WEIGHTED POLYNOMIAL APPROXIMATION AND THE HAMBURGER MOMENT PROBLEM

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Dedicated to Matts Essén on the occasion of his 65-th birthday

0. Introduction.

In this paper we deal with the following problem: when are the polynomials complete in $L^p(\mu)$ spaces with discrete measures μ on the real line? In the general setting this problem was investigated by S. Bernstein and M. Riesz, and later by N. Akhiezer, L. de Branges, L. Carleson, T. Hall, P. Malliavin, S. Mergelyan, H. Pollard and many others (for an extensive discussion see survey papers [2, 17] and the book [12, Chapter VI]). The particular problem we are interested in appears when one deals with the indeterminate case of the Hamburger moment problem [3, 4, 5].

Consider a (positive Borel) measure μ on the real line such that

$$\int_{\mathbb{R}} |t|^n \, d\mu(t) < \infty, \qquad n \geq 0.$$

We associate with this measure its moment sequence

$$s_n = \int_{\mathbb{R}} t^n \, d\mu(t), \qquad n = 0, 1, 2, \dots .$$

The Hamburger moment problem consists in finding, given a sequence of numbers $\{s_n\}_{n\geq 0}$, a positive Borel measure μ with moments s_n. If the solution is not unique we say that the moment problem is *indeterminate*. Furthermore, measures μ solving such problems are called indeterminate ($\mu \in$ (indet)). In other words, for a measure μ to be indeterminate means that there exists another measure ν, $\nu \neq \mu$, with the same moments,

$$\int_{\mathbb{R}} t^n \, d\mu(t) = \int_{\mathbb{R}} t^n \, d\nu(t), \qquad n = 0, 1, 2, \dots .$$

Otherwise, the measure μ is said to be determinate ($\mu \in (\mathrm{det})$).

R. Nevanlinna described in [18] (see also [3, Sections 2.4, 3.2]) the set of all solutions to an indeterminate moment problem. He parametrized this set using the class $(\mathcal{N})$ of functions φ holomorphic in the upper half-plane $\mathbb{C}_+$ and such that

$$\operatorname{Im}\varphi(z) \ge 0 \ \text{ for } \ \operatorname{Im} z > 0.$$

This class includes real constants, and we add formally the constant ∞ function. As a consequence of the Riesz–Herglotz formula, every function f in this class possesses an integral representation (see, for instance, [3, Section 3.1])

$$f(z) = az + b + \int_{\mathbb{R}} \left(\frac{1}{u-z} - \frac{u}{1+u^2} \right) d\mu(u)\,, \tag{0.1}$$

for $z \in \mathbb{C}_+$, where a and b are real numbers, $a \ge 0$ and μ is a positive Borel measure such that

$$\int_{\mathbb{R}} \frac{d\mu(u)}{1+u^2} < \infty\,.$$

If f is extended to the lower half-plane $\mathbb{C}_-$ by $f(z) = \overline{f(\overline{z})}$, $z \in \mathbb{C}_-$, then formula (0.1) holds for $z \in \mathbb{C}\setminus\mathbb{R}$. (Generally speaking, this is not an analytic continuation.)

For a fixed indeterminate moment problem there exists an entire matrix-function

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}, \qquad AD - BC \equiv 1, \tag{0.2}$$

whose elements A, B, C, and D are real entire functions (entire functions with real coefficients) such that for every $t \in \mathbb{R} \cup \{\infty\}$,

$$-\frac{C(z)t + D(z)}{A(z)t + B(z)} \in (\mathcal{N}).$$

The Nevanlinna formula

$$v(z,\nu) = -\frac{C(z)\varphi(z) + D(z)}{A(z)\varphi(z) + B(z)}, \qquad \varphi \in (\mathcal{N}), \tag{0.3}$$

gives a bijection between the class $(\mathcal{N})$ and the set of the Stieltjes transforms

$$v(z,\nu) = \int_{\mathbb{R}} \frac{d\nu(t)}{t-z}$$

of all the solutions to the indeterminate moment problem.

A solution μ of an indeterminate moment problem is called *canonical* if it corresponds to $\varphi(z) \equiv t$, $t \in \mathbb{R} \cup \{\infty\}$, in formula (0.3). Every canonical measure is a discrete measure with masses on the zero set of the corresponding entire function $A(z)t + B(z)$, $t \in \mathbb{R} \cup \{\infty\}$.

