-4}, \qquad n \neq l. \tag{4.5}$$

C. For $0 \le c \le 2^{-3}$ put $g_l^c = f_l$, $l > 1$, $g_1^c(z) = f_1(z) - c$, $|z| < 1$.

Suppose that $\left\|\sum_{1\le n\le N} g_n^c h_n\right\| \le 1$, $N < \infty$. Since all h_n are analytic in a neighborhood of $\operatorname{Clos}\mathbb{D}$, for some $k \ge N$ we have

$$\lambda(a_{k+1,1}, h_n) \le 2^{k+3}, \qquad 1 \le n \le N. \tag{4.6}$$

Put $h_s = 0$, $N < s \le k$. Then by (iii)

$$1 \ge \Big\| \sum_{1\le n\le k} g_n^c h_n \Big\| \ge \Big\| \chi_{k,l} \sum_{1\le n\le k} g_n^c h_n \Big\|, \qquad 1 \le l \le k.$$

Therefore, for every l, $1 \le l \le k$,

$$\|\chi_{k,l} Z_{k,l} h_l\| 2^{-k} \le 1 + \Big\| \chi_{k,l} \sum_{1\le n\le k,\, n\ne l} g_n^c h_n \Big\| + \big\| \chi_{k,l} \big(g_l^c - Z_{k,l} e^{-k}\big) h_l \big\|.$$

By (v) we obtain

$$\lambda(a_{k,l}, h_l) \|\chi_{k,l} Z_{k,l}\| 2^{-k} \le 1 + \sum_{1\le n\le k,\, n\ne l} \big\| \chi_{k,l} g_n^c h_n \big\| + \big\| \chi_{k,l} \big(g_l^c - Z_{k,l} e^{-k}\big) h_l \big\|.$$

Furthermore, by (v), (4.4) and (4.5) we have

$$\begin{aligned}
\lambda(a_{k,l}, h_l)\|\chi_{k,l} Z_{k,l}\| 2^{-k} \le 1 + &\sum_{1\le n\le k,\, n\ne l} \Big[\|\chi_{k,l} h_n\| 2^{-2n} + \sum_{p\ge n} \big\|\chi_{k,l} Z_{p,n} h_n\big\| 2^{-p}\Big] + \\
&+ \|\chi_{k,l} h_l\| 2^{-2l} + \sum_{p\ge l,\, p\ne k} \big\|\chi_{k,l} Z_{p,l} h_l\big\| 2^{-p} \le \\
\le 1 + \sum_{1\le n\le k,\, n\ne l} &\lambda(b_{k,l}, h_n) \Big[\|\chi_{k,l}\| 2^{-2n} + \sum_{p\ge n} \big\|\chi_{k,l} Z_{p,n}\big\| 2^{-p}\Big] + \\
&+ \lambda(b_{k,l}, h_l) \Big[\|\chi_{k,l}\| 2^{-2l} + \sum_{p\ge l,\, p\ne k} \big\|\chi_{k,l} Z_{p,l}\big\| 2^{-p}\Big] \le \\
\le 1 + \sum_{1\le n\le k,\, n\ne l} &\lambda(b_{k,l}, h_n) 2^{-2k-2|n-l|-4} + \lambda(b_{k,l}, h_l) 2^{-2k-4}.
\end{aligned} \tag{4.7}$$

Here we use estimates like

$$\Big\| \sum_{k\ge 1} F_k \Big\|_{B_0} \le \sum_{k\ge 1} \|F_k\|_{B_0}$$

in the situation where $\lim_{s\to\infty} \|\sum_{k\ge s} F_k\|_{B_0} = 0$. That is why we do not need to assume B_0 to be complete.

