

# KREIN'S ENTIRE FUNCTIONS AND THE BERNSTEIN APPROXIMATION PROBLEM

ALEXANDER BORICHEV AND MIKHAIL SODIN

ABSTRACT. We extend two theorems of Krein concerning entire functions of Cartwright class, and give applications for the Bernstein weighted approximation problem.

**1. The Krein class and functions of bounded type.** We start with two classical theorems of Krein concerning entire functions. An entire function  $f$  belongs to *the Cartwright class* if  $f$  has at most exponential type, that is

$$\log |f(z)| = O(|z|), \quad |z| \rightarrow \infty,$$

and the logarithmic integral converges:

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1+x^2} dx < \infty.$$

**Theorem A.** (Krein [14]) *An entire function  $f$  belongs to the Cartwright class if and only if the function  $\log^+ |f|$  has (positive) harmonic majorants in both the upper and the lower half-planes.*

An entire function  $f$  belongs to *the Krein class* if its zeros  $\lambda_n$  are (simple and) real, and  $1/f$  is represented as an absolutely convergent sum of simple fractions

$$\frac{1}{f(z)} = \sum_n \frac{1}{f'(\lambda_n)(z - \lambda_n)}, \quad \sum_n \frac{1}{|f'(\lambda_n)|} < \infty. \quad (1.1)$$

**Theorem B.** (Krein [14]) *The Krein class is contained in the Cartwright class.*

For the proofs see also [19]. These two results have numerous applications in operator theory and harmonic analysis (see, for example, [15], [16], [7], [8, Chapter IV], [13, Section VI F]). Later, they were generalized in different directions (cf. [19, Section 26.4], [9, Section VI.2]).

---

*Date:* February 21, 2001.

The second named author was supported by the Israel Science Foundation of the Israel Academy of Sciences and Humanities under Grant No. 93/97-1.

Let  $E$  be a non-empty closed subset of the real line. In what follows we assume that  $E$  is regular for the Dirichlet problem in  $\mathbb{C} \setminus E$ . A function  $f$  analytic in  $\mathbb{C} \setminus E$  is said to be *of bounded type* if  $\log^+ |f|$  has a harmonic majorant in  $\mathbb{C} \setminus E$ . It is well known that if  $f$  and  $g$  are of bounded type, and  $f/g$  is analytic in  $\mathbb{C} \setminus E$ , then  $f/g$  is also of bounded type there (cf. [21, Chapter VII], [22, Theorem 19, p. 181]). It is worth to mention that any function  $\varphi$  lower semicontinuous in the plane, which has a positive harmonic majorant in  $\mathbb{C} \setminus E$ , satisfies the inequality

$$\int_E \varphi^+(x) \omega(i, dx, \mathbb{C} \setminus E) < \infty. \quad (1.2)$$

We would like to know when the assertions in two Krein's theorems can be improved to guarantee that  $f$  is of bounded type in  $\mathbb{C} \setminus E$ . Note that every polynomial is of bounded type in  $\mathbb{C} \setminus E$ . Indeed, our conditions on  $E$  imply that  $E$  has positive capacity, and therefore the identity function omits in  $\mathbb{C} \setminus E$  values from a set of positive capacity. Hence, by the Frostman theorem, (see [21, Chapter X, Section 2.8] for the case of the unit disc, and use the uniformization argument in the general case), the identity function is of bounded type in  $\mathbb{C} \setminus E$ , and the statement for polynomials follows immediately.

For every regular set  $E \subset \mathbb{R}$ , we denote by  $\mathcal{M}_E(z)$  the symmetric Martin function for  $\mathbb{C} \setminus E$  with singularity at infinity, that is, a positive harmonic function in  $\mathbb{C} \setminus E$  which vanishes on  $E$  and satisfies the equality  $\mathcal{M}_E(\bar{z}) = \mathcal{M}_E(z)$ . A uniqueness theorem proved by Benedicks [3, Theorems 2 and 3] and Levin [18, Theorem 3.2] asserts that  $\mathcal{M}_E$  exists and is unique up to a positive multiplicative constant. The function  $\mathcal{M}_E$  extended by zero on  $E$  is subharmonic in  $\mathbb{C}$ , and has at most order one and mean type:  $\mathcal{M}_E(z) = O(|z|)$ ,  $|z| \rightarrow \infty$ .

Our first result describes the sets  $E$  such that every Cartwright class function is of bounded type in  $\mathbb{C} \setminus E$ . We say that a set  $E \subset \mathbb{R}$  is an *Akhiezer–Levin set* if the function  $\mathcal{M}_E$  is of mean type with respect to order 1, that is,

$$\sigma_{\mathcal{M}_E} \stackrel{\text{def}}{=} \limsup_{|z| \rightarrow \infty} \frac{\mathcal{M}_E(z)}{|z|} > 0.$$

It is worth mentioning that in this case the limit

$$\sigma_{\mathcal{M}_E} = \lim_{|y| \rightarrow \infty} \mathcal{M}_E(iy)/|y|$$

exists, and  $\mathcal{M}_E(z) \geq \sigma_{\mathcal{M}_E} |\operatorname{Im} z|$ . The function  $\mathcal{M}_E$  normalized by the condition  $\sigma_{\mathcal{M}_E} = 1$  is sometimes called the Phragmén–Lindelöf function.

The class of Akhiezer–Levin sets was introduced in [2]. Let us present two equivalent conditions. A set  $E \subset \mathbb{R}$  is an Akhiezer–Levin set if and only if either of the following two properties holds:

1. (Koosis [13, Section VIII A.2])  $\int_{\mathbb{R}} G(t, z) dt < \infty$ , where  $G$  is the Green function for  $\mathbb{C} \setminus E$ ,  $z \in \mathbb{C} \setminus E$ .
2. (Benedicks [3, Theorem 4])  $\int_{\mathbb{R}} \beta_E(t)/(1+|t|) dt < \infty$ , where  $\beta_E(t)$  is the harmonic measure  $\omega(t, \partial S_t, S_t \setminus E)$  of the boundary of the square  $S_t = \{z = x + iy : |x - t| < t/2, |y| < t/2\}$  with respect to the domain  $S_t \setminus E$  at the point  $t$ .

Next we present three metric tests:

1. (Akhiezer–Levin [2, Section 3.VII], Kargaev [12, Theorem 6(a)]) If  $\int_{\mathbb{R} \setminus E} dx/(1+|x|) < \infty$ , then  $E$  is an Akhiezer–Levin set.
2. (Schaeffer [23, Lemma 1]) If  $E$  is relatively dense with respect to Lebesgue measure  $dm$  (that is, for some  $a, b$ , and for every  $x \in \mathbb{R}$ , we have  $m(E \cap [x, x+a]) \geq b$ ), then  $E$  is an Akhiezer–Levin set.
3. (Kargaev [12, Theorem 4]) If  $E$  is an Akhiezer–Levin set, then  $\int_{\mathbb{R}} \text{dist}(x, E)/(1+x^2) dx < \infty$ .

Given a positive symmetric harmonic function  $h$  on  $\mathbb{C} \setminus E$ , set

$$C = \max\{c \geq 0 : h - c\mathcal{M}_E \text{ is non-negative on } \mathbb{C} \setminus E\},$$

and define the function  $PI_{E,h}$  (the Poisson integral of a non-negative measure with support on  $E$ ) which is non-negative, symmetric, and harmonic on  $\mathbb{C} \setminus E$ , and satisfies the equality

$$h = PI_{E,h} + C\mathcal{M}_E; \quad (1.3)$$

clearly,

$$\text{there is no } \varepsilon > 0 \text{ such that } PI_{E,h} \geq \varepsilon\mathcal{M}_E \text{ on } \mathbb{C} \setminus E. \quad (1.4)$$

We use a lemma which is possibly known. Since we were unable to find the appropriate reference, we give its proof in Section 3.

