AN ACCESS THEOREM FOR CONTINUOUS FUNCTIONS

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ABSTRACT. Let f be a continuous function on an open subset Ω of $\mathbb{R}^2$ such that for every $x \in \Omega$ there exists a continuous map $\gamma : [-1,1] \to \Omega$ with $\gamma(0) = x$ and $f \circ \gamma$ increasing on $[-1,1]$. Then for every $y \in \Omega$ there exists a continuous map $\gamma : [0,1) \to \Omega$ such that $\gamma(0) = y$, $f \circ \gamma$ is increasing on $[0,1)$, and for every compact subset K of Ω, $\max\{t : \gamma(t) \in K\} < 1$. This result gives an answer to a question posed by M. Ortel. Furthermore, an example shows that this result is not valid in higher dimensions.

1. INTRODUCTION

The following statement is part of a theorem, due to W. K. Hayman and M. Ortel, on the topological properties of real analytic functions [9].

Theorem A. *Suppose that f is analytic on an open subset Ω of $\mathbb{R}^n$, and every $x \in \Omega$ is an $f\uparrow$ point, that is there exists a continuous map $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $f \circ \gamma$ (strictly) increasing on $[0,1]$. Then for every $y \in \Omega$ there exists a continuous map $\gamma : [0,1) \to \Omega$ such that $\gamma(0) = y$, $f \circ \gamma$ is increasing on $[0,1)$, and for every compact subset K of Ω, $\max\{t : \gamma(t) \in K\} < 1$.*

Ortel calls this statement an access theorem by analogy with a similar result on subharmonic functions given in [4], Section 10.3 (see also [6]). It seems natural to ask whether these results do actually depend on analyticity (subharmonicity) of the functions under consideration, or they are valid just for continuous functions satisfying certain local properties. In other words, how different are the asymptotical behavior of (real) analytic functions and that of continuous functions? For a somewhat similar situation see [2].

An example constructed by Ortel [8] shows that Theorem A does not extend to C^∞-smooth functions even for $n = 2$, $\Omega = \mathbb{R}^2$. However,

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in his example the set of $f\!\downarrow$ points is not connected. A point $x \in \Omega$ is an $f\!\downarrow$ point if there exists a continuous map $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, and $f \circ \gamma$ decreases on $[0,1]$. The following question was posed in [8]: whether the claim of Theorem A holds for $f \in C^\infty(\mathbb{R}^2)$, with the additional condition that all the points in $\mathbb{R}^2$ are $f\!\downarrow$ points?

In our paper we answer this question in the positive for functions which are just continuous on (open) subsets of $\mathbb{R}^2$ (Theorem 1). Furthermore, we show (Theorem 2) that the answer to Ortel's question is negative in higher dimensions.

NOTATION: Given a function f continuous on a subset Ω of $\mathbb{R}^n$ and an interval I on $\mathbb{R}$ we use the following terminology (cf. [9]).

(a) A continuous map $\gamma : I \to \Omega$ is an (f) path if $f \circ \gamma$ is increasing on I.
(b) A continuous map $\gamma : I \to \Omega$ is a weak (f) path if $f \circ \gamma$ is non-decreasing on I.
(c) For a map $\gamma : I \to \Omega$, denote by $\gamma(I)$ the image of γ,

$$\mathrm{Cluster}_\Omega(\gamma) = \bigcap_K \operatorname*{Clos}_\Omega \gamma(I \setminus K),$$

where the intersection is taken by all the compact subintervals K of I.
(d) For $x \in \mathbb{R}^n$ denote $|x| = \mathrm{dist}\,(0,x)$. For $\delta > 0$ denote

$$D(x,\delta) = \{y \in \mathbb{R}^2 : |x-y| < \delta\},\ B(x,\delta) = \{y \in \mathbb{R}^3 : |x-y| < \delta\},$$
$$T = \{x \in \mathbb{R}^3 : |x| = 1\}.$$

(e) Denote by $\mathfrak{M}_\pm(f,\Omega)$ the set of $x \in \Omega$ such that there exists an (f) path $\gamma : [-1,1] \to \Omega$ with $\gamma(\mp 1) = x$; $\mathfrak{M}(f,\Omega) = \mathfrak{M}_+(f,\Omega) \cap \mathfrak{M}_-(f,\Omega)$.

2. AN ACCESS THEOREM

Theorem 1. *Let Ω be an open subset of $\mathbb{R}^2$, and let $f \in C(\Omega)$, $\Omega = \mathfrak{M}(f,\Omega)$. Then for every $y \in \Omega$ there exists an (f) path $\gamma : [0,1) \to \Omega$ such that $\gamma(0) = y$, and* $\mathrm{Cluster}_\Omega(\gamma) = \emptyset$.