By (4.1), $\|\chi_{k,l} Z_{k,l}\| \ge \frac{1}{2}$, and we obtain from (4.7) that

$$\lambda(a_{k,l}, h_l) \le 2^{k+1} \Big(1 + 2^{-2k-4} \sum_{1\le n\le k} 2^{-2|n-l|} \lambda(b_{k,l}, h_n) \Big).$$

Finally, inequality (4.6), the monotonicity of $\lambda(\cdot, h)$ and property (P_1) imply that

$$\lambda(a_{k,l}, h_l) \le 2^{k+1}\Big(1 + 2^{-2k-4} \cdot 2^{k+3} \cdot \frac{5}{3}\Big) \le 2^{k+2}, \qquad 1 \le l \le k.$$

Arguing by induction, we get

$$\lambda(a_{u,l}, h_l) \le 2^{u+1}\Big(1 + 2^{-2u-4} \sum_{1\le n\le k} 2^{-2|n-l|}\lambda(b_{u,l}, h_n)\Big) \le$$
$$\le 2^{u+1}\Big(1 + 2^{-2u-4} \sum_{1\le n\le u} 2^{-2|n-l|}\lambda(a_{u+1,n}, h_n) + 2^{-2u-4} \sum_{u+1\le n\le k} 2^{-2|n-l|}\lambda(a_{n,n}, h_n)\Big) \le$$
$$\le 2^{u+1}\Big(1 + 2^{-2u-4} \sum_{1\le n\le u} 2^{-2|n-l|+(u+3)} + 2^{-2u-4} \sum_{u+1\le n\le k} 2^{2(u-n)+(n+2)}\Big) \le$$
$$\le 2^{u+2}, \qquad 1 \le l \le u \le k.$$

As a result,

$$\lambda(a_{u,u}, h_u) \le 2^{u+2}, \qquad 1 \le u \le k.$$

Finally, by (v),

$$|(g_m^c h_m)(0)| \le 2^{-2m}|h_m(0)| \le 2^{-m+2}, \qquad m \ge 1,$$

and this bound does not depend on the sequence $\{h_n\}$.

D. As a result we obtain

$$\operatorname{ind}\big(\vee_{l\le k}[g_l^c]\big) = k, \qquad 1 \le k \le +\infty,\ 0 \le c \le 2^{-3}.$$

Put $X = \{z \in \mathbb{D} : f_2(z) = 0\}$, $Y = \{f_1(z), z \in X\}$. Since X and Y are countable, we can find $b \in [0, 2^{-3}] \setminus Y$. Now the invariant subspaces $\vee_{l\le k}[g_l^b]$ are of index k, and have no common zeros in the unit disc, $2 \le k \le +\infty$. ∎

5. Weighted Banach spaces of sequences

We deal with the Banach spaces

$$\ell_A^p(\{\gamma_n\}_{n\ge0}) = \Big\{f = \sum_{n\ge0} a_n z^n : \|f\|^p_{\ell_A^p(\{\gamma_n\}_{n\ge0})} = \sum_{n\ge0} \gamma_n^p |a_n|^p < \infty\Big\}, \qquad 1 \le p < \infty,$$
$$\ell_A^\infty(\{\gamma_n\}_{n\ge0}) = \Big\{f = \sum_{n\ge0} a_n z^n : \|f\|_{\ell_A^\infty(\{\gamma_n\}_{n\ge0})} = \sup_{n\ge0} |\gamma_n a_n| < \infty\Big\},$$
$$c_A(\{\gamma_n\}_{n\ge0}) = \Big\{f = \sum_{n\ge0} a_n z^n : \lim_{n\to\infty} \gamma_n |a_n| = 0,\ \|f\|_{c_A(\{\gamma_n\}_{n\ge0})} = \|f\|_{\ell_A^\infty(\{\gamma_n\}_{n\ge0})}\Big\}.$$

where sequences of positive numbers $\{\gamma_n\}_{n\ge0}$ satisfy the following conditions:

$$\sup_n \frac{\gamma_{n+1}}{\gamma_n} < \infty, \qquad \liminf_{n\to\infty} {\gamma_n}^{1/n} = 1. \tag{5.1}$$

If the sequence $\{\gamma_n\}_{n\ge 0}$ is bounded and $\liminf_{n\to\infty}\gamma_n > 0$, then $\ell^p_A(\{\gamma_n\}_{n\ge 0})$ is just the usual unweighted space ℓ^p_A. From now on we assume that

$$\liminf_{n\to\infty}\gamma_n = 0. \tag{5.2}$$

Our spaces satisfy property (i) of the Introduction. Furthermore, if $\{\gamma_n\}_{n\ge 0}$ is non-increasing, then the class of spaces $\ell^2_A(\{\gamma_n\}_{n\ge 0})$ considered here is contained in the class $\mathcal{H}$ defined in Introduction.