**Lemma 1.1.** *For every positive symmetric harmonic function  $h$  on  $\mathbb{C} \setminus E$ ,*

$$PI_{E,h}(iy) = o(\mathcal{M}_E(iy)), \quad |y| \rightarrow \infty.$$

Now, we present an extension of Theorem A:

**Theorem 1.2.** *If  $E \subset \mathbb{R}$  is an Akhiezer–Levin set, then every function  $f$  in the Cartwright class is of bounded type in  $\mathbb{C} \setminus E$ . Conversely, let  $f$  be an entire function of non-zero exponential type belonging to*

the Cartwright class. If  $f$  is of bounded type in  $\mathbb{C} \setminus E$ , then  $E$  is an Akhiezer–Levin set.

*Proof.* Let  $E$  be an Akhiezer–Levin set, and let  $f$  be in the Cartwright class, and of exponential type  $\sigma \geq 0$ . First we suppose that  $|f(x)| \leq 1$ ,  $x \in \mathbb{R}$ . Applying the Phragmén–Lindelöf principle to the function  $\log |f| - \sigma_1 \mathcal{M}_E$  with  $\sigma_1 > \sigma \sigma_{\mathcal{M}_E}^{-1}$ , in the upper and in the lower half-planes, we conclude that  $\log |f| - \sigma_1 \mathcal{M}_E$  is non-positive everywhere in  $\mathbb{C}$ , and therefore,  $\sigma_1 \mathcal{M}_E$  is a positive harmonic majorant for  $\log |f|$  in  $\mathbb{C} \setminus E$ .

In the general case, we use the Beurling–Malliavin multiplier theorem [5]: there exists a function  $g$  in the Cartwright class with  $(1 + |f(x)|)|g(x)| \leq 1$ ,  $x \in \mathbb{R}$ . Applying the previous argument, we obtain that  $g$  and  $fg$ , and hence,  $f$ , are of bounded type in  $\mathbb{C} \setminus E$ .

Now, let  $f$  be an entire function of non-zero exponential type belonging to the Cartwright class. Suppose that  $f$  is of bounded type in  $\mathbb{C} \setminus E$ . Then the function  $\log |f(z)|$  has a positive harmonic majorant  $h(z)$ ; without loss of generality we may assume that  $h$  is symmetric,  $h(z) = h(\bar{z})$ . By formula (1.3),  $h = PI_{E,h} + C\mathcal{M}_E$ . Since the function  $f$  has non-zero exponential type, Lemma 1.1 implies that  $\mathcal{M}_E(iy) \geq c|y|$  for large  $|y|$ . This implies that  $E$  is an Akhiezer–Levin set.  $\square$

Our next result extends Theorem B. Now we assume that  $E$  is the union of disjoint closed intervals  $I_m = [a_m, b_m]$  with  $\text{dist}(0, I_m) \rightarrow \infty$ . Given an interval  $I$  we denote its length by  $|I|$ .

**Theorem 1.3.** *If  $f$  is a Krein class function with zeros  $\lambda_n$  on  $E = \cup I_m$ , and  $|I_m| \geq c \text{dist}(0, I_m)^{-M}$ ,  $c > 0$ ,  $M < \infty$ , then  $f$  is of bounded type in  $\mathbb{C} \setminus E$ .*

*Proof.* We are to prove that the function  $\sum_n 1/[f'(\lambda_n)(z - \lambda_n)]$  is of bounded type in  $\mathbb{C} \setminus E$ . Multiplying, if necessary,  $f$  by a polynomial with real zeros, we obtain

$$\sum_n \frac{1 + |\lambda_n|^M}{|f'(\lambda_n)|} < \infty. \quad (1.5)$$

Without loss of generality, we may assume that  $f'(\lambda_n)$  are real. Furthermore,

$$\sum_n \frac{1}{f'(\lambda_n)(z - \lambda_n)} = \sum_{j=1}^2 g_j(z) = \sum_{j=1}^2 \sum_n \frac{c_{n,j}}{z - \lambda_n}, \quad (1.6)$$

where  $c_{n,1} \geq 0$ ,  $c_{n,2} \leq 0$ , and

$$\sum_{j=1}^2 \sum_n (1 + |\lambda_n|^M) |c_{n,j}| < \infty.$$

It suffices to verify that each  $g_j$  in (1.6) is a function of bounded type in  $\mathbb{C} \setminus E$ . Take one of them, say  $g_1$ , and represent it as the sum of two functions,

$$g_1(z) = g_-(z) + g_+(z) = \sum_{\lambda_n \in E_-} \frac{c_{n,1}}{z - \lambda_n} + \sum_{\lambda_n \in E_+} \frac{c_{n,1}}{z - \lambda_n},$$

where  $E_- = \bigcup [a_m, (a_m + b_m)/2)$ ,  $E_+ = \bigcup [(a_m + b_m)/2, b_m]$ ,  $E = E_+ \cup E_-$ . Let us verify that  $g_+$  is of bounded type in  $\mathbb{C} \setminus E$ . Indeed, this function is analytic in  $\mathbb{C} \setminus E$  and satisfies the following properties:

$$\frac{\operatorname{Im} g_+(z)}{\operatorname{Im} z} < 0, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

and, for  $x \in \mathbb{R} \setminus E$ ,

$$g_+(x) \geq - \sum_{\lambda_n \in E_+, \lambda_n > x} \frac{2c_{n,1}}{b_m - a_m} \geq -\frac{2}{c} \sum_n c_{n,1} (1 + |\lambda_n|^M) = \tau > -\infty.$$

Thus, the image  $g_+(\mathbb{C} \setminus E)$  omits the ray  $(-\infty, \tau)$ . Applying the Frostman theorem, we get that the function  $g_+$  is of bounded type in  $\mathbb{C} \setminus E$ . (Alternatively, we could just consider the functions  $\theta(z) = (z-1)/(z+1)$ ,  $\theta_1(z) = (1+z)/(1-z)$ ,  $\psi = \theta(\sqrt{g_+ - \tau})$ ;  $\psi$  has absolute values bounded by one in  $\mathbb{C} \setminus E$ . Hence,  $g_+ = [\theta_1(\psi)]^2 + \tau$  is of bounded type there.) The same argument works for  $g_-$ , and we conclude that  $g_1 = g_- + g_+$  is of bounded type in  $\mathbb{C} \setminus E$ ,  $g_2$ , and finally,  $1/f$  are of bounded type in  $\mathbb{C} \setminus E$ .  $\square$

*Remark 1.4.* Using more information on the Krein class function  $f$  we can further weaken our conditions on  $|I_m|$ : the assertion of Theorem 1.3 holds if for some  $c > 0$ ,  $M < \infty$ , and for every zero  $\lambda_n$  of  $f$ ,

$$|I(\lambda_n)| \geq \frac{c}{(1 + |\lambda_n|^M) |f'(\lambda_n)|},$$

where  $I(\lambda_n)$  is the interval of  $E$  containing the point  $\lambda_n$ . It looks plausible that the assertion of Theorem 1.3 holds for any system of intervals of “non-quasianalytically decaying lengths”. Namely, we may conjecture that if  $f$  is an entire function in the Krein class, of positive exponential type, with zeros  $\lambda_n$ , and if  $\varphi$  is a positive Lip 1 function on

$\mathbb{R}$ , then  $f$  is of bounded type in  $\mathbb{C} \setminus E$ , with

$$E = \bigcup_n \left[ \lambda_n - \frac{1}{\varphi(\lambda_n)}, \lambda_n + \frac{1}{\varphi(\lambda_n)} \right]$$

if and only if  $\int_{\mathbb{R}} \varphi(x)/(1+x^2)dx < \infty$ . In some special cases, when the zero set of  $f$  is regularly distributed, and  $\varphi$  satisfies additional regularity assumptions, this statement follows from Benedicks' results [3, Theorem 5].