Proof. Fix $y_0 = y \in \Omega$, and consider

$$c_0 = \sup\bigl\{f(\gamma(1)) - f(y_0),\quad \text{where } \gamma : [0,1] \to \Omega \text{ are } (f) \text{ paths with } \gamma(0) = y_0\bigr\} > 0.$$

Then, pick an (f) path γ_0 such that $\gamma_0(0) = y_0$, $f(\gamma_0(1)) - f(y_0) > \min\{1, c_0/2\}$, and put $y_1 = \gamma_0(1)$. We repeat this procedure:

$$c_k = \sup\{f(\gamma(1)) - f(y_k),$$
$$\text{where } \gamma : [0,1] \to \Omega \text{ are } (f) \text{ paths with } \gamma(0) = y_k\} > 0, \quad (2.1)$$

γ_k is an (f) path such that

$$\gamma_k(0) = y_k,\ f(\gamma_k(1)) - f(y_k) > \min\{1, c_k/2\},$$
$$y_{k+1} = \gamma_k(1), \qquad k \geq 1.$$

The concatenation of γ_k is an (f) path $\gamma : [0, \infty) \to \Omega$, $\gamma(k + s) = \gamma_k(s)$, $k \in \mathbb{Z}_+$, $0 \leq s < 1$. There are two possibilities: either (A) $\Gamma = \text{Cluster}_\Omega(\gamma) = \emptyset$ or (B) $\Gamma \neq \emptyset$ and

$$\lim_{t\to\infty} f(\gamma(t)) = c < \infty. \quad (2.2)$$

In case (A) the (f) path δ, $\delta(x) = \gamma(x/(1-x))$, $0 \leq x < 1$, gives the conclusion of the theorem. Therefore, from now on we deal with case (B). Then Γ is a non-empty closed set. Furthermore, condition (2.2) implies that $\lim_{k\to\infty} c_k = 0$.

Next we show that Γ has more than one point. Indeed, if

$$\text{Cluster}_\Omega(\gamma) = \{z\},$$

then $\lim_{t\to\infty} \gamma(t) = z$. Consider an (f) path $\gamma^* : [0,1] \to \Omega$ with $\gamma^*(0) = z$. Now we take k such that $c_k < f(\gamma^*(1)) - f(z)$, and construct an (f) path γ_k^*,

$$\gamma_k^*(x) = \begin{cases} \gamma(k + x/(1-2x)), & 0 \leq x < 1/2, \\ \gamma^*(2x - 1), & 1/2 \leq x \leq 1. \end{cases}$$

We have $\gamma_k^*(0) = y_k$, $f(\gamma_k^*(1)) - f(y_k) > c_k$, that contradicts to (2.1).

Since Γ is connected, it intersects every suitably small circle centered at one of its points. Hence, Γ is uncountable. Furthermore, f is constant on Γ, and without loss of generality we may assume $f\big|\Gamma = 0$. For every $x \in \Gamma$ fix an (f) path $\gamma_x : [-1,1] \to \Omega$ with $\gamma_x(0) = x$. Then there exist $\varepsilon > 0$ and an uncountable subset $\Gamma_1 \subset \Gamma$ such that $|f(\gamma_x(\pm 1))| > \varepsilon$, $x \in \Gamma_1$. Therefore, by continuity of f, there exist $\delta > 0$ and an uncountable subset $\Gamma_2 \subset \Gamma_1$ such that for every $x \in \Gamma_2$ and for every $y \in D(\gamma_x(-1), \delta) \cup D(\gamma_x(1), \delta)$,

$$|f(y)| > \varepsilon/2.$$

Finally, since Γ_2 is uncountable, for some point u_+ with rational coordinates the set $\Gamma_3 = \{x \in \Gamma_2 : u_+ \in D(\gamma_x(1), \delta)\}$ is uncountable, and for some point u_- with rational coordinates the set $\Gamma_4 = \{x \in$

$\Gamma_3 : u_- \in D(\gamma_x(-1), \delta)\}$ is uncountable; we pick three different points $x_1, x_2, x_3 \in \Gamma_4$ and get

$$u_\pm \in \mathcal{O}_\pm = \bigcap_{i=1}^{3} D(\gamma_{x_i}(\pm 1), \delta).$$

Put $K = K_+ \cup K_- \cup \{x_1, x_2, x_3\}$, where

$$K_+ = \bigcup_{i=1}^{3} \big(D(\gamma_{x_i}(1), \delta) \cup \gamma_{x_i}(0, 1]\big), \; K_- = \bigcup_{i=1}^{3} \big(D(\gamma_{x_i}(-1), \delta) \cup \gamma_{x_i}[-1, 0)\big).$$