The following statement is rather standard.

LEMMA 5.1. *Let $\{\beta_n\}_{n\ge 0}$ be the maximal logarithmically convex minorant of the sequence $\{\gamma_n\}_{n\ge 0}$. Then $\beta_n \le \gamma_n$, $n \ge 0$, for some infinite subset $\mathcal{T}$ of $\mathbb{N}$ we have $\beta_n = \gamma_n$, $n \in \mathcal{T}$, and*

$$\beta_n^2 \le \beta_{n-1}\beta_{n+1}, \quad n \ge 1, \qquad \lim_{n\to\infty}\beta_n = 0, \qquad \lim_{n\to\infty}\beta_n^{1/n} = 1. \tag{5.3}$$

LEMMA 5.2. (a) *Given a decreasing sequence $\{\beta_n\}_{n\ge 0}$ satisfying properties* (5.3), *and* p, $1 \le p \le \infty$, *there exists a positive continuous function $\varphi_p : [0,1) \to (0,\infty)$ which is summable in the case $1 \le p < \infty$, which is decreasing on $[0,1)$ if $p=\infty$, $\lim_{t\to 1}\varphi_\infty(t) = 0$, and such that the sequence $\{\beta^*_n\}_{n\ge 0}$,*

$$\beta^*_n = \Big(\int_0^1 r^{pn}\varphi_p(r)dr\Big)^{1/p}, \qquad n \ge 0, 1 \le p < \infty,$$
$$\beta^*_n = \sup_{0<r<1} r^n\varphi_\infty(r), \qquad n \ge 0, p = \infty, \tag{5.4}$$

is equivalent to the sequence $\{\beta_n\}_{n\ge 0}$, that is for some $c > 0$,

$$c\beta_n \le \beta^*_n \le \beta_n/c.$$

(b) *If*

$$\beta^*_n(R) = \sup_{R<r<1} r^n\varphi_\infty(r), \qquad n \ge 0, 0 \le R < 1,$$

then $0 < \beta^*_n(R) \le \beta^*_n$, $\lim_{R\to 1}\beta^*_n(R) = 0$, $n \ge 0$, and for every $R < 1$ there exists $N = N(R)$ such that $\beta^*_n(R) = \beta^*_n$, $n \ge N(R)$.

PROOF: (a) In the case $1 \le p < \infty$ the claim of the lemma is just a discrete version of [**6**, Proposition B.1] (for $p = 2$ a detailed argument describing adaptation of [**6**, Proposition B.1] is given in [**9**, Lemma 5.2]).

In the case $p = \infty$ just take

$$\varphi_\infty(r) = \inf_n \beta_n r^{-n}.$$

Then equality (5.4) and the part (b) of the lemma follow from standard properties of the Legendre transform. ∎

We use the following notation. Given two Banach spaces A, B of holomorphic functions in the unit disc, such that A and B coincide as subsets of $\mathrm{Hol}(\mathbb{D})$, and for some $c > 0$,

$$c\|\cdot\|_A \le \|\cdot\|_B \le \frac{1}{c}\|\cdot\|_A,$$

we write $A{\sim}B$.