Combining Theorems 1.2 and 1.3, we get a sufficient condition for  $E$  to be an Akhiezer–Levin set.

**Corollary 1.5.** *If  $f$  is a Krein class function of positive exponential type, with zeros  $\lambda_n$ , and if  $M$  is a constant, then the set*

$$E = \bigcup_n \left[ \lambda_n - \frac{1}{(1 + |\lambda_n|^M)|f'(\lambda_n)|}, \lambda_n + \frac{1}{(1 + |\lambda_n|^M)|f'(\lambda_n)|} \right]$$

*is an Akhiezer–Levin set.*

**2. The Bernstein problem on subsets of the real line.** Fix a weight  $W$ , that is a lower semicontinuous function  $W : \mathbb{R} \rightarrow [1, +\infty]$  such that  $\lim_{|x| \rightarrow \infty} |x|^n/W(x) = 0$  for  $n \geq 0$ . Consider the space  $C(W)$  of functions  $f$  continuous on  $\mathbb{R}$  and such that  $\lim_{|x| \rightarrow \infty} |f(x)|/W(x) = 0$ . Put

$$\|f\|_{C(W)} = \sup_{x \in \mathbb{R}} \frac{|f(x)|}{W(x)}.$$

The Bernstein problem consists in finding out whether the set  $\mathcal{P}$  of all polynomials is dense in  $C(W)$ . From now on, we suppose that *the weight  $W$  is finite on a subset of  $\mathbb{R}$  having a finite limit point*. In this case the polynomials are simultaneously dense or not dense in every  $C(W_r)$ ,  $W_r(x) = W(x)(1 + |x|)^r$ ,  $r \in \mathbb{R}$  (see [20, subsection 24]). Denote by  $X_W$  the set of polynomials  $P$  such that  $\|P\|_{C(W)} \leq 1$ . We define *the Hall majorant*  $M_W$  as

$$M_W(z) = \sup\{|P(z)| : P \in X_W\}.$$

Since the function  $\varphi$ ,  $\varphi \equiv 1$ , belongs to  $X_W$ , we have  $M_W(z) \geq 1$ ,  $z \in \mathbb{C}$ . Furthermore,  $M_W(x) \leq W(x)$ ,  $x \in \mathbb{R}$ ,  $\log M_W$  is lower semicontinuous in the plane.

General criteria of density of polynomials in weighted spaces were obtained by Akhiezer–Bernstein, Pollard and Mergelyan at the beginning of the 50-ies (cf. [1, 20, 13]). We shall use

**Theorem C.** *The polynomials are dense in  $C(W)$  if and only if one of three equivalent conditions holds:*

(1) (Akhiezer–Bernstein)

$$\sup_{P \in X_W} \int_{\mathbb{R}} \frac{\log^+ |P(x)|}{1+x^2} dx = +\infty;$$

(2) (Mergelyan)

$$\int_{\mathbb{R}} \frac{\log M_W(x)}{1+x^2} dx = +\infty;$$

(3) (Mergelyan)  $M_W(z) = +\infty$  for some (all)  $z \in \mathbb{C} \setminus \mathbb{R}$ .

Another criterion was proposed in [6].

**Theorem D.** (de Branges) *The polynomials are not dense in  $C(W)$  if and only if there exists an entire function  $F$  of zero exponential type,  $F \notin \mathcal{P}$ , with (simple) real zeros  $\lambda_n$ , such that*

$$\sum_n \frac{W(\lambda_n)}{|F'(\lambda_n)|} < +\infty.$$

*Such a function  $F$  belongs to the Krein class; for such  $F$  and for every polynomial  $P$ ,*

$$\sum_n \frac{P(\lambda_n)}{F'(\lambda_n)} = 0.$$

The weight  $W$  is assumed to be continuous in [6]; however, the result holds for lower semicontinuous  $W$  as well, see [24].

From now on, we suppose that  $W(x) = \infty$  for  $x \in \mathbb{R} \setminus E$ , where  $E$  is a subset of  $\mathbb{R}$  of the kind considered in the previous section,  $E = \cup I_m$ , where  $I_m$  are disjoint closed intervals on  $\mathbb{R}$ ,  $\text{dist}(0, I_m) \rightarrow \infty$  as  $m \rightarrow \infty$ . Following Benedicks [4] and Koosis [13, Section VIIIA] we try to solve the Bernstein problem for  $C(W)$  in terms of  $M_W|_E$ , replacing the form  $(1+x^2)^{-1}dx$  by the harmonic measure  $\omega_E(dx) = \omega(i, dx, \mathbb{C} \setminus E)$ .

**Theorem 2.1.** *Suppose that  $|I_m| \geq c(\text{dist}(0, I_m))^{-M}$  for some  $c > 0$ ,  $M < \infty$ . The polynomials are dense in  $C(W)$  if and only if*

$$\int_E \log M_W(x) \omega_E(dx) = +\infty. \quad (2.1)$$

*Furthermore, if the polynomials are not dense in  $C(W)$ , then the function  $\log M_W$  has a (positive) harmonic majorant in  $\mathbb{C} \setminus E$ .*

*Proof.* To obtain the result in one direction, we prove that

$$\log |P(z)| \leq \int_E \log^+ |P(x)| \omega(z, dx, \mathbb{C} \setminus E), \quad P \in \mathcal{P}. \quad (2.2)$$

First, observe that our assumptions on  $E$  imply that

$$\log |y| = o(\mathcal{M}_E(iy)), \quad y \rightarrow \infty. \quad (2.3)$$

Indeed, take a sequence of points  $x_k \in E$  tending to  $\infty$  sufficiently rapidly (for example,  $|x_{k+1}| > 2|x_k|$  suffices), and consider the entire function  $B(z) = \prod_k (1 - z/x_k)$ . Then  $B$  is in the Krein class, and  $\log |B(iy)| / \log |y| \rightarrow \infty$  as  $y \rightarrow \infty$ . By Theorem 1.3,  $\log |B|$  has a positive harmonic majorant  $h$  in  $\mathbb{C} \setminus E$ . Using the representation (1.3) for  $h$  together with Lemma 1.1 we obtain (2.3).

Then, applying a standard Phragmén–Lindelöf argument to the subharmonic functions

$$\log |P(z)| - \int_E \log^+ |P(x)| \omega(z, dx, \mathbb{C} \setminus E) - c\mathcal{M}_E(z), \quad c > 0,$$

in the domain  $\mathbb{C} \setminus E$ , we obtain (2.2).

By (2.2),

$$\log M_W(i) \leq \int_E \log M_W(x) \omega_E(dx).$$

If the last integral is finite, then  $M_W(i) < \infty$ , and by Theorem C, the polynomials are not dense in  $C(W)$ .