The function f is positive on K_+ and negative on K_-. Now we show that if $c_k < \varepsilon$, then $\gamma([k, \infty)) \cap \gamma_{x_i}([-1, 1]) = \emptyset$, $i = 1, 2, 3$. Indeed, suppose that for some $t \ge k$, $-1 \le s \le 1$ we have $\gamma_{x_i}(s) = \gamma(t)$. Since $f(\gamma_{x_i}(s)) = f(\gamma(t)) < 0$, we get $s < 0$. Consider the (f) path γ_t^*:

$$\gamma_t^*(x) = \begin{cases} \gamma(k + (t-k)x/(1-x)), & 0 \le x < 1/2, \\ \gamma_{x_i}(1 - 2(1-s)(1-x)), & 1/2 \le x \le 1. \end{cases}$$

Then $\gamma_t^*(0) = y_k$, $f(\gamma_t^*(1)) > \varepsilon > c_k > c_k + f(y_k)$, and we get a contradiction to (2.1).

Furthermore, if $f(\gamma(t)) > -\varepsilon/2$, then $\gamma([t, \infty)) \cap D(\gamma_{x_i}(\pm 1), \delta) = \emptyset$. Thus, for some t_0, $\gamma([t_0, \infty)) \cap K = \emptyset$.

Since $\mathcal{O}_\pm$ are convex, there exist two points $a, b \in K$ and three simple Jordan arcs $\beta_i : [-1, 1] \to K$, $i = 1, 2, 3$, such that $\beta_i(-1) = a$, $\beta_i(1) = b$, $\beta_i(0) = x_i$, the sign of $f(\beta_i(t))$ coincides with the sign of t, $t \in [-1, 1]$, $i = 1, 2, 3$.

Denote by s_1 the minimal number $s \in (0, 1]$ such that $\beta_1(s) \in \beta_2([0, 1]) \cup \beta_3([0, 1])$. Without loss of generality we assume that $\beta_1(s_1) = \beta_2(s_{21})$ for some $0 < s_{21} \le 1$. Next, denote by s_3 the minimal number $s \in (0, 1]$ such that $\beta_3(s) \in \beta_2([0, 1])$. Clearly, the sets $\beta_1([0, s_1))$, $\beta_2([0, 1])$ and $\beta_3([0, s_3))$ are disjoint. There are three alternative possibilities: $\beta_3(s_3) = \beta_2(s_{23})$ for some $0 < s_{23} < s_{21}$, $\beta_3(s_3) = \beta_2(s_{23})$ for some $s_{21} < s_{23} \le 1$, $\beta_3(s_3) = \beta_2(s_{21}) = \beta_1(s_1)$.

In the first case put $a' = \beta_3(s_3)$ and define

$$\beta_1'(x) = \begin{cases} \beta_1(2 s_1 x), & 0 \le x \le 1/2, \\ \beta_2((2-2x)s_{21} + (2x-1)s_{23}), & 1/2 \le x \le 1, \end{cases}$$

$$\beta_2'(x) = \beta_2(s_{23} x), \qquad \beta_3'(x) = \beta_3(s_3 x), \qquad 0 \le x \le 1,$$

In the second case put $a' = \beta_1(s_1)$ and define

$$\beta_1'(x) = \beta_1(s_1 x), \qquad \beta_2'(x) = \beta_2(s_{21} x), \qquad 0 \le x \le 1,$$

$$\beta_3'(x) = \begin{cases} \beta_3(2 s_3 x), & 0 \le x \le 1/2, \\ \beta_2((2-2x)s_{23} + (2x-1)s_{21}), & 1/2 \le x \le 1, \end{cases}$$

In the third case put $a' = \beta_1(s_1)$ and define

$$\beta_1'(x) = \beta_1(s_1 x), \quad \beta_2'(x) = \beta_2(s_{21} x), \quad \beta_3'(x) = \beta_3(s_3 x), \quad 0 \le x \le 1.$$

In all three cases, $\beta_i' : [0,1] \to K$ are simple Jordan arcs, the point a' is the only point of intersection of $\beta_i'([0,1])$, and f is strictly positive on $\beta_i'((0,1])$, $i = 1,2,3$. In an analogous way we define b' and β_i' on $[-1,0]$.