LEMMA 5.3. *If p, $\{\beta_n\}_{n\geq 0}$ and φ_p are as in Lemma 5.2, then*

$$\begin{aligned} &\ell^p_A(\{\beta_n\}_{n\geq 0})\sim F^p(\varphi_p(x)dx), \qquad 1\leq p<\infty,\\ &\ell^\infty_A(\{\beta_n\}_{n\geq 0})\sim F(\varphi_\infty),\\ &c_A(\{\beta_n\}_{n\geq 0})\sim F_0(\varphi_\infty). \end{aligned}$$

PROOF: First,

$$\Big\|\sum_{n\geq 0} a_n z^n\Big\|^p_{F^p(\varphi_p(x)dx)} = \int_0^1 \|\mathcal{F}(\sum_{n\geq 0} a_n r^n z^n)\|^p_p \varphi_p(r)dr =$$

$$= \int_0^1 \sum_{n\geq 0} |a_n|^p r^{np}\varphi_p(r)dr = \sum_{n\geq 0} |a_n|^p(\beta^*_n)^p, \qquad 1\leq p<\infty.$$

Furthermore,

$$\Big\|\sum_{n\geq 0} a_n z^n\Big\|_{F(\varphi_\infty)} = \sup_{0<r<1}\Big[\|\mathcal{F}(\sum_{n\geq 0} a_n r^n z^n)\|_\infty \varphi_\infty(r)\Big] =$$

$$= \sup_{0<r<1}\sup_n\big[|a_n|r^n\varphi_\infty(r)\big] = \sup_n\big[|a_n|\beta^*_n\big].$$

Finally,

$$\sum_{n\geq 0} a_n z^n \in F_0(\varphi_\infty) \Longleftrightarrow \lim_{R\to 1}\sup_{R<r<1}\big[\big[\sup_n |a_n|r^n\big]\varphi_\infty(r)\big] = 0 \Longleftrightarrow$$

$$\Longleftrightarrow \lim_{R\to 1}\sup_n\big[|a_n|\big[\sup_{R<r<1} r^n\varphi_\infty(r)\big]\big] = 0.$$

By Lemma 5.2 (b),

$$\lim_{R\to 1}\big[\sup_n |a_n|\beta^*_n(R)\big] = 0 \Longleftrightarrow \lim_{n\to\infty}|a_n|\beta^*_n = 0 \Longleftrightarrow \sum_{n\geq 0} a_n z^n \in c_A(\{\beta^*_n\}_{n\geq 0}). \quad \blacksquare$$

THEOREM 5.4. *Let A be one of the spaces $\ell^p_A(\{\gamma_n\}_{n\geq 0})$, $1\leq p\leq\infty$, $c_A(\{\gamma_n\}_{n\geq 0})$, where $\{\gamma_n\}_{n\geq 0}$ is a sequence of positive numbers satisfying conditions* (5.1) *and* (5.2). *Then A contains invariant subspaces of index n, $2\leq n\leq +\infty$, without common zeros in the unit disc.*

PROOF: We apply Lemma 5.1 and define a Banach space of sequences B in the following way: if $A=\ell^p_A(\{\gamma_n\}_{n\geq 0})$, then $B=\ell^p_A(\{\beta_n\}_{n\geq 0})$, and if $A=c_A(\{\gamma_n\}_{n\geq 0})$, then $B=c_A(\{\beta_n\}_{n\geq 0})$.

Introduce linear subspaces T, T_0 of $\mathrm{Hol}(\mathbb{D})$,

$$T=\Big\{\sum_{n\in\mathcal{T}} a_n z^n \in \mathrm{Hol}(\mathbb{D})\Big\}, \qquad T_0=\Big\{\sum_{n\in\mathcal{T}\cup\{0\}} a_n z^n \in \mathrm{Hol}(\mathbb{D})\Big\}.$$

Then $\|f\|_B = \|f\|_A$, $f \in A \cap T = B \cap T$,

$$\|f\|_B \le \|f\|_A, \qquad f \in A. \tag{5.5}$$

By Lemma 5.3 there exists a Banach space F of analytic functions of one of the classes considered in Theorem 2.2 (d) and (e) such that $B \sim F$. Then, the proof of Theorem 2.1 shows that there exist elements $g_n \in F$ and positive numbers c_n such that the inequality

$$\Big\| \sum_{1 \le n \le N} g_l h_l \Big\|_F \le 1,$$

where h_l are polynomials implies that

$$|h_n(0)| \le c_n, \qquad n \ge 1.$$

Furthermore, g_1 and g_2 have no common zeros in the unit disc.