Now we suppose that the polynomials are not dense in  $C(W)$ , and as a consequence, are not dense in  $C(W_r)$  where  $r$  is to be chosen later on. We apply Theorem D, and get an entire function  $F$  in the Krein class, of zero exponential type,  $F \notin \mathcal{P}$ , with zeros  $\lambda_n \in E$ , such that

$$\sum_n \frac{W_r(\lambda_n)}{|F'(\lambda_n)|} \leq 1, \quad (2.4)$$

and for every polynomial  $P$ , for every  $z \in \mathbb{C}$ ,

$$\sum_n \frac{P(z) - P(\lambda_n)}{(z - \lambda_n)F'(\lambda_n)} = 0.$$

Using relation (1.1) we get

$$\frac{P(z)}{F(z)} = \sum_n \frac{P(\lambda_n)}{(z - \lambda_n)F'(\lambda_n)}.$$

Theorem 1.3 implies that the function  $F$  is of bounded type in  $\mathbb{C} \setminus E$ , and hence, by (1.2),

$$0 \leq h(z) \stackrel{\text{def}}{=} \int_E \log^+ |F(t)| \omega(z, dt, \mathbb{C} \setminus E) < \infty.$$



Using this fact and relation (2.2), we obtain

$$\begin{aligned}\log |P(z)| &\leq h(z) + \int_E \log^+ \left| \sum_n \frac{P(\lambda_n)}{(t - \lambda_n)F'(\lambda_n)} \right| \omega(z, dt, \mathbb{C} \setminus E), \\ \log M_W(z) &\leq h(z) + \int_E \log^+ \sup_{P \in X_W} \left| \sum_n \frac{P(\lambda_n)}{(t - \lambda_n)F'(\lambda_n)} \right| \omega(z, dt, \mathbb{C} \setminus E),\end{aligned}$$

for  $z \in \mathbb{C} \setminus E$ , where  $X_W$  is, as above, the set of polynomials  $P$  such that  $\|P\|_{C(W)} \leq 1$ . Therefore, to complete the proof of the theorem we need only to verify that

$$\int_E \log^+ \sup_{P \in X_W} \left| \sum_n \frac{P(\lambda_n)}{(t - \lambda_n)F'(\lambda_n)} \right| \omega_E(dt) < \infty. \quad (2.5)$$

Then, using Harnack's inequality, we conclude that  $\log M_W$  has a harmonic majorant in  $\mathbb{C} \setminus E$ , and by (1.2) we get that the condition (2.1) does not hold.

Let us return to (2.5). By Jensen's inequality, for every  $d > 0$  we have

$$\begin{aligned}&\frac{1}{d} \int_E \log^+ \sup_{P \in X_W} \left| \sum_n \frac{P(\lambda_n)}{(t - \lambda_n)F'(\lambda_n)} \right|^d \omega_E(dt) \\ &\leq \frac{1}{d} \log^+ \int_E \sup_{P \in X_W} \left| \sum_n \frac{P(\lambda_n)}{(t - \lambda_n)F'(\lambda_n)} \right|^d \omega_E(dt).\end{aligned}$$

Furthermore, by (2.4), for  $0 < d < 1$  we get

$$\begin{aligned}\sup_{P \in X_W} \left| \sum_n \frac{P(\lambda_n)}{(t - \lambda_n)F'(\lambda_n)} \right|^d &\leq \sup_{P \in X_W} \sum_n \left| \frac{P(\lambda_n)}{W_r(\lambda_n)} \right|^d \left| \frac{W_r(\lambda_n)}{(t - \lambda_n)F'(\lambda_n)} \right|^d \\ &\leq \sum_n \frac{1}{(1 + |\lambda_n|)^{rd} |t - \lambda_n|^d}.\end{aligned}$$

Therefore, to prove (2.5) it remains to check that for some  $0 < d < 1$ ,

$$\sum_n \int_E \frac{1}{(1 + |\lambda_n|)^{rd} |t - \lambda_n|^d} \omega_E(dt) < \infty. \quad (2.6)$$

Since  $\lambda_n$  are the zeros of an entire function of zero exponential type,  $1 + |\lambda_n| \geq cn$  for some  $c > 0$ . Thus, inequality (2.6) follows from the estimate

$$\int_E \frac{1}{|t - \lambda|^d} \omega_E(dt) \leq C(1 + |\lambda|)^{rd-2} \quad (2.7)$$

with  $C$  independent of  $\lambda \in E$ , and with some  $0 < d < 1$ .

Our conditions on  $E$  imply that for some  $c > 0$ ,  $M < \infty$ , and every  $\lambda \in E$ , there exists  $\delta$ ,  $c(1 + |\lambda|)^{-M} \leq \delta \leq 10c(1 + |\lambda|)^{-M}$  such that for  $I = [\lambda - \delta, \lambda + \delta]$  we have

$$E \cap I = J_1 \cup J_2$$

where intervals  $J_k = [a_k, b_k]$  satisfy the condition  $|J_k| \geq c(1 + |\lambda|)^{-M}$ ,  $k = 1, 2$ . The following elementary estimate of harmonic measure,

$$\omega_E(dt) = \omega(i, dt, \mathbb{C} \setminus E) \leq \omega(i, dt, \mathbb{C} \setminus I_k) \leq \frac{C dt}{\sqrt{(b_k - t)(t - a_k)}}, \quad t \in J_k,$$

shows that for  $d = 1/4$ ,

$$\begin{aligned} & \int_{J_k} \frac{1}{|t - \lambda|^d} \omega_E(dt) \\ & \leq \int_{a_k}^{b_k} \frac{C dt}{(b_k - t)^{1/2}(t - a_k)^{1/2}|t - \lambda|^{1/4}} \leq C(1 + |\lambda|)^{M/4}, \quad k = 1, 2. \end{aligned}$$

Furthermore,

$$\int_{E \setminus I} \frac{1}{|t - \lambda|^{1/4}} \omega_E(dt) \leq \sup_{t \in E \setminus I} \frac{1}{|t - \lambda|^{1/4}} \leq C(1 + |\lambda|)^{M/4}.$$

Thus, estimate (2.7) is true for  $r \geq M + 8$ ,  $d = 1/4$ .  $\square$

*Remark 2.2.* The same proof shows that the polynomials are not dense in  $C(W)$  as soon as (2.1) fails and there exists a sequence  $m_k \rightarrow \infty$  such that  $|I_{m_k}| \geq c(\text{dist}(0, I_{m_k}))^{-M}$  for some  $c > 0$ ,  $M < \infty$ .

*Remark 2.3.* In the general setting, if the polynomials are not dense in  $C(W)$ , then every function  $f$  from the closure of the polynomials  $\text{Clos}_{C(W)}\mathcal{P}$  has analytic continuation into the whole complex plane, and  $|f(z)| \leq M_W(z)||f||_{C(W)}$ ,  $z \in \mathbb{C}$ . Therefore, in the assumptions of Theorem 2.1, every function from  $\text{Clos}_{C(W)}\mathcal{P}$  is of bounded type in  $\mathbb{C} \setminus E$ .

*Remark 2.4.* As in Theorem C, condition (2.1) is equivalent to

$$\sup_{P \in X_W} \int_E \log^+ |P(x)| \omega_E(dx) = +\infty.$$

The following examples demonstrate that the assertions of Theorem 2.1 are not valid if the condition  $|I_m| \geq c(\text{dist}(0, I_m))^{-M}$  is not fulfilled.