Canonical measures correspond to self-adjoint extensions (without extension of space) of symmetric operators with indices (1,1) associated with Jacobi matrices, see details in [3, Chapter 4]. These measures enjoy important extremal properties (see, for example, [3, Theorem 3.4.1]).

Fix a canonical measure μ. Since the matrix-functions

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

and

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

correspond to the same indeterminate moment problem, without loss of generality we can assume that the support of μ coincides with the zero set of B. We denote this zero set by Λ_B, $\Lambda_B \subset \mathbb{R}$.

By a theorem of M. Riesz [3, Theorem 2.4.3], the elements of the matrix-function (0.2) describing the solutions of an indeterminate moment problem are entire functions of zero exponential type. Furthermore, we have

$$\sum_{\lambda\in\Lambda_B} \frac{|\lambda|^n}{|B'(\lambda)|} \le \sum_{\lambda\in\Lambda_B} |\lambda|^{n+1} \sqrt{\frac{D(\lambda)}{B'(\lambda)}} \sqrt{\frac{1}{D(\lambda)B'(\lambda)(1+\lambda^2)}}$$
$$\le \left[\sum_{\lambda\in\Lambda_B} \lambda^{2n+2} \left(\frac{D(\lambda)}{B'(\lambda)}\right)\right]^{1/2} \left[\sum_{\lambda\in\Lambda_B} \left(\frac{1}{D(\lambda)B'(\lambda)}\right) \frac{1}{1+\lambda^2}\right]^{1/2} < \infty\,, \qquad n \ge 0.$$

Let us explain the last inequlaity. We consider the (positive) measure ν_0 whose Stieltjes transform is equal to $-D/B \in (\mathcal{N})$. Then the signs of $B'(\lambda)$ and $D(\lambda)$ coincide for every $\lambda \in \Lambda_B$, and the sum of the series in the first square brackets is just the moment of order $2n+2$ of ν_0. Furthermore, we use that the function $f = -A/B$ belongs to the class $(\mathcal{N})$, is meromorphic in the plane and has poles only on the real line, hence $A(\lambda)/B'(\lambda) > 0$ and, as an immediate consequence of formula (0.1),

$$\sum_{\lambda\in\Lambda_B} \frac{A(\lambda)}{B'(\lambda)(1+\lambda^2)} < \infty\,. \tag{0.4}$$

Since $A(\lambda)D(\lambda) = 1$, $\lambda \in \Lambda_B$, this gives us the convergence of the sum in the second square brackets.

Definition. *The Hamburger class $\mathfrak{H}$ consists of all transcendental real entire functions B of zero exponential type with only real (and simple) zeros Λ_B such that*

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_B}} \frac{|\lambda|^n}{|B'(\lambda)|} = 0, \qquad n \ge 0.$$

Without loss of generality, we always assume that the origin does not belong to the zero set Λ_B. A Hamburger class function is uniquely determined (up to a multiplicative constant) by its zero set.

Thus, entire functions involved in the Nevanlinna formula (0.3) belong to the Hamburger class. Furthermore, if

$$\mu = \sum_{\lambda\in\Lambda_B} \mu_\lambda \delta_\lambda$$

is a canonical measure, where δ_λ is the unit point mass measure at the point λ,

$$\mu_\lambda = \frac{D(\lambda)}{B'(\lambda)} = \frac{1}{A(\lambda)B'(\lambda)}, \qquad \lambda \in \Lambda_B,$$

then the functions A and D can be reconstructed by formulas

$$\frac{A(z)}{B(z)} = \alpha z + \beta + \sum_{\lambda\in\Lambda_B} \frac{A(\lambda)}{B'(\lambda)} \Big[\frac{1}{\lambda - z} - \frac{1}{\lambda}\Big] = \alpha z + \beta + \sum_{\lambda\in\Lambda_B} \frac{1}{\mu_\lambda [B'(\lambda)]^2} \Big[\frac{1}{\lambda - z} - \frac{1}{\lambda}\Big],$$

$$\frac{D(z)}{B(z)} = \gamma z + \delta + \sum_{\lambda\in\Lambda_B} \frac{D(\lambda)}{B'(\lambda)} \frac{1}{\lambda - z} = \gamma z + \delta + \sum_{\lambda\in\Lambda_B} \frac{\mu_\lambda}{\lambda - z},$$

where $\alpha, \gamma \ge 0$, $\beta, \delta \in \mathbb{R}$. Estimate (0.4) ensures here that

$$\sum_{\lambda\in\Lambda_B} \frac{1}{\mu_\lambda [B'(\lambda)]^2 (1+\lambda^2)} < \infty.$$

In 1944, Hamburger claimed the following statement to be true.