As a result, we get three simple Jordan arcs $\beta_i' : [-1,1] \to K$, such that $\beta_i'(0) = x_i$, and the points a', b' are the only points of intersection of $\beta_i'([-1,1])$. By the Jordan theorem (see [7, Chapter 10]),

$$\mathbb{R}^2 \setminus \Big(\bigcup_{i=1}^{3} \beta_i'([-1,1]) \Big) = U_1 \cup U_2 \cup U_3,$$

where U_1, U_2, U_3 are disjoint connected open subsets of $\mathbb{R}^2$, and

$$\partial U_i = \beta_j'([-1,1]) \cup \beta_k'([-1,1]), \qquad \{i,j,k\} = \{1,2,3\}.$$

Since $\gamma([t_0,\infty))$ is connected and does not intersect K, it is a subset of one of the sets U_1, U_2, U_3, $\gamma([t_0,\infty)) \subset U_i$. Then $\Gamma = \mathrm{Cluster}_\Omega(\gamma) \subset \mathrm{Clos}\, U_i$, $\Gamma \cap \beta_i'((-1,1)) = \emptyset$, hence $x_i \notin \Gamma$, and we get a contradiction. Thus, case (B) is impossible, and the theorem is proved. □

3. An example

Our proof of Theorem 1 relies on the fact that the dimension of Ω is 2 (we use the Jordan theorem on the plane). That is why it is natural to ask whether an analogous result holds for functions defined on domains in $\mathbb{R}^3$. In the example by M. Ortel [8], where $\Omega = \mathbb{R}^2$, no (f) path can go through certain barriers consisting of points that are not $f{\downarrow}$ points; these barriers cut the plane into a union of bounded components. It turns out that in the space $\mathbb{R}^3$ a different kind of barriers may appear. They are 2-dimensional surfaces in $\mathbb{R}^3$. For every such surface V the complement $\mathbb{R}^3 \setminus V$ is the union of two open disjoint domains $V_\pm$, $\partial V_+ = \partial V_- = V$, such that no (f) path connects points in V_- and V, and $V \subset \mathfrak{M}(f, V \cup V_+)$.

Theorem 2. *There exist a connected open proper subset Ω of $\mathbb{R}^3$, and a function $f \in C^\infty(\mathbb{R}^3)$ such that $\Omega = \mathfrak{M}(f,\Omega)$, and for every $x \in \Omega$ there exists a compact subset $K(x)$ of Ω such that there are no weak (f) paths $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$, $\gamma(1) \in \Omega \setminus K(x)$.*

To produce our example, we, like in [8], first define f on a thin subset of Ω, and then extend it piecewise harmonically. After that, we smooth f up. We need several technical lemmas. The following statement is contained in [9, pp. 2216–2217].

Lemma 1. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, and let f be (real) analytic on $\mathcal{O}$. Suppose that $x \in \mathcal{O}$ is not a point of local maximum of f. Then there exists an (f) path $\gamma : [0,1] \to \mathcal{O}$ with $\gamma(0) = x$.*

We are going to use special domains in $\mathbb{R}^3$ with smooth boundary. We call a subset $\mathcal{B}$ of $\mathbb{R}^3$ a *pseudoball* if for some $\varepsilon > 0$ there exists a C^∞-smooth diffeomorphism φ from $B(0, 1+\varepsilon)$ to $\mathbb{R}^3$ with $\varphi(B(0,1)) = \mathcal{B}$.

Let us consider the following situation. The boundary of an open subset $\mathcal{O}$ of $\mathbb{R}^3$ is a finite union of disjoint sets $\partial\mathcal{B}_j$, $\mathcal{B}_j$ being pseudoballs. A function f is C^∞-smooth on $\partial\mathcal{O}$ (that is, it extends C^∞-smoothly to a neighborhood of $\partial\mathcal{O}$).

Lemma 2. *In this situation f extends harmonically to a function C^∞-smooth in* Clos $\mathcal{O}$.

This follows from standard results on boundary regularity of solutions to elliptic equations. See, for example, [3, Theorem 8.14], [1, Section 9, Theorem 9.9], [10, Chapter 27].

Let us consider the Dirichlet problem in a cylinder $C = \{(x,y,z) : -1 < x < 1, y^2 + z^2 < R^2\}$ with boundary values f satisfying the following properties: $f(\pm 1, 0, 0) = 0$, $\max_{\partial C} |f| \le 1$,

$$\sup_{y^2+z^2<R^2} \left|\nabla^s_{y,z} f(\pm 1, y, z)\right| \le \frac{1}{R}, \qquad s = 1, 2. \tag{3.1}$$

Then the solution F to the Dirichlet problem satisfies the estimate

$$\sup_{-1<x<1} \left|\frac{\partial}{\partial x} F(x,0,0)\right| = o(1), \qquad R \to \infty. \tag{3.2}$$