Since $\mathcal{T}$ is infinite, the proof of Theorem 2.1 can be easlily modified in such a way that the functions g_n are chosen in $F \cap T_0$. Then $g_n \in B \cap T_0$, $g_n \in A$. Put

$$E_k = \vee_{l \le k} [g_l]_A.$$

It remains to verify that

$$\operatorname{ind} E_k = k, \qquad 1 \le k \le +\infty. \tag{5.6}$$

Suppose that

$$\Big\| \sum_{1 \le n \le N} g_l h_l \Big\|_A \le 1,$$

with some polynomials h_l. By (5.5),

$$\Big\| \sum_{1 \le n \le N} g_l h_l \Big\|_B \le 1,$$

and since $B \sim F$,

$$\Big\| \sum_{1 \le n \le N} g_l h_l \Big\|_F \le c,$$

for some c depending only on B and F. Therefore,

$$|h_n(0)| \le c \cdot c_n, \qquad n \ge 1,$$

and by Lemma 3.1, (5.6) is proved. ∎

Corollary 5.5. *Let A be one of the spaces $\ell^p_A(\{\gamma_n\}_{n \ge 0})$, $1 \le p \le \infty$, $c_A(\{\gamma_n\}_{n \ge 0})$, where $\{\gamma_n\}_{n \ge 0}$ is a sequence of positive numbers satisfying conditions*

$$\sup_n \frac{\gamma_{n+1}}{\gamma_n} < \infty, \qquad \liminf_{n \to \infty} \gamma_n = 0, \qquad \gamma = \liminf_{n \to \infty} {\gamma_n}^{1/n} > 0.$$

Then A contains invariant subspaces of index n, $2 \le n \le +\infty$, without common zeros in the disc $\gamma\mathbb{D}$.

6. Final remarks

Example 6.1: The construction in the proof of Theorem 2.1 gives us a possibility to produce concrete examples of subspaces of arbitrary index. In particular, for the Bergman space B^2 put

$$f_l(z) = 2^{-2l} + \sum_{k\geq l} z^{2^{2k^3+6kl}} 2^{k^3+3kl-k}, \qquad l \geq 1.$$

Then

$$f_l \in B^2, \qquad l \geq 1,$$
$$\operatorname{ind}\big(\vee_{l\leq k}[f_l]\big) = k, \qquad 1 \leq k \leq +\infty.$$

Remark 6.2: The claim of Theorem 2.1 holds also for F-spaces, that is linear spaces that possess a complete invariant metric, if this metric is homogeneous of degree p, $0 < p < 1$, and if natural analogs of conditions (i)–(v) ((i)–(iv) and (vi)) are fulfilled. This class includes the spaces $B^{p,q}(\mu)$, $0 < p < 1$, $1 \leq q \leq \infty$.

Remark 6.3: An easy corollary of our results is that every space B satisfying the conditions of Theorem 2.1 contains an uncountable family of invariant subspaces with trivial pairwise intersections, and the same is true for every space B listed in Theorem 5.4 . (Indeed, let f_1 and f_2 be two elements of B constructed in the proof of Theorem 2.1, $\operatorname{ind}\big([f_1] \vee [f_2]\big) = 2$. Since

$$[f_1] \vee [f_2] = [f_1 + \alpha f_2] \vee [f_1 + \beta f_2], \qquad \alpha, \beta \in \mathbb{C}, \alpha \neq \beta,$$

we get

$$\operatorname{ind}\big([f_1 + \alpha f_2] \vee [f_1 + \beta f_2]\big) = 2,$$

and then (see (1.2))

$$[f_1 + \alpha f_2] \cap [f_1 + \beta f_2] = \{0\}, \qquad \alpha, \beta \in \mathbb{C}, \alpha \neq \beta.)$$

Earlier, Abakumov [**1**] proved this property for ℓ^p_A and weighted ℓ^p_A spaces. His methods and precise conditions on weights and exponents p are different from ours.