**Proposition 2.5.** (a) *There exist a weight  $W$  and a subset  $E$  of  $\mathbb{R}$  such that  $W(x) = \infty$  for  $x \in \mathbb{R} \setminus E$ , the polynomials are dense in  $C(W)$  and*

$$\int_E \log M_W(x) \omega_E(dx) < \infty.$$

(b) *There exist a weight  $W$  and a subset  $E$  of  $\mathbb{R}$  such that  $W(x) = \infty$  for  $x \in \mathbb{R} \setminus E$ , the polynomials are not dense in  $C(W)$ , and*

$$\int_E \log M_W(x) \omega_E(dx) = +\infty.$$

*Proof.* (a) Consider a set of disjoint intervals  $I_n$ , such that  $|I_n| \leq 1$ ,  $\exp n \in I_n$ ,  $n \geq 1$ , and  $\omega(i, I_n, \mathbb{C} \setminus (I_1 \cup I_n)) < n^{-2} \exp(-n)$  for  $n > 1$ . We define a weight  $W$  as follows:  $W|_{I_n} \equiv \exp \exp n$ ,  $n \geq 1$ , and  $W(x) = +\infty$  elsewhere. By Theorem D, the polynomials are not dense in  $C(W)$ . Indeed, no entire function  $F$  of zero exponential type, with real zeros  $\lambda_n \rightarrow \infty$ , satisfies the condition

$$|F'(\lambda_n)| \geq c \exp \lambda_n.$$

Finally, since  $M_W(x) \leq W(x)$ ,  $x \in \mathbb{R}$ , we get

$$\begin{aligned} \int_E \log M_W(x) \omega_E(dx) &\leq \int_{I_1} \log W(x) \omega_E(dx) + \sum_{n>1} \int_{I_n} \log W(x) \omega_E(dx) \\ &\leq C + \sum_{n>1} \omega(i, I_n, \mathbb{C} \setminus (I_1 \cup I_n)) \sup_{I_n} \log W \leq C + \sum_{n>1} n^{-2} e^{-n} e^n < \infty. \end{aligned}$$

(b) We use an auxiliary estimate of harmonic measure.

**Lemma 2.6.** *Let  $I_n$  be intervals of length 1 such that  $\text{dist}(0, I_n) = (1 + o(1)) \exp n$ ,  $n \rightarrow +\infty$ ,  $E = \cup I_n$ . Then for some  $\varepsilon > 0$  and  $C > 0$ , independent of  $E$ ,*

$$\omega(i, I_n, \mathbb{C} \setminus E) \geq C \cdot e^{(\varepsilon-1)n}.$$

We postpone the proof of this lemma till the next section.

Fix  $\rho$  with  $1 - \varepsilon < 2\rho < 1$ , and consider the canonical product

$$B_\rho(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^{1/\rho}}\right),$$

and the entire function  $F_\rho$  of zero exponential type,  $F_\rho(z) = B_\rho(z^2)$ . Denote by  $\Lambda$  the zero set of  $F_\rho$ ,  $\Lambda = \{\lambda_{\pm n} = \pm n^{1/(2\rho)}, n \geq 1\}$ . It follows from the results of G. H. Hardy in [10] that the expression

$$\frac{B_\rho(z)}{z^{-1/2} \sin(\pi z^\rho) \exp((\pi \cot \pi \rho) z^\rho)}$$

tends to a finite non-zero limit for  $|z| \rightarrow \infty$  with  $|\arg z| \leq \pi/2$ . Put  $f(z) = \sin(\pi z^\rho)$ ,  $g = B_\rho/f$ . Since  $B'_\rho(\lambda) = g(\lambda)f'(\lambda)$ ,  $\lambda = n^{1/\rho}$ ,  $n \geq 1$ , we get for some  $c > 0$ :

$$\begin{aligned} |B'_\rho(\lambda)| &= (c + o(1))\lambda^{\rho-3/2} \exp((\pi \cot \pi \rho)\lambda^\rho), & \lambda = n^{1/\rho}, n \rightarrow \infty, \\ |F'_\rho(\lambda)| &= (2c + o(1))|\lambda|^{2\rho-2} \exp((\pi \cot \pi \rho)|\lambda|^{2\rho}), & \lambda \in \Lambda, |\lambda| \rightarrow \infty, \end{aligned}$$

whence

$$\begin{aligned} \log |F'_\rho(\pm \exp t)| &= \log(2c) + o(1) + (2\rho - 2)t + (\pi \cot \pi \rho) \exp(2\rho t), \\ &\exp t \in \Lambda, t \rightarrow \infty. \end{aligned}$$

We define an even weight  $W$  by the formula

$$\log W(\pm \exp t) = (\pi \cot \pi \rho) \exp(2\rho t). \quad (2.8)$$

Since

$$\sum_{\lambda \in \Lambda} \frac{W(\lambda)}{|\lambda|^2 |F'_\rho(\lambda)|} < \infty,$$

Theorem D implies that the polynomials are not dense in  $C(W_r)$ . The even log-convex function  $W$  is increasing on the positive half-line. Let us verify that

$$\log W(x) = (1 + o(1)) \log M_W(x), \quad x \rightarrow +\infty. \quad (2.9)$$

First, without loss of generality, we assume that  $W$  is  $C^2$ -smooth. By convexity of  $\varphi = \log W \circ \exp$ , for every  $r, s$  we have  $\varphi(s) + \varphi'(s)(r-s) \leq \varphi(r)$ . Now we fix  $t = \log x$ . If  $\varphi'(r) = n \leq \varphi'(t) < n+1$ , then  $\varphi(t) \geq n(t-r)$ . Furthermore, for some  $\xi \in (r, t)$  we have  $\varphi(t) - \varphi(r) = \varphi'(\xi)(t-r)$ . Since  $\varphi'(\xi) \leq \varphi'(t) < n+1 = \varphi'(r)+1$ , we get  $\varphi(t) - \varphi(r) - \varphi'(r)(t-r) \leq t-r$ , and as a result,  $\varphi(r) + \varphi'(r)(t-r) \geq (1 - 1/n)\varphi(t)$ . Therefore,  $x^n \exp(\varphi(r) - rn) \geq W(x)^{1-1/n}$ ; for every  $y > 0$ ,  $r \in \mathbb{R}$ , we have  $W(y) \geq y^n \exp(\varphi(r) - rn)$ , thus the estimate (2.9) is proved.

Relations (2.8) and (2.9) imply that

$$\log M_W(x) = (1 + o(1))(\pi \cot \pi \rho)|x|^{2\rho}, \quad |x| \rightarrow \infty. \quad (2.10)$$

Choose a sequence of intervals  $I_n$  with  $|I_n| = 1$ ,  $\text{dist}(0, I_n) = (1 + o(1)) \exp n$ ,  $n \rightarrow +\infty$ , and set  $E = \cup I_n$ . Apply Lemma 2.6, and add to  $E$  a union  $E'$  of small intervals such that  $\Lambda \subset E'$ , and

$$\omega(i, I_n, \mathbb{C} \setminus (E \cup E')) \geq C_1 \cdot e^{(\varepsilon-1)n}.$$

Then by (2.10),

$$\begin{aligned} & \int_{E \cup E'} \log M_W(x) \omega(i, dx, \mathbb{C} \setminus (E \cup E')) \\ & \geq \sum_n \omega(i, I_n, \mathbb{C} \setminus (E \cup E')) \inf_{I_n} \log M_W \geq \sum_n C_2 \cdot e^{(\varepsilon-1)n} e^{2\rho n} = +\infty. \end{aligned}$$