Statement (Hamburger [11], [3, Addenda and Problems to Chapter 4]). *A positive measure μ is a canonical solution to an indeterminate moment problem if and only for some function $B \in \mathfrak{H}$ we have*

$$\text{(i)} \quad \mu = \sum_{\lambda\in\Lambda_B} \mu_\lambda \delta_\lambda; \qquad \sum_{\lambda\in\Lambda_B} |\lambda|^n \mu_\lambda < \infty, \quad n \ge 0,$$

$$\text{(ii)} \quad \sum_{\lambda\in\Lambda_B} \frac{1}{\mu_\lambda [B'(\lambda)]^2 (1+\lambda^2)} < \infty,$$

$$\text{(iii)} \quad \sum_{\lambda\in\Lambda_B} \frac{1}{\mu_\lambda [B'(\lambda)]^2} = +\infty.$$

In particular, for masses $\mu_\lambda = [B'(\lambda)]^{-2}$, $\lambda \in \Lambda_B$, conditions (i)–(iii) are fulfilled, and as a result, the zero set Λ_B of an arbitrary entire functions in $\mathfrak{H}$ should be the support of a canonical measure.

In 1989, a gap in the proof of Hamburger's Statement was found by Berg and Pedersen. Soon Koosis [13] constructed a counterexample to Hamburger's Statement.

What was the source of Hamburger's mistake? We have already pointed out that if μ is a canonical measure, then conditions (i) and (ii) should hold. On the other hand, if μ is a measure satisfying conditions (i) and (ii), then $\mu \in$ (indet), see [3, Addenda and Problems to Chapter 4, Lemma 2]. Furthermore, a well-known theorem by M. Riesz [3, Sections 2.3, 2.4], claims that the following conditions are equivalent:

(a) The space of polynomials $\mathcal{P}$ is dense in $L^2(\mu)$,

$$\mathrm{Clos}_{\,L^2(\mu)}\mathcal{P} = L^2(\mu). \tag{0.5}$$

(b) Either $\mu \in$ (det) or μ is a canonical measure.

Thus, a measure μ is canonical if and only if conditions (i) and (ii) are fulfilled together with (0.5). Hamburger believed that when conditions (i) and (ii) are fulfilled, condition (iii) is necessary and sufficient for completeness of polynomials in $L^2(\mu)$. It is indeed necessary. Consider a function c defined by $c(\lambda) = [\mu_\lambda B'(\lambda)]^{-1}$, $\lambda \in \Lambda_B$. If

$$\sum_{\lambda\in\Lambda_B} \frac{1}{\mu_\lambda [B'(\lambda)]^2} < \infty,$$

then the function c is an element of $L^2(\mu)$, and it is orthogonal to $\mathcal{P}$:

$$\sum_{\lambda\in\Lambda_B} c(\lambda)p(\lambda)\mu_\lambda = \sum_{\lambda\in\Lambda_B} \frac{p(\lambda)}{B'(\lambda)} = 0\,, \qquad \forall p \in \mathcal{P}$$

(the last equality follows directly from the definition of the Hamburger class, see e.g. [3, Addenda and Problems to Chapter IV, Lemma 1] or [19, p.298]).

However, condition (iii) is not sufficient for completeness of polynomials. In [13], an entire function $B \in \mathfrak{H}$ is constructed such that for the measure $\mu = \sum_{\lambda\in\Lambda_B}[B'(\lambda)]^{-2}\delta_\lambda$,

$$\mathrm{Clos}_{\,L^2(\mu)}\mathcal{P} \neq L^2(\mu),$$

and hence, μ is not canonical.

The above described situation was the reason for writing this paper. Here we consider the following problem.