To verify this estimate we consider first the (bounded) solution F_1 to the Dirichlet problem in the half-space $x < 1$ with boundary values $f(1,y,z)\chi_{\{y^2+z^2<1\}}$,

$$F_1(x,y,z) = \frac{1}{2\pi} \int_{y_1^2+z_1^2<1} \frac{(1-x) f(1, y_1, z_1)}{\left[(1-x)^2 + (y_1 - y)^2 + (z_1 - z)^2\right]^{3/2}} \, dy_1 \, dz_1.$$

Condition (3.1) implies that

$$\sup_{-1<x<1, y^2+z^2<1} |F_1(x,y,z)| = o(1), \qquad R \to \infty. \tag{3.3}$$

$$\sup_{0\le x<1} \left|\frac{\partial F_1}{\partial x}(x,0,0)\right| = o(1), \qquad R \to \infty. \tag{3.4}$$

Furthermore, elementary estimates of the harmonic measure in C show that

$$\sup_{-1<x<1, y^2+z^2<1} |F(x,y,z)| = o(1), \qquad R \to \infty. \tag{3.5}$$

The function $F - F_1$ vanishes on $\{(1, y, z) : y^2 + z^2 < 1\}$ and, consequently, extends harmonically to $\{(x, y, z) : -1 < x < 3, y^2 + z^2 < 1\}$. Therefore, estimates (3.3) and (3.5) imply that

$$\sup_{0 \le x < 1} \left| \frac{\partial F}{\partial x}(x, 0, 0) - \frac{\partial F_1}{\partial x}(x, 0, 0) \right| = o(1), \qquad R \to \infty. \tag{3.6}$$

Finally, estimates (3.4) and (3.6) together with analogous estimates for $-1 < x \le 0$ give us (3.2).

As a consequence, after scaling, we get the following statement:

Lemma 3. *For every $\varepsilon > 0$ there exists $M(\varepsilon)$ satisfying the following property: if $\delta > 0$,*

$$C = \{(x, y, z) : |x| < \delta/M(\varepsilon), y^2 + z^2 < \delta^2\},$$

and a function f harmonic in C and C^2-smooth up to the boundary of C satisfies the relations

$$\sup_C |f| \le 1, \;\; f(\delta/M(\varepsilon), 0, 0) - f(-\delta/M(\varepsilon), 0, 0) = \varepsilon,$$
$$\sup_{y^2+z^2<\delta^2} \left|\nabla^s_{y,z} f(\pm\delta/M(\varepsilon), y, z)\right| \le 1, \qquad s = 1, 2,$$

then

$$\frac{\partial f}{\partial x}(x, 0, 0) > 0, \qquad |x| \le \delta/M(\varepsilon).$$

Given a point $x \in \mathbb{R}^3$, a point $v \in \mathbb{R}^3 \setminus \{0\}$, and a number $\delta > 0$, denote by $D(x, \delta, v)$ the disc of radius δ centered at x that is contained in the plane $L \subset \mathbb{R}^3$ such that $x \in L$, L is orthogonal to the line $\mathbb{R}v$.

To describe the situation where we are going to apply the previous lemma, we introduce the following notion. We say that two (disjoint) pseudoballs $\mathcal{B}_1, \mathcal{B}_2$ are *a-joined* (by a cylinder C) at the points x_1, x_2 ($x_i \in \partial\mathcal{B}_i$, $i = 1, 2$), if $|x_1 - x_2| < a$ and for some $\delta > M(a)|x_1 - x_2|$,

$$D(x_i, \delta, x_1 - x_2) \subset \partial C \cap \partial\mathcal{B}_i, \quad i = 1, 2, \qquad C \cap (\mathcal{B}_1 \cup \mathcal{B}_2) = \emptyset.$$

Lemma 4. *Suppose that the boundary of a pseudoball $\mathcal{B}$ contains three disjoint discs $D_j = D(x_j, \delta_j, v_j)$, $j = 1, 2, 3$, and a function f is defined on $\{x_1, x_2, x_3\}$, say $f(x_1) = f(x_2) = 1$, $f(x_3) = 0$. Then f extends C^∞-smoothly to $\partial\mathcal{B}$ in such a way that $0 \le f \le 1$,*

$$\partial\mathcal{B} \setminus \{x_1, x_2\} \subset \mathfrak{M}_+(f, \partial\mathcal{B}), \qquad \partial\mathcal{B} \setminus \{x_3\} \subset \mathfrak{M}_-(f, \partial\mathcal{B}),$$
$$|\nabla^s f| \le 1 \;\; \text{on} \;\; D^0_j = D(x_j, \delta_j/2, v_j), \quad j = 1, 2, 3, \;\; s = 1, 2,$$

and there exist (f) paths $\gamma_{31}, \gamma_{32} : [0, 1] \to \partial\mathcal{B}$ such that $\gamma_{3j}(0) = x_3$, $\gamma_{3j}(1) = x_j$, $j = 1, 2$.