Remark 6.4: The space $\ell^\infty_A(\{\gamma_n\}_{n\geq 0})$ introduced in Section 5 is dual to $\ell^1_A(\{1/\gamma_n\}_{n\geq 0})$, and we can consider the corresponding weak* topology on this space. Furthermore, we consider the weak* topology on $A^\infty(\varphi)$ induced by the weak* topology on $L^\infty(\varphi)$, the space of the complex-valued functions f on $\mathbb{D}$ with $\sup_{z\in\mathbb{D}} |f(z)|\varphi(|z|) < \infty$. (For an analogous situation see [**7**]). A sequence $\{f_k\}_{k\geq 1}$ of elements in one of the spaces $A = A^\infty(\varphi)$, $\ell^\infty_A(\{\gamma_n\}_{n\geq 0})$, satisfying property (i), converges weakly* to 0 if and only if $\sup_{k\geq 1} \|f_k\|_A < \infty$, and $f_k(z) \to 0$, $k \to \infty$, uniformly on compact subsets of $\mathbb{D}$. Like in the proof of Theorem 5.4, consider $\mathcal{T} \subset \mathbb{N}$ and F in the case $A = \ell^\infty_A(\{\gamma_n\}_{n\geq 0})$. If $A = A^\infty(\varphi)$, just put $\mathcal{T} = \mathbb{N}$ and $F = A$. For $A = A^\infty(\varphi)$ put $\mathrm{MF} = H^\infty(\mathbb{D})$, for $A = \ell^\infty_A(\{\gamma_n\}_{n\geq 0})$ put $\mathrm{MF} = \mathcal{F}\ell^\infty = \{f \in \mathrm{Hol}(\mathbb{D}) : \sup_{0<r<1} \|\mathcal{F}(f_r)\|_\infty < \infty\}$. The notion of index extends to the linear topological spaces A.

Now, for a countable index set $\mathcal{N}$ consider functions f_n in $\mathrm{Hol}(\mathbb{D})$ of the type

$$f_n = \sum_{s\geq 1} Z_{k_s^{(n)}}, \qquad n \in \mathcal{N}.$$

Here $k_1^{(1)} = 0$. If the sequences $\{k_s^{(n)}\}_{s\geq 1}$, $n \geq 1$, grow sufficiently rapidly, then $f_n \in A$. An argument like in [**7**] using estimates like that in the proof of Theorem 2.1 shows that for sufficiently rapidly growing $\{k_s^{(n)}\}_{s\geq 1}$, $k_s^{(n)} \in \mathcal{T}$, the spaces

$$V = V(\{f_n\}_{n\in\mathcal{N}}) = \{g = \sum_{n\in\mathcal{N}} f_n h_n \in A : \|g\|_V = \sup_{n\in\mathcal{N}} \|h_n\|_{\mathrm{MF}} < \infty\}$$

($\sum_{n\in\mathcal{N}}$ means here the sum in $\mathrm{Hol}(\mathbb{D})$) and $V_0 = V(\{zf_n\}_{n\in\mathcal{N}})$ are weakly* closed invariant subspaces of A.

Indeed, we need only to verify that V and zV are weakly* sequentially closed. It is clear that for some $c > 0$, $c^2\|g\|_V \leq c\|g\|_F \leq \|g\|_A$, $g \in V$. Furthermore, if $\varphi_p = \sum_{n\in\mathcal{N}} f_n h_{n,p} \in V$, $\|\varphi_p\|_A \leq c_1$, and φ_p converge uniformly on compact subsets of the unit disc to a function $g \in A$, then $\|\varphi_p\|_V \leq c_2$, $\|h_{n,p}\|_{\mathrm{MF}} \leq c_2$. Taking a subsequence $\{p_k\}$ we can assume that h_{n,p_k} converge uniformly on compact subsets of the unit disc to $h_n \in \mathrm{MF}$, $\|h_n\|_{\mathrm{MF}} \leq c_2$. As a result, $g = \sum_{n\in\mathcal{N}} f_n h_n \in V$. The same argument works for $V_0 = \operatorname{Clos} zV$.