Problem. *Let $B \in \mathfrak{H}$, $1 \le p < \infty$, and let $\mu = \sum_{\lambda\in\Lambda_B} \mu_\lambda \delta_\lambda$ be a positive measure such that $\mathcal{P} \subset L^p(\mu)$. When is*

$$\text{Clos}_{\,L^p(\mu)}\mathcal{P} = L^p(\mu)?$$

In Section 1 we give a solution to this Problem and obtain a correct version of Hamburger's statement, and in Section 2 we discuss relations to de Branges spaces of entire functions.

1. An l^p-counterpart of de Branges' solution to the Bernstein problem.

Fix $B \in \mathfrak{H}$. Let us consider a function w on the zero set of B such that $w(\lambda) > 0$,

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_B}} |\lambda|^n w(\lambda) = 0, \qquad n \ge 0.$$

We introduce the Banach spaces $\ell^p(w)$, $1 \le p < \infty$, of functions a on Λ_B, with norm

$$\|a\|^p_{\ell^p(w)} = \sum_{\lambda\in\Lambda_B} |a(\lambda)|^p [w(\lambda)]^p.$$

Theorem A. *The polynomials are dense in $\ell^p(w)$ if and only if for every function $F \in \mathfrak{H}$ such that $\Lambda_F \subset \Lambda_B$, we have for $p > 1$*

$$\sum_{\lambda\in\Lambda_F} \left| \frac{1}{w(\lambda)F'(\lambda)} \right|^{p/(p-1)} = +\infty, \tag{1.1}$$

and for $p = 1$,

$$\liminf_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_F}} F'(\lambda) w(\lambda) = 0. \tag{1.2}$$

As the limit case for $p = +\infty$ one can consider the space $c_0(w)$ of functions a on Λ_B such that

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_B}} |a(\lambda)| w(\lambda) = 0,$$

with norm

$$\|a\|_{c_0(w)} = \sup_{\lambda\in\Lambda_B} |a(\lambda)| w(\lambda).$$

In this case (1.1) is to be replaced by

$$\sum_{\lambda\in\Lambda_F} \frac{1}{w(\lambda)|F'(\lambda)|} = +\infty.$$

This is a special case of a remarkable theorem by de Branges [6, 12] which gives one of the solutions of the Bernstein problem on weighted polynomial approximation (see, for instance, [12]). The original proof of (the general case of) de Branges' theorem uses extensively geometric properties of the dual space to $c_0(w)$. Another proof found recently [21] uses the ideas which go back to P. Chebyshev and A. Markov. None of these proofs seems work for the spaces $\ell^p(w)$. However, in the special case under consideration, when the weight is defined on the zero set of a function in the class $\mathfrak{H}$, the polynomial approximation problem in the spaces $\ell^p(w)$ can be reduced to that in the space $c_0(w)$. The details of the proof will appear elsewhere.

Corollary 1.1. *To make Hamburger's Statement correct, condition* (iii) *should be replaced by the following condition:*

(iii′) *for every $F \in \mathfrak{H}$ such that $\Lambda_F \subset \Lambda_B$, we have*

$$\sum_{\lambda\in\Lambda_F} \frac{1}{\mu_\lambda [F'(\lambda)]^2} = +\infty.$$

Corollary 1.2. *If ν is a canonical solution to an indeterminate moment problem, $B \in \mathfrak{H}$, $\operatorname{supp}\nu = \Lambda_B$, then the measure $\mu = \sum_{\lambda\in\Lambda_B} [B'(\lambda)]^{-2}\delta_\lambda$ is also a canonical solution to an indeterminate moment problem.*

Proof. If μ is not a canonical solution to an indeterminate moment problem, then by Corollary 1.1 for some divisor $F \in \mathfrak{H}$ of B we have

$$\sum_{\lambda\in\Lambda_F} \left|\frac{B'(\lambda)}{F'(\lambda)}\right|^2 < \infty.$$

Clearly, B/F is not a constant function. Pick a zero w of B/F and consider $F_0(z) = F(z)(z - w)$. If $\nu = \sum_{\lambda\in\Lambda_B} \nu_\lambda \delta_\lambda$ is a canonical solution, then by condition (ii) of Hamburger's Statement, for some C we have

$$\frac{1}{\nu_\lambda} \le C[B'(\lambda)]^2(1+\lambda^2)\,.$$

Now,

$$\sum_{\lambda\in\Lambda_F} \frac{1}{\nu_\lambda [F_0'(\lambda)]^2} \le C \sum_{\lambda\in\Lambda_F} \left|\frac{B'(\lambda)}{F'(\lambda)}\right|^2 \frac{1+\lambda^2}{|\lambda - w|^2} < \infty\,,$$

and again by Corollary 1.1 we obtain that ν cannot be a canonical solution to an indeterminate moment problem.