Proof. Consider a C^∞-smooth diffeomorphism Φ mapping $\partial\mathcal{B}$ onto T such that

$$\Phi(x_1)=(\sqrt{3}/2,-1/2,0),\ \Phi(x_2)=(-\sqrt{3}/2,-1/2,0),\ \Phi(x_3)=(0,1,0).$$

Define f_0 on T by $f_0(x,y,z)=x^2-(y+1)^2$. It is immediately verified that

$$T\setminus\{(\sqrt{3}/2,-1/2,0),(-\sqrt{3}/2,-1/2,0)\}\subset\mathfrak{M}_+(f_0,T),$$
$$T\setminus\{(0,1,0)\}\subset\mathfrak{M}_-(f_0,T),$$

and $\gamma_\pm:[0,1]\to T$ defined by $\gamma_\pm(t)=(\pm\sqrt{3t-(9t^2/4)},1-3t/2,0)$ are (f_0) paths. Put $f_1=(2f_0\circ\Phi+8)/9$. Then $f_1\in C^\infty(\partial\mathcal{B})$, $0\le f_1\le 1$, $f_1(x_1)=f_1(x_2)=1$, $f_1(x_3)=0$, $\Phi^{-1}\circ\gamma^\pm$ are (f_1) paths.

To make the gradient small on D_j^0, we use the maps

$$\varphi_j(x_j+x)=\begin{cases}x_j+\psi_j(|x|)x, & x_j+x\in D_j,\\ x_j+x, & x_j+x\in\partial\mathcal{B}\setminus D_j,\end{cases}$$

where ψ_j are non-decreasing C^∞-smooth functions on $[0,\infty)$ with

$$\psi_j(t)=\begin{cases}\varepsilon, & 0\le t\le\delta_j/2,\\ 1, & t\ge 2\delta_j/3,\end{cases}\tag{3.7}$$

for sufficiently small ε. Put $f=f_1\circ\varphi_1\circ\varphi_2\circ\varphi_3$. By (3.7) we have $|\nabla^s f|\le 1$ on D_j^0, $j=1,2,3$, $s=1,2$. □

Fix an increasing function $\rho:(1/2,3/2)\to(1/2,3/2)$ such that $\rho\in C^\infty(1/2,3/2)$, $\rho^{-1}\in C(1/2,3/2)\cap C^\infty((1/2,3/2)\setminus\{1\})$, $\rho(x)=x$ outside $(3/4,5/4)$, $\rho(1)=1$, and $\rho^{(n)}(1)=0$, $n\ge 1$.

Lemma 5. *Let φ be a C^∞-smooth diffeomorphism of the domain $A=\{x\in\mathbb{R}^3:1/2<|x|<3/2\}$ onto a domain $\mathcal{Q}\subset\mathbb{R}^3$. Denote $\Gamma=\varphi(T)$. Suppose that f is a function C^∞-smooth on $\varphi(\operatorname{Clos}B(0,1)\cap A)$ and on $\varphi(A\setminus B(0,1))$. Let F be defined on $\mathcal{Q}$ by the formula*

$$F(\varphi(ry))=f(\varphi(\rho(r)y)),\qquad y\in T,\ 1/2<r<3/2,$$

Then $F\in C^\infty(\mathcal{Q})$, $F|\Gamma=f|\Gamma$.

Proof. It is sufficient to prove that $G=F\circ\varphi\in C^\infty(A)$. Denote $g=f\circ\varphi$, $g\in C^\infty(\operatorname{Clos}B(0,1)\cap A)\cap C^\infty(A\setminus B(0,1))$. Since the map $\Psi:ry\to\rho(r)y$ is a C^∞-smooth diffeomorphism from $A\setminus T$ to $A\setminus T$, we need only to verify that all the derivatives of $g\circ\Psi$ are continuous at T. Fix a point x on T and consider a C^∞-smooth diffeomorphism Λ of a neighborhood U of the point $(1,0,0)\in\mathbb{R}^3$ onto a neighborhood of

x such that $\Lambda(1,0,0)=x$, $\Lambda(r,y,z)=r\lambda(y,z)$, $\lambda(y,z)\in T$. We verify that all the derivatives of $g_1=g\circ\Psi\circ\Lambda$ are continuous in U as follows:

$$\lim_{r\to 1}\max_{y,z}\Big|\frac{\partial^s}{\partial y^s}\frac{\partial^t}{\partial z^t}g_1(r,y,z)-\frac{\partial^s}{\partial y^s}\frac{\partial^t}{\partial z^t}g_1(1,y,z)\Big|=0,\qquad s\ge 0,\ t\ge 0,$$

$$\lim_{r\to 1}\max_{y,z}\Big|\frac{\partial^k}{\partial r^k}\frac{\partial^s}{\partial y^s}\frac{\partial^t}{\partial z^t}g_1(r,y,z)\Big|=0,\qquad k>0,\ s\ge 0,\ t\ge 0.$$