□

As corollaries to Theorem 2.1 we obtain some results of Levin and Benedicks. First, suppose that  $W|_E = \widetilde{W}|_E$  for an even log-convex function  $\widetilde{W}$  increasing on the positive half-line,  $W|_{\mathbb{R} \setminus E} \equiv +\infty$ . Applying Theorem 2.1 and using relation (2.9) for  $\widetilde{W}$ , we arrive at a statement, essentially equivalent to that proved by Levin in [17, Theorem 3.23]:

**Theorem E.** *If  $W$  is a weight as above, and a set  $E$  satisfies the conditions of Theorem 2.1, then the polynomials are dense in  $C(W)$  if and only if*

$$\int_E \log W(x) \omega_E(dx) = +\infty.$$

Benedicks investigated in [4] (see also the discussion in [13, Section VIII A.4]) the weighted polynomial approximation on the sets

$$E = \cup_{n \in \mathbb{Z}} [|n|^p \operatorname{sgn} n - \delta, |n|^p \operatorname{sgn} n + \delta] \quad (2.11)$$

for  $p > 1$ ,  $\delta < 1/2$ . He announced the following result:

**Theorem F.** *Suppose that  $E$  is a set of the form (2.11), and  $W$  is a weight such that  $W(x) = +\infty$  for  $x \in \mathbb{R} \setminus E$ . The polynomials are dense in  $C(W)$  if and only if*

$$\sup_{P \in X_W} \int_E \frac{\log |P(x)|}{1 + |x|^{1+1/p}} dx = +\infty.$$

In [4] Benedicks gave a proof of the “only if” part of this theorem based on his estimate of the harmonic measure  $\omega_E(dx) = \omega(i, dx, \mathbb{C} \setminus E)$  for sets  $E$  of the form (2.11):

**Lemma 2.7.** *If  $E$  has the form (2.11), then*

$$\begin{aligned} & \frac{c}{1 + |x|^{1+1/p}} \frac{1}{\sqrt{\delta^2 - (x - |n|^p \operatorname{sgn} n)^2}} \\ & \leq \frac{\omega_E(dx)}{dx} \leq \frac{C}{1 + |x|^{1+1/p}} \frac{1}{\sqrt{\delta^2 - (x - |n|^p \operatorname{sgn} n)^2}}, \end{aligned} \quad (2.12)$$

for  $x \in [|n|^p \operatorname{sgn} n - \delta, |n|^p \operatorname{sgn} n + \delta]$ , where  $c$  and  $C$  are positive constants that do not depend on  $x$  and  $n$ .

A more accessible reference for the upper bound in (2.12) is [13, Section VIII A.4] where the reader may find a sketch of the proof. A proof for the lower bound is given in the next section.

Our Theorem 2.1 together with the lower bound in (2.12) immediately yields the “if” part of Theorem F:

**Corollary 2.8.** *Suppose that  $W$  is a weight and  $E$  is a set of the form (2.11) such that  $W(x) = \infty$  for  $x \in \mathbb{R} \setminus E$ . If the polynomials are not dense in  $C(W)$ , then*

$$\int_E \frac{\log M_W(x)}{1 + |x|^{1+1/p}} dx < \infty.$$

**3. Harmonic estimation in slit domains.** Here, we prove Lemmas 1.1 and 2.6, and the lower estimate in Lemma 2.7.

*Proof of Lemma 1.1.* Suppose that for a sequence  $y_k \rightarrow +\infty$  we have

$$\infty > PI_{E,h}(iy_k) \geq \mathcal{M}_E(iy_k).$$

By Harnack’s inequality, for some positive  $c_1, c_2$  independent of  $k$ ,

$$\left. \begin{aligned} PI_{E,h}(z) &\geq c_1 PI_{E,h}(iy_k) \\ \mathcal{M}_E(z) &\leq c_2 \mathcal{M}_E(iy_k) \end{aligned} \right\}, \quad |z - iy_k| < y_k/2. \quad (3.1)$$

For some positive  $c_3$ , consider the function  $u = c_3 \mathcal{M}_E - PI_{E,h}$  harmonic on  $\mathbb{C} \setminus E$ . Then

$$u(z) \leq -(c_1 - c_2 c_3) \mathcal{M}_E(iy_k), \quad |z - iy_k| < y_k/2. \quad (3.2)$$

Next, we use the following fact (cf. Lemma 1 of [23], Lemma 6 of [3]):

$$\text{the function } y \rightarrow \mathcal{M}_E(x + iy) \text{ is increasing for } y \geq 0. \quad (3.3)$$

For the sake of completeness, we give an argument from [18]. Since  $\mathcal{M}_E$  is positive, harmonic and symmetric in  $\mathbb{C} \setminus E$ , is subharmonic in the plane, and has at most order one and mean type there, the subharmonic

version of the Hadamard representation (see [11, Section 4.2]) implies that for a finite positive measure  $\mu$  on  $\mathbb{R}$ ,

$$\begin{aligned} \mathcal{M}_E(z) &= \int_{|t| \geq 1} \left( \log \left| 1 - \frac{z}{t} \right| + \frac{\operatorname{Re} z}{t} \right) d\mu(t) \\ &\quad + \int_{|t| \leq 1} \log |t - z| d\mu(t) + c_1 + c_2 \operatorname{Re} z, \quad z \in \mathbb{C} \setminus E. \end{aligned}$$

Now, the property (3.3) follows immediately.

Using (3.3) and the second estimate in (3.1), we get

$$u(z) \leq c_3 \mathcal{M}_E(z) \leq c_2 c_3 \mathcal{M}_E(iy_k), \quad |\operatorname{Re} z| = y_k/2, |\operatorname{Im} z| < y_k. \quad (3.4)$$

Denote by  $H$  the union of two horizontal sides of the domain  $S = \{z \in \mathbb{C} : |\operatorname{Re} z| < y_k/2, |\operatorname{Im} z| < y_k\}$ , and by  $V$  the union of its two vertical sides. An estimate of harmonic measure in  $S \setminus E$  gives us that for some positive  $C$  independent of  $k$  and  $E$ ,

$$\frac{\omega(z, H, S \setminus E)}{\omega(z, V, S \setminus E)} \geq \frac{1}{C}, \quad |z| < y_k/5. \quad (3.5)$$

To obtain this estimate we use an easy generalization of a lemma of Benedicks [3, Lemma 7] (see also [13, p.436]). This generalization claims that for every square  $S_{x,t} = \{z \in \mathbb{C} : |\operatorname{Re} z - x| < t, |\operatorname{Im} z| < t\}$  with horizontal sides  $H_{x,t}$  and vertical sides  $V_{x,t}$ , the following inequality holds:

$$\omega(x + iy, H_{x,t}, S_{x,t} \setminus E) \geq \omega(x + iy, V_{x,t}, S_{x,t} \setminus E), \quad x + iy \in S_{x,t}. \quad (3.6)$$

To verify (3.6) we note first that by symmetry, on the diagonals of the square  $S_{x,t}$  we have  $\omega(\cdot, H_{x,t}, S_{x,t}) = \omega(\cdot, V_{x,t}, S_{x,t})$ . Therefore, applying the maximum principle to the difference of these functions, we get

$$\begin{aligned} \omega(r, H_{x,t}, S_{x,t}) &\leq \omega(r, V_{x,t}, S_{x,t}), \quad r \in (x - t, x + t), \\ \omega(x + iy, H_{x,t}, S_{x,t}) &\geq \omega(x + iy, V_{x,t}, S_{x,t}), \quad x + iy \in S_{x,t}. \end{aligned}$$