Thus, Koosis' example [13] shows that there are $B \in \mathfrak{H}$ for which no canonical measure μ exists with $\operatorname{supp}\mu = \Lambda_B$. This implies, in particular, that not every function in $\mathfrak{H}$ can be an element of a matrix-function in (0.2) parametrizing the set of solutions for an indeterminate moment problem. Our discussion from Introduction shows that the description of canonical solutions to the Hamburger moment problem and the description of the first row of Nevanlinna matrices parametrizing all solutions are basically equivalent problems. It is worth to mention that Krein [14] and de Branges [7, Chapter 2] described (in different terms) the first row of *an arbitrary* Nevanlinna matrix, see also [20].

To apply Theorem A one needs to verify condition (1.1) (or (1.2)) for a rather large family of "Hamburger divisors" F. Nevertheless, we show below that it can be efficiently applied (compare with recent applications [19] of the original de Branges theorem).

In a recent paper [8], Fryntov considered the situation when $\Lambda_B \subset \mathbb{R}_+$ is an (R)-set in the sense of Levin [15, Section 2.1]: for the counting function n of the set Λ_B the following limit exists

$$\lim_{t\to\infty} \frac{n(t)}{t^\rho} = \Delta, \qquad 0 < \Delta < \infty, \qquad 0 < \rho < 1/2\,,$$

and the following separation condition is fulfilled,

$$|\lambda - \lambda'| \ge C\lambda^{1-\rho}, \qquad \lambda, \lambda' \in \Lambda_B,\ \lambda \ne \lambda'.$$

Then the function B is of completely regular growth in the sense of Levin–Pfluger, and the following limit exists

$$\lim_{\substack{\lambda\to\infty \\ \lambda\in\Lambda_B}} \frac{\log|B'(\lambda)|}{\lambda^\rho} = \pi\Delta\,\cot\pi\rho > 0\,.$$

Therefore, in this case $B \in \mathfrak{H}$.

Theorem (Fryntov [8]). *For the entire function B described above,*

$$\mathrm{Clos}_{\,\ell^2(\frac{1}{|B'|})}\mathcal{P} = \ell^2\Big(\frac{1}{|B'|}\Big)\,;$$

i.e. the measure $\mu = \sum_{\lambda\in\Lambda_B}[B'(\lambda)]^{-2}\delta_\lambda$ is a canonical solution to the corresponding indeterminate moment problem.

A similar situation was considered by H. Hamburger in [11], where he produced a false statement. A correct formulation (without proof) is contained in [3, Addenda and Problems to Chapter 4, Subsection 5] where the credit is given to B. Ya.Levin.

However, the late Professor Levin told the second-named author that in his proof he had used the Hamburger statement (see above). Fryntov's proof of this theorem was rather ingenious and involved. We show that Fryntov's result follows easily from Theorem A.

Sketch of the proof. Let B be the same function as in the Fryntov theorem and let F be a "Hamburger divisor" of B: i.e. F belongs to the Hamburger class and $\Lambda_F \subset \Lambda_B$. Assume that

$$\sum_{\lambda\in\Lambda_F} \left|\frac{B'(\lambda)}{F'(\lambda)}\right|^2 < +\infty\,. \tag{1.3}$$

Then, for some positive constants c_0, c_1

$$|F'(\lambda)| \ge c_0|B'(\lambda)| \ge c_1 \exp\left[(c-\varepsilon)\lambda^\rho\right]\,, \quad \lambda\in\Lambda_F,\ c=\pi\Delta\,\cot\pi\rho = h_B(0),\ \varepsilon>0,$$

where h_B is the Phragmén-Lindelöf indicator of B. This inequality yields a lower bound for F on a sequence of circles. Making use of an elementary property of canonical products of genus zero with positive zeros and then the Harnack inequality, we obtain that there is a sequence $r_n \to +\infty$ such that

$$|F(r_n e^{i\theta})| \ge |F(r_n)| \ge c_2 \exp\left[(c-\varepsilon)r_n^\rho\right]\,, \quad c_2>0\,. \tag{1.4}$$