Now, clearly, $f_k \in V$, the vectors $\{f_k + zV\}$ are independent in V/zV, $\{f_k + zV\} \notin \operatorname{Clos} \mathcal{L}_{l\in\mathcal{N}\setminus\{k\}}\{f_l + zV\}$, $k \in \mathcal{N}$, and

$$\dim V/V_0 = \operatorname{card} \mathcal{N}.$$

As a result we obtain the following statement. Let A be one of the spaces $A^\infty(\varphi)$, $\ell_A^\infty(\{\gamma_n\}_{n\geq 0})$, where φ and $\{\gamma_n\}_{n\geq 0}$ satisfy, correspondingly, conditions of Theorem 2.2 (a) and Theorem 5.4. Suppose that the space A is equipped with the weak* topology as described above. Then A contains invariant subspaces of index n, $2 \leq n \leq +\infty$, without common zeros in the unit disc.

REMARK 6.5: The division property extends in the following way to deal with the case $\operatorname{ind} E > 1$.

Let B be a Banach space of analytic functions in the unit disc, satisfying property (i) (BSAF), and let the operators of multiplication by $z - \lambda$, $\lambda \in \mathbb{D}$, be bounded from below. Take an invariant subspace E of B and a point $\lambda \in \mathbb{D}$ such that for some $f \in E$, $f(\lambda) \neq 0$.

Then $\operatorname{ind} E \leq n$, $1 \leq n < \infty$, if and only if for every $f_1, \ldots, f_n \in E$ such that $f_k(\lambda) = 0$, $1 \leq k \leq n$, there exist $a_1, \ldots, a_n \in \mathbb{C}$, $\sum_{1\leq k\leq n} |a_k| > 0$, such that

$$\sum_{1\leq k\leq n} a_k f_k \in (z - \lambda)E.$$

PROOF: Put $E_0 = \{f \in E : f(\lambda) = 0\}$. Clearly, $(z-\lambda)E \subset E_0 \subset E$, $\dim E/E_0 = 1$. Therefore $\dim E/(z-\lambda)E \leq n \iff \dim E_0/(z-\lambda)E \leq n - 1$. The last inequality means just that there exists a nontrivial linear combination of f_k belonging to $(z - \lambda)E$. ∎

REMARK 6.6: The space $A^\infty(\varphi)$ considered in Theorem 2.2 (a) is non-separable. Let us show that this space contains invariant subspaces E (without common zeros) with

$$\dim(E/zE) = \mathfrak{c}, \tag{6.1}$$

that is for every complete set $\{f_\gamma\}_{\gamma\in\Gamma}$ in E/zE, $\operatorname{card}\Gamma = \mathfrak{c}$.

PROOF: We begin with the space $H^\infty = A^\infty(1)$. One can easily produce a Blaschke sequence $\{z_{k,l}\}_{k,l\ge1}$, such that the points $\theta_k = \lim_{l\to\infty} z_{k,l} \in \mathbb{T}$ are dense on $\mathbb{T}$, and the Blaschke product $B = \prod_{k,l\ge1} b_{k,l}$, where the Blaschke factors $b_{k,l}$ correspond to the points $z_{k,l}$, satisfies the property

$$|(B/b_{k,l})(z_{k,l})| \ge 1/2, \qquad k,l \ge 1.$$

Furthermore, one associates to every $\alpha \in [0,1]$ a subset S_α of $\mathbb{N}$ such that for every finite family $\alpha_0,\dots,\alpha_n \in [0,1]$, $\alpha_0 \ne \alpha_i$, $1 \le i \le n$,

$$\operatorname{card}\Big(S_{\alpha_0} \setminus \bigcup_{1\le i\le n} S_{\alpha_i}\Big) = \infty.$$

Put

$$B_\alpha = \prod_{k\ge1,\, l\in\mathbb{N}\setminus S_\alpha} b_{k,l}, \qquad \alpha \in [0,1].$$