Finally,

$$\begin{aligned} &\omega(x + iy, H_{x,t}, S_{x,t} \setminus E) \\ &= \omega(x + iy, H_{x,t}, S_{x,t}) - \int_E \omega(r, H_{x,t}, S_{x,t}) \omega(x + iy, dr, S_{x,t} \setminus E) \\ &\geq \omega(x + iy, V_{x,t}, S_{x,t}) - \int_E \omega(r, V_{x,t}, S_{x,t}) \omega(x + iy, dr, S_{x,t} \setminus E) \\ &= \omega(x + iy, V_{x,t}, S_{x,t} \setminus E), \quad x + iy \in S_{x,t}. \end{aligned}$$

To deduce (3.5) note that the function  $\omega(z, H, S \setminus E)$  is positive and continuous in  $S \setminus E$ . Therefore, for  $A = \{x \pm iy_k/5 : |x| \leq 2y_k/5\}$  we

have  $\min_{z \in A} \omega(z, H, S \setminus E) > 0$ . Hence, for sufficiently big  $C$ ,

$$\varphi(z) \stackrel{\text{def}}{=} C\omega(z, H, S \setminus E) - \omega(z, V, S \setminus E) \geq 1, \quad z \in A.$$

For every  $z = x + iy$  with  $|z| < y_k/5$  consider the square  $S_{x,t}$  with  $t = y_k/5$ , and note that  $H_{x,t} \subset A$ ,  $S_{x,t} \subset S$ . Therefore,  $\varphi|_{H_{x,t}} \geq 1$ ,  $\varphi|_E \equiv 0$ ,  $\varphi|_{V_{x,t}} \geq -1$ , and the estimate (3.6) implies that  $\varphi(x+iy) \geq 0$ . Thus, the property (3.5) is proved.

Now, if  $c_3$  is sufficiently small, then applying the theorem on two constants to the symmetric harmonic function  $u$  in the domain  $S \setminus E$ , and using (3.2), (3.4), (3.5), and the property

$$\limsup_{z \rightarrow w} u(z) = -\liminf_{z \rightarrow w} PI_{E,h}(z) \leq 0, \quad w \in E,$$

we obtain

$$u(z) \leq 0, \quad |z| < y_k/5.$$

Thus,  $c_3 \mathcal{M}_E(z) - PI_{E,h}(z) = u(z) \leq 0$ ,  $z \in \mathbb{C} \setminus E$ . Applying the property (1.4) we come to a contradiction.  $\square$

*Proof of the lower estimate in Lemma 2.7.* Step A. For  $t \geq 0$  denote by  $K_t$  the square  $\{z \in \mathbb{C} : |\operatorname{Re} z - t| \leq t/2, |\operatorname{Im} z| \leq t/2\}$ . In what follows we use a function  $u(z) = \log |z + \sqrt{z^2 - 1}|$ ; it is positive and harmonic in  $\mathbb{C} \setminus [-1, 1]$ , vanishes on  $[-1, 1]$ , and  $u(z) = \log |z| + O(1)$ ,  $|z| \rightarrow \infty$ .

The function  $v(z) = u(Ct^{1-1/p} \sin \pi z^{1/p})$  vanishes on a closed set  $F$ ,  $K_t \cap E \subset F$ , for some  $C$  independent of  $t$ . Furthermore,  $v$  is non-negative on  $K_t$ , and harmonic on  $K_t \setminus F$ . We estimate the function  $z \rightarrow |t^{1-1/p} \sin \pi z^{1/p}|$  as follows:

$$\begin{aligned} |t^{1-1/p} \sin \pi z^{1/p}| &\leq \exp(Ct^{1/p}), \quad z \in K_t, \\ |t^{1-1/p} \sin \pi t^{1/p}| &\geq Cn^{p-1}, \quad t = (n + 1/2)^p, n \geq 0. \end{aligned}$$

The asymptotical relation  $u(z) = \log |z| + O(1)$ ,  $|z| \rightarrow \infty$ , implies now that  $v(z) \leq ct^{1/p}$ ,  $z \in K_t$ , and  $v((n + 1/2)^p) \geq c \log n$ ,  $n \geq 0$ . Next, we choose  $t = (n + 1/2)^p$ , and apply the theorem on two constants to the function  $v(z)$  in  $K_t \setminus F$ :

$$c \log n \leq v((n + 1/2)^p) \leq \omega(t, \partial K_t, K_t \setminus F) \sup_{\partial K_t} v \leq cn \omega(t, \partial K_t, K_t \setminus F).$$

Hence, for  $t = (n + 1/2)^p$ ,  $n > 1$ ,

$$\omega(t, \partial K_t, K_t \setminus E) \geq \omega(t, \partial K_t, K_t \setminus F) \geq c \frac{\log n}{n}.$$



Let  $H_t$  be the union of two horizontal sides of  $K_t$ . The lemma of Benedicks mentioned in the proof of Lemma 1.1 claims that

$$\omega(t, H_t, K_t \setminus E) \geq \frac{1}{2} \omega(t, \partial K_t, K_t \setminus E).$$

Therefore, for  $n > 1$ ,

$$\omega((n+1/2)^p, H_{(n+1/2)^p}, K_{(n+1/2)^p} \setminus E) \geq c \frac{\log n}{n}. \quad (3.7)$$

Step B. The Green function  $G(z, i)$  for  $\mathbb{C} \setminus E$  is positive, bounded and harmonic on  $\{z : |\operatorname{Im} z| > 2\}$ . Therefore, applying the Poisson formula in the half-planes  $\{z : \pm \operatorname{Im} z > 2\}$  we get

$$\begin{aligned} G(z, i) &\geq \frac{1}{\pi} \int_{-\infty}^{\infty} G(x \pm 2i, i) \frac{|\operatorname{Im} z| - 2}{(|\operatorname{Im} z| - 2)^2 + x^2} dx \\ &\geq \frac{c}{|\operatorname{Im} z|}, \quad 3|\operatorname{Im} z| \geq |\operatorname{Re} z| \geq 10. \end{aligned} \quad (3.8)$$

The inequalities (3.7) and (3.8) imply that for  $n > 1$ ,

$$G((n+1/2)^p, i) \geq \omega((n+1/2)^p, H_{(n+\frac{1}{2})^p}, K_{(n+\frac{1}{2})^p} \setminus E) \inf_{H_{(n+\frac{1}{2})^p}} G \geq c \frac{\log n}{n^{p+1}}.$$

Set  $I_n = [n^p - \delta, n^p + \delta]$ . Since  $G(z, i)$  is positive and harmonic on  $\{z : |z - n^p| \leq n^{p-1}\} \setminus I_n$ , by Harnack's inequality we get

$$G(z, i) \geq c \frac{\log n}{n^{p+1}}, \quad |z - n^p| = n^{p-1}.$$

Step C. Finally, we consider an auxiliary function  $w(z) = u((z - n^p)/\delta)$ . It is harmonic on  $\mathbb{C} \setminus I_n$ , vanishes on  $I_n$ , and  $w(z) \leq c \log n$  for  $|z - n^p| = n^{p-1}$ . Therefore, for  $n > 1$ ,

$$G(z, i) \geq \frac{c}{n^{p+1}} w(z), \quad |z - n^p| \leq n^{p-1},$$

and

$$\begin{aligned} \frac{\omega(i, dx, \mathbb{C} \setminus E)}{dx} &= \frac{1}{\pi} \frac{\partial G(x + iy, i)}{\partial y} \Big|_{y=0} \\ &\geq \frac{c}{1 + |x|^{1+1/p}} \frac{1}{\sqrt{\delta^2 - (x - |n|^p \operatorname{sgn} n)^2}} dx, \quad x \in I_n. \end{aligned}$$