Now, we show that this last estimate yields a lower bound for F everywhere in the complex plane outside small exceptional discs around Λ_F. Let H be an arbitrary entire function of completely regular growth in the Levin-Pfluger sense (with respect to the same order ρ) with the constant Phragmén-Lindelöf indicator function $h_H(\theta)\equiv c-2\varepsilon$. Then we consider the sequence of integrals

$$\frac{1}{2\pi i}\int_{|\zeta|=r_n} \frac{H(\zeta)}{F(\zeta)}\,\frac{d\zeta}{z-\zeta} \to 0\,, \qquad n\to\infty$$

(the integrals converge to zero because of (1.4) and the choice of $H(z)$), and the residue theorem yields the Lagrange interpolation formula

$$\frac{H(z)}{F(z)} = \sum_{\lambda\in\Lambda_F} \frac{H(\lambda)}{F'(\lambda)(z-\lambda)}\,.$$

Hence,

$$\left|\frac{H(z)}{F(z)}\right| \le c_3$$

outside exceptional discs $\{z:\,|z-\lambda|<1\}$, $\lambda\in\Lambda_F$. The entire function $G=B/F$ is of finite type with respect to order ρ, and has an upper bound

$$|G(z)| \le c_3\left|\frac{B(z)}{H(z)}\right| \le C_\epsilon \exp\left[(h_B(\theta)+3\varepsilon-h_B(0))r^\rho\right]\,, \qquad z=re^{i\theta}$$

(by the maximum principle, the last estimate holds also inside the exceptional discs). Thus, $h_G(0) \le 0$. Since $\rho < 1/2$, G has zero type with respect to order ρ.

Let us recall that G is bounded on Λ_B. Therefore, we can use an argument due to Ganapathy Iyer. The Lagrange interpolation formula applies to G^n and B for every integer $n \ge 1$:

$$\frac{G^n(z)}{B(z)} = \sum_{\lambda\in\Lambda_B} \frac{G^n(\lambda)}{B'(\lambda)(z-\lambda)}, \qquad z \in \mathbb{C}\setminus\Lambda_B.$$

This formula implies that G is bounded on the whole complex plane, and, as a consequence, G is a constant function. This contradicts to the assumption (1.3).

2. Completeness of polynomials in the de Branges spaces of entire functions.

Consider an entire function E in the Hermite–Biehler class (see [15, Chapter 7]), that is $|E(x+iy)| > |E(x-iy)|$, $y>0$, and $E(x) \ne 0$, $x \in \mathbb{R}$. Equivalently, if $E = A - iB$, where A and B are real entire functions, then $B/A \in (\mathcal{N})$, and the functions A and B have no common zeros on $\mathbb{R}$.

Then the corresponding de Branges space $\mathcal{H}(E)$ [7] consists of all entire functions F such that $F/E, F^*/E \in H^2_+$. Here $F^*(z) = \overline{F(\overline{z})}$, and $H^2_\pm$ are the Hardy spaces correspondingly in the upper and the lower half-planes $\mathbb{C}_\pm$. Equipped with the inner product

$$\langle F, G\rangle_{\mathcal{H}(E)} = \int_{\mathbb{R}} F(t)\overline{G(t)}\frac{dt}{|E(t)|^2},$$

$\mathcal{H}(E)$ becomes a Hilbert space. Note that the point evaluation functionals $F \to F(z)$, $z \in \mathbb{C}$, are continuous on $\mathcal{H}(E)$. An equivalent definition of this space is given by the orthogonal expansion

$$L^2(|E|^{-2}) = E^* H^2_+ \oplus \mathcal{H}(E) \oplus E H^2_-,$$

obtained earlier in a special case by Akhiezer [1].

If $E_\alpha = e^{i\alpha}E$, $0 \le \alpha < 2\pi$, then the spaces $\mathcal{H}(E_\alpha)$ coincide, and it is easily seen that the function $A\cos\alpha + B\sin\alpha$ can belong to $L^2(|E|^{-2})$ for at most one value of α. From now on let us assume that the function B does not belong to the space $\mathcal{H}(E)$.

We consider the system of functions

$$\Phi_\lambda(z) = \sqrt{\frac{A(\lambda)}{\pi B'(\lambda)}}\cdot\frac{B(z)}{z-\lambda}, \qquad \lambda \in \Lambda_B.$$

Theorem (de Branges [7, Theorem 22]). *The system of functions* $\{\Phi_\lambda(z)\}_{\lambda\in\Lambda_B}$ *is an orthonormal basis in the space* $\mathcal{H}(E)$.