Note that for every finite family $\alpha_0,\dots,\alpha_n$, $\alpha_0 \ne \alpha_i$, $1 \le i \le n$, there exists a sequence of natural numbers $l_p \to \infty$, $p \to \infty$, such that

$$\begin{aligned} &|B_{\alpha_0}(z_{k,l_p})| \ge 1/2, \qquad k,p \ge 1,\\ &|B_{\alpha_i}(z_{k,l_p})| = 0, \qquad 1 \le i \le n, \quad k,p \ge 1. \end{aligned} \tag{6.2}$$

We claim that $E = \vee_{\alpha\in[0,1]}[B_\alpha]$ satisfies property (6.1). Otherwise, we approximate vectors f_γ by finite linear combinations of B_α,

$$\|f_\gamma - \sum_{i\in I} a_i B_i\|_{E/zE} < \varepsilon. \tag{6.3}$$

Denote by $A(\gamma,\varepsilon)$ the intersection of all finite subsets $I \subset [0,1]$ such that inequality (6.3) holds for some a_i. Then for every $\gamma \in \Gamma$, the set $A(\gamma) = \cup_{\varepsilon>0} A(\gamma,\varepsilon)$ is at most countable. Take $\alpha \in [0,1] \setminus \cup_{\gamma\in\Gamma} A(\gamma)$.

Suppose that there exists a finite linear combination of f_{γ_i}, $\gamma_i \in \Gamma$, $1 \le i \le n$, approximating B_α in E/zE:

$$\|B_\alpha - \sum_{1\le i\le n} c_i f_{\gamma_i}\|_{E/zE} < 1/4.$$

Every f_{γ_i} is approximated in E/zE by finite linear combinations $\sum c_{i,j} B_{\alpha_{i,j}}$, $\alpha_{i,j} \ne \alpha$, and as a result we get elements $\alpha_i \ne \alpha$ and coefficients a_i, $1 \le i \le m$, such that

$$\|\sum_{1\le i\le n} c_i f_{\gamma_i} - \sum_{1\le i\le m} a_i B_{\alpha_i}\|_{E/zE} < 1/4.$$

Finally, for a finite set of elements $\beta_i \neq \alpha$, $1 \leq i \leq s$, and polynomials $p_0, \dots, p_s$ we obtain

$$\|(1+zp_0)B_\alpha - \sum_{1\leq i\leq s} p_i B_{\beta_i}\|_{H^\infty} < 1/2.$$

However, this inequality, relation (6.2) and the fact that θ_k are dense on $\mathbb{T}$ imply that $\|1+zp_0\|_{H^\infty} < 1$ which is impossible.

A similar example of a subspace in H^∞ of index 2 is given in [**16**, Example 3.11].

To deal with general $A^\infty(\varphi)$ we need a function $F = \sum Z_{n_k} \in A^\infty(\varphi)$ of the type used in Remark 6.4: for a sequence $r_n \to 1$, $n \to \infty$,

$$1/2 \leq |F(r_n e^{i\theta})|/\varphi(r_n) \leq 2, \qquad -\pi < \theta \leq \pi.$$

Then we repeat the previous construction with $\{z_{k,l}\} \subset \cup_{n\geq 1} r_n\mathbb{T}$, and consider $E = \vee_{\alpha\in[0,1]}[B_\alpha F]$. ∎

Question 6.7: Using the construction of Remark 6.4, for every space $A^\infty(\varphi)$ satisfying the conditions of Theorem 2.2 (a) equipped with the weak* topology we can produce an invariant subspace $E = [f] \vee [g]$ of index 2 such that

$$\|pf + qg\|_{A^\infty(\varphi)} \asymp \|p\|_{H^\infty} + \|q\|_{H^\infty}, \qquad p, q \in H^\infty.$$

Then E is a linear topological space of analytic functions in the unit disc having invariant subspaces of indices $0, 1, 2$ only. It would be interesting to find a BSAF (rotation-invariant, if possible) possessing invariant subspaces only up to index n, $2 \leq n < +\infty$.

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A. A. Borichev, Laboratoire de Mathématiques Pures de Bordeaux, UPRESA 5467 CNRS, Université Bordeaux I, 351, cours de la Libération, 33405 Talence, FRANCE
E-mail: borichev@math.u-bordeaux.fr