The estimate for  $n \leq 1$  is obtained in an analogous way.  $\square$

*Proof of Lemma 2.6.* As in the previous proof, let us consider the Green function  $G(z, i)$  for  $\mathbb{C} \setminus E$ . It is positive, harmonic and bounded in

$\mathbb{C} \setminus (E \cup \{z : |z - i| < 1\})$ . Denote

$$h_n = \int_0^{\exp n} G(x, i) dx, \quad n \geq 0.$$

Applying the Poisson formula in the lower half-plane, we get

$$G(-ie^n, i) \geq \frac{1}{\pi} \int_0^{\exp n} \frac{e^n}{x^2 + e^{2n}} G(x, i) dx \geq c \cdot e^{-n} h_n, \quad n \geq 0.$$

By Harnack's inequality, at least on one half of the length of the interval  $[e^n, e^{n+1}]$ , we have  $G(x, i) \geq c \cdot G(-ie^n, i) \geq c \cdot e^{-n} h_n$ . Therefore, for some  $c > 0$ , we have  $h_{n+1} > (1 + c)h_n$ ,  $n \geq 0$ . Consequently,  $h_{n+1} > c(1 + c)^n$ ,  $n \geq 0$ . Once again, by the Poisson formula, for some  $\varepsilon > 0$ ,

$$G(-ix, i) > c \cdot x^{\varepsilon-1}, \quad x \geq 1.$$

For every  $n$  we denote by  $c_n$  the center of  $I_n$ . Arguing as in step C of the previous proof, we compare  $G(z, i)$  with  $w(z) = u(2(z - c_n))$  in  $\{z : |z - c_n| \leq e^{n-1}\} \setminus I_n$ , and deduce

$$\omega(i, I_n, \mathbb{C} \setminus E) \geq c \cdot e^{(\varepsilon-1)n}/n, \quad n \geq 1.$$

□

*Remark 3.1.* To estimate harmonic measure  $\omega_E$  from above, we may use Theorem 1.3 or Theorems D and E. In particular, in the conditions of Lemma 2.6, for some positive  $c$  we have

$$\omega(i, I_n, \mathbb{C} \setminus E) \leq \exp(-c\sqrt{n}), \quad n \geq 1.$$

**Acknowledgment.** The authors thank Alexander Fryntov for very useful discussions.

## REFERENCES

- [1] N. I. Akhiezer, *On the weighted approximation of continuous functions by polynomials on the entire real axis*, Uspekhi Mat. Nauk **11** (1956), 3–43; English translation in American Mathematical Society Translations (ser. 2) **22** (1962), 95–137.
- [2] N. I. Akhiezer, B. Ja. Levin, *Generalization of S. N. Bernstein's inequality for derivatives of entire functions*, Issledovaniya po sovremennym problemam teorii funktsii kompleksnogo peremennogo, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow 1960, 111–165. French translation in Fonctions d'une variable complexe. Problèmes contemporains (ed. A. Marcouchevitch), Gauthier-Villars, Paris, 1962, 109–161.
- [3] M. Benedicks, *Positive harmonic functions vanishing on the boundary of certain domains in  $\mathbb{R}^n$* , Arkiv för Matematik **18** (1980), 53–72.
- [4] M. Benedicks, *Weighted polynomial approximation on subsets of the real line*, Preprint 1981:11, Uppsala University, Mathematical Department (1981), 12 pp.

- [5] A. Beurling, P. Malliavin, *On Fourier transforms of measures with compact support*, Acta Mathematica **107** (1962), 291–309.
- [6] L. de Branges, *The Bernstein problem*, Proceedings of the American Mathematical Society **10** (1959), 825–832.
- [7] L. de Branges, *Hilbert spaces of entire functions*, Prentice-Hall, 1968.
- [8] I. C. Gohberg, M. G. Krein, *Introduction to the theory of linear non-selfadjoint operators*, Translations of Mathematical Monographs **18**, American Mathematical Society, 1969.
- [9] A. A. Goldberg, I. V. Ostrovskii, *The distribution of values of meromorphic functions*, Nauka, Moscow, 1970 (in Russian).
- [10] G. H. Hardy, *On the function  $P_\rho(x)$* , Quart. Journ. Math. Ser. 2 **37** (1906), 146–172; Complete works, vol. 4.
- [11] W. K. Hayman, P. B. Kennedy, *Subharmonic functions*, Vol. I, Academic Press, London et al, 1976.
- [12] P. P. Kargaev, *Existence of the Phragmén–Lindelöf function and some conditions for quasianalyticity*, Zap. Nauchn. Sem. LOMI **126** (1983), 97–108; English translation in J. Soviet Math. **27** (1984), 2486–2495.
- [13] P. Koosis, *The Logarithmic Integral*, Vol. I, Cambridge University Press, Cambridge, 1988.
- [14] M. G. Krein, *A contribution to the theory of entire functions of exponential type*, Izv. Akad. Nauk SSSR **11** (1947), 309–326 (in Russian).
- [15] M. G. Krein, *Fundamental aspects of the representation theory of Hermitian operators with deficiency index  $(m, m)$* , Ukr. Math. Zh. **1** (1949), no. 2, 3–66; English translation in American Mathematical Society Translations (ser. 2), **97** (1970), 75–143.
- [16] M. G. Krein, *On the indeterminate case of the Sturm–Liouville problem in the interval  $(0, \infty)$* , Izv. Akad. Nauk SSSR **16** (1952), 293–324 (in Russian).
- [17] B. Ya. Levin, *Density of systems of functions, quasianalyticity and subharmonic majorants*, Zap. Nauchn. Seminar LOMI **170** (1989), 102–156; English translation in J. Soviet Math. **63** (1993), 171–201.
- [18] B. Ya. Levin, *Majorants in classes of subharmonic functions, II, The relation between majorants and conformal mapping, III, The classification of the closed sets on  $\mathbb{R}$  and the representation of the majorants*, Teor. Funktsii Funktsional. Anal. i Prilozhen. **52** (1989), 3–33; English translation in J. Soviet Math. **52** (1990), 3351–3372.
- [19] B. Ya. Levin, *Lectures on entire functions*, Translations of Mathematical Monographs **150**, American Mathematical Society, 1996.
- [20] S. N. Mergelyan, *Weighted approximation by polynomials*, Uspekhi Mat. Nauk **11** (1956), 107–152; English translation in American Mathematical Society Translations (ser. 2) **10** (1958), 59–106.
- [21] R. Nevanlinna, *Analytic functions*, Springer-Verlag, New York–Berlin, 1970.
- [22] M. Parreau, *Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann*, Ann. Inst. Fourier Grenoble **3** (1951), 103–197.
- [23] A. C. Schaeffer, *Entire functions and trigonometric polynomials*, Duke Math. Journ. **20** (1953), 77–88.
- [24] M. Sodin and P. Yuditskii, *Another approach to de Branges' theorem on weighted polynomial approximation*, in: Proceedings of the Ashkelon Workshop on Complex Function Theory (May 1996), L. Zalcman, ed., Israel Mathematical

Conferences Proceedings **11**, American Mathematical Society, Providence RI, 1997, pp. 221-227.

A. Borichev

Laboratoire de Mathématiques Pures de Bordeaux

UPRESA 5467 CNRS, Université Bordeaux I

351, cours de la Libération, 33405 Talence

FRANCE

borichev@math.u-bordeaux.fr

M. Sodin

School of Mathematical Sciences

Tel-Aviv University

Ramat-Aviv, 69978

ISRAEL

sodin@math.tau.ac.il