□

Lemma 6. *Let f be a function continuous on* Clos $B(0,1)$ *and C^∞-smooth on $B(0,1)$, $f|T=0$, $f(B(0,1))\subset[-1,1]$. Then there exists a C^∞-smooth increasing function $\psi:[-1,1]\to[-1,1]$ such that $\psi^{(k)}(0)=0$, $k\ge 0$, and*

$$\left.\begin{array}{c}\psi\circ f \text{ vanishes with all the derivatives at } T,\\ \psi\circ f\in C^\infty(\text{Clos } B(0,1)).\end{array}\right\}\tag{3.8}$$

Proof. Applying elementary rules for differentiating the composite function we obtain that the properties (3.8) hold if ψ satisfies the system of differential inequalities

$$|\psi^{(k)}(x)|<\varepsilon_k(|x|),\qquad 0\le|x|\le 2^{-k},\ \ k\ge 0,\tag{3.9}$$

where ε_k are continuous functions determined by f, $\varepsilon_k(x)>0$, $x>0$, $k\ge 0$. Furthermore, one can easily produce a solution to this system of inequalities which is C^∞-smooth and increasing on $[-1,1]$. For example, if $h\in C^\infty((0,\infty))$, $h(x)>0$ for $x\in(1/2,1)$, $h(x)=0$ for $x\notin(1/2,1)$, positive numbers c_n tend to 0 sufficiently rapidly, $\psi(0)=0$,

$$\psi'(x)=\sum_{n\ge 0}c_n h(2^n|x|),$$

then ψ satisfies (3.9). □

Proof of Theorem 2. Step A. Let us describe the plan of our construction. The difficult part (Steps B–D) is to define f on the set Clos $\mathcal{O}$,

$$\mathcal{O}=B(0,10)\setminus\big(\text{Clos } B(0,1)\cup\text{Clos } B(5e,1)\big),$$

where $e=(1,0,0)$, and to verify its properties there. The values of f on $\partial\mathcal{O}$ are as follows: $f\big|10T=1$, $f\big|T=0$, $f\big|\partial B(5e,1)=-1$. After that, the simple part (Step E) is to extend f into $\mathbb{R}^3$.

We start (on Step B) with a locally finite family of pseudoballs $\mathcal{B}_j$ in $\mathcal{O}$ with disjoint closures, fix some points $x^\pm_{jk}$ and discs $D(x^\pm_{jk},\delta^\pm_{jk},v^\pm_{jk})$ on their boundaries and prescribe the values of a function f_0 at these points, $f_0(x^+_{jk})>f_0(x^-_{jm})$. Furthermore, these points are grouped into pairs (x^+_{jk},x^-_{ms}), $f_0(x^+_{jk})<f_0(x^-_{ms})$ in such a way that $\mathcal{B}_j,\mathcal{B}_m$ are

$\big(f_0(x^-_{ms}) - f_0(x^+_{jk})\big)$-joined (by some cylinders C_{jm}) at points x^+_{jk}, x^-_{ms}. Every $\partial\mathcal{B}_j$ contains two or three of these points $x^{\pm}_{jk}$.

Using Lemma 4 (or its natural analog for two distinguished points) we extend f_0 to a function $f_1 \in \cap_j C^\infty(\partial\mathcal{B}_j)$ such that

$$\partial\mathcal{B}_j \setminus \{x^+_{jk}\} \subset \mathfrak{M}_+(f_1, \partial\mathcal{B}_j), \qquad \partial\mathcal{B}_j \setminus \{x^-_{jk}\} \subset \mathfrak{M}_-(f_1, \partial\mathcal{B}_j). \quad (3.10)$$

Since the domains $\mathcal{B}_j$, $\mathcal{O} \setminus \cup_j \operatorname{Clos} \mathcal{B}_j$ satisfy the cone condition (see, for example, [5, Theorem 2.11]), we can solve the Dirichlet problem with boundary values f_1 on $\partial\mathcal{B}_j$, 1 on $10T$, 0 on T, -1 on $\partial B(5e, 1)$, separately in $\mathcal{B}_j$ and in $\mathcal{O} \setminus \cup_j \operatorname{Clos} \mathcal{B}_j$. Denote the solution by f_2. Since the boundary values are continuous, we have $f_2 \in C(\operatorname{Clos} \mathcal{O})$, $f_2\big| \cup_j \partial\mathcal{B}_j = f_1$.