The expansion of a function $F\in\mathcal{H}(E)$ by this system is given by the Lagrange interpolation series,

$$F(z)=\sum_{\lambda\in\Lambda_B}\frac{F(\lambda)}{B'(\lambda)}\cdot\frac{B(z)}{z-\lambda}=\sum_{\lambda\in\Lambda_B}\sqrt{\frac{\pi B'(\lambda)}{A(\lambda)}}\cdot\frac{F(\lambda)}{B'(\lambda)}\,\Phi_\lambda(z)=\sum_{\lambda\in\Lambda_B}c_\lambda(F)\Phi_\lambda(z)\,,$$

$$c_\lambda(F)=\sqrt{\frac{\pi}{A(\lambda)B'(\lambda)}}\,F(\lambda)\,,$$

and

$$\begin{aligned}\|F\|^2_{\mathcal{H}(E)}&=\sum_{\lambda\in\Lambda_B}|c_\lambda(F)|^2=\pi\sum_{\lambda\in\Lambda_B}\frac{|F(\lambda)|^2}{A(\lambda)B'(\lambda)}\\&=\sum_{\lambda\in\Lambda_B}|F(\lambda)|^2\mu_\lambda=\|F\|^2_{L^2(\mu)}\end{aligned}\tag{2.1}$$

where $\mu=\sum_{\lambda\in\Lambda_B}\mu_\lambda\delta_\lambda$, $\mu_\lambda=\dfrac{\pi}{A(\lambda)B'(\lambda)}$.

Since $-A/B\in\mathcal{N}$, we have

$$\sum_{\lambda\in\Lambda_B}\frac{A(\lambda)}{B'(\lambda)}\frac{1}{1+\lambda^2}<\infty,$$

that is

$$\sum_{\lambda\in\Lambda_B}\frac{1}{\mu_\lambda[B'(\lambda)]^2(1+\lambda^2)}<\infty.\tag{2.2}$$

This is just condition (ii) in Hamburger's Statement.

Assuming additionally that E is of zero exponential type and $\mathcal{P}\subset\mathcal{H}(E)$, we obtain that both functions A and B belong to the Hamburger class $\mathfrak{H}$. Indeed, as a consequence of (2.1), $\mathcal{P}\subset L^2(\mu)$, and thus

$$\sum_{\lambda\in\Lambda_B}|\lambda|^n\mu_\lambda=\pi\sum_{\lambda\in\Lambda_B}\frac{|\lambda|^n}{A(\lambda)B'(\lambda)}<\infty\,,\qquad n\geq 0.\tag{2.3}$$

Therefore, like in Introduction, the Cauchy–Schwarz–Bunyakovskiĭ inequality applied to (2.2) and (2.3) gives us that

$$\sum_{\lambda\in\Lambda_B}\frac{|\lambda|^n}{|B'(\lambda)|}<\infty,\qquad n\geq 0.$$

Now let us move in the opposite direction. We begin with a function $B \in \mathfrak{H}$ and a measure μ satisfying condition (ii) in Hamburger's Statement. Define a real entire function A by the equality

$$\frac{A(z)}{B(z)} = \pi \sum_{\lambda \in \Lambda_B} \frac{1}{\mu_\lambda [B'(\lambda)]^2} \left[\frac{1}{\lambda - z} - \frac{1}{\lambda} \right], \tag{2.4}$$

and put $E(z) = A(z) - iB(z)$. The argument in the proof of de Branges' Theorem 22 permits us to verify that $B \notin \mathcal{H}(E)$ with this choice of A.

If $c \in L^2(\mu)$, then the Lagrange series

$$F(z) = \sum_{\lambda \in \Lambda_B} \frac{F(\lambda)}{B'(\lambda)} \cdot \frac{B(z)}{z - \lambda}$$

$$= \sum_{\lambda \in \Lambda_B} c_\lambda \sqrt{\frac{\pi}{A(\lambda) B'(\lambda)}} \sqrt{\frac{A(\lambda)}{\pi B'(\lambda)}} \frac{B(z)}{z - \lambda} = \sum_{\lambda \in \Lambda_B} c_\lambda \sqrt{\mu_\lambda} \Phi_\lambda(z),$$

is norm convergent in the space $\mathcal{H}(E)$, and $||F||^2_{\mathcal{H}(E)} = ||c||^2_{L^2(\mu)}$.

As a result, we obtain that the restriction operator $F \to F\big|_{\Lambda_B}$ gives an isometric isomorphism between the spaces $\mathcal{H}(E)$ and $L^2(\mu)$ which preserves polynomials. Therefore, the properties $\operatorname{Clos}_{L^2(\mu)} \mathcal{P} = L^2(\mu)$ and $\operatorname{Clos}_{\mathcal{H}(E)} \mathcal{P} = \mathcal{H}(E)$ hold simultaneously, and the main problem discussed in our paper can be reformulated as follows:

Problem. *Let E be an entire function of zero exponential type with no zeros in the closed upper half-plane such that*

$$\int_{\mathbb{R}} \frac{|t|^n}{|E(t)|^2} dt < \infty, \qquad n = 0, 1, 2, \dots .$$

(Then E belongs to the Hermite–Biehler class.) *What additional conditions on E imply that*

$$\operatorname{Clos}_{\mathcal{H}(E)} \mathcal{P} = \mathcal{H}(E)\,?$$

This problem was investigated by Akhiezer [1] and V. P. Gurariĭ [9]. Akhiezer proved that the polynomials are complete when the zeros of E belong to a half-strip $\{z : |\operatorname{Re} z| < h, \operatorname{Im} z < 0\}$. His result was improved by Gurariĭ who proved the completeness of the polynomials when E is an even entire function of convergence class,

$$\sum_{\lambda \in \Lambda_E} \frac{1}{|\lambda|} < \infty,$$

with zeros in the angle $\{z : -3\pi/4 \le \arg z \le -\pi/4\}$.

It is worth to note that for $\mu = \sum_{\lambda\in\Lambda_B} [B'(\lambda)]^{-2}\delta_\lambda$, $B \in \mathfrak{H}$, we have $E = B' - iB$. In this situation, if the zeros of B have zero uniform density,

$$\sup_{x\in\mathbb{R}} \operatorname{card}\big\{\Lambda_B \cap [x-r, x+r]\big\} = o(r), \qquad r \to \infty,$$

then a simple estimate of the imaginary part of the logarithmic derivative of B (which (the imaginary part) equals to just π times the Poisson integral of the measure $\sum_{\lambda\in\Lambda_B} \delta_\lambda$) demonstrates that

$$\lim_{|y|\to\infty} \sup_{-\infty<x<\infty} \left| \operatorname{Im}\left[\frac{B'(x+iy)}{B(x+iy)}\right]\right| = 0.$$

Therefore, in this case the zeros of E lie in a strip $\{z : -h < \operatorname{Im} z < 0\}$.

The Koosis counterexample shows that in the general situation the polynomials are no longer automatically complete in $\mathcal{H}(E)$ (it is clear from [9, p.449] that Levin and Gurariĭ were aware of this already in 1962, however, they did not observe a relation of this fact to Hamburger's Statement). Theorem A gives a criterion for polynomials to be complete in $\mathcal{H}(E)$. It looks plausible that a more natural criterion (and hence another form of Hamburger's Statement) could be obtained in terms of E instead of B and μ.

It is worth to mention that Hamburger's condition (iii) means that the sequence $\{A(\lambda)\}_{\lambda\in\Lambda_B}$ does not belong to the space $L^2(\mu)$, and therefore $\mathcal{H}(E)$ does not contain any linear combination (with complex coefficients) $\alpha A + \beta B$. A result of de Branges ([7, Theorem 29]) implies now that the operator of multiplication by z is densely defined in $\mathcal{H}(E)$. The corresponding reformulation of our condition (iii′) permits us to produce the following result: *the polynomials are dense in $\mathcal{H}(E)$, $E = A - iB$, if and only if for each Hamburger divisor B_0 of B the operator of multiplication by z is densely defined in the space $\mathcal{H}(A_0 - iB_0)$ where A_0 is defined by B_0 and $\mu\,|\Lambda_{B_0}$ like in* (2.4).

Unfortunately, we were unable to find a reasonable interpretation of Hamburger divisors of B in the framework of de Branges' theory of Hilbert spaces of entire functions.

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