Furthermore, f_2 is harmonic, and, as a consequence, is real analytic, and has no points of local maximum (minimum) in $\mathcal{O} \setminus \cup_j\partial\mathcal{B}_j$. By Lemma 1, $\mathcal{O} \setminus \cup_j\partial\mathcal{B}_j = \mathfrak{M}(f_2, \mathcal{O} \setminus \cup_j\partial\mathcal{B}_j)$. By Lemma 2, f_2 is C^∞-smooth in $\operatorname{Clos} \mathcal{B}_j$ and in $\mathcal{O} \setminus \cup_j\mathcal{B}_j$. By Lemma 4 and by the properties of $\mathcal{B}_j$ to be given on Step B, the values of f_1 ($= f_2$) on ∂C_{jm} satisfy the conditions of Lemma 3. Therefore, $x^{\pm}_{jk} \in \mathfrak{M}_{\pm}(f_2, \mathcal{O})$. Together with (3.10) this gives us that $\mathcal{O} \subset \mathfrak{M}(f_2, \mathcal{O})$.

On Step C we verify that

$$\mathcal{O} \cup T = \mathfrak{M}(f_2, \mathcal{O} \cup T), \quad (3.11)$$

and on Step D we verify that

$$\left.\begin{array}{c}\text{no weak } (f_2) \text{ path } \gamma : [-1, 1] \to \operatorname{Clos} \mathcal{O} \text{ with } \gamma(0) \in \mathcal{O} \\ \text{can reach } \partial\mathcal{O} \setminus T.\end{array}\right\} \quad (3.12)$$

Using Lemma 5, we produce an invertible map $\Phi \in C^\infty(\mathcal{O} \to \mathcal{O}) \cap C(\operatorname{Clos} \mathcal{O} \to \operatorname{Clos} \mathcal{O})$, $\Phi^{-1} \in C(\operatorname{Clos} \mathcal{O} \to \operatorname{Clos} \mathcal{O})$, with $\Phi(x) = x$, $x \in \cup_j\partial\mathcal{B}_j \cup \partial\mathcal{O}$, such that

$$f_3 = f_2 \circ \Phi \in C^\infty(\mathcal{O}) \cap C(\operatorname{Clos} \mathcal{O}).$$

Since (weak) (f_2) paths γ correspond to (weak) (f_3) paths $\Phi^{-1} \circ \gamma$, and $f_3\big| \cup_j\partial\mathcal{B}_j \cup \partial\mathcal{O} = f_2\big| \cup_j\partial\mathcal{B}_j \cup \partial\mathcal{O}$, the function f_3 satisfies properties (3.11) and (3.12).

Next, using Lemma 6, we find an increasing function φ, $\varphi \in C^\infty([-1, 1] \to [-1, 1])$ with $\varphi(0) = 0$, $\varphi(\pm 1) = \pm 1$ such that

$$f = \varphi \circ f_3 \in C^\infty(\operatorname{Clos} \mathcal{O}), \quad (3.13)$$

$f|T = 0$, $f|10T = 1$, $f|\partial B(5e, 1) = -1$, and all the derivatives of f vanish at $\partial\mathcal{O}$. Since every (f_3) path is an (f) path and vice versa, we obtain that $\mathcal{O} \cup T = \mathfrak{M}(f, \mathcal{O} \cup T)$, and property (3.12) is valid for f.

Finally, on Step E we extend f into $\mathbb{R}^3$.

Step B. On this step we define disjoint pseudoballs $\mathcal{B}_j \subset \mathcal{O}$ mentioned on Step A, fix some points on their boundaries and values of f_0 at these points.

We consider two subsets $E^\pm$ of T, $E^\pm = \{x \in T : |x \pm e| > 1\}$, and two sequences of (different) points $a^i_{k,j} \in \mathbb{R}^3$, $i = 1, 2$, $k \geq 0$, $1 \leq j \leq 2^k$, such that

$$\left.\begin{aligned} |a^i_{k,j}| &= 1 + 2^{-k}, \\ |a^i_{k,j} - a^i_{k+1,2j-s}| &< 10 \cdot 2^{-k}, \quad s = 0, 1, \end{aligned}\right\} \qquad 1 \leq j \leq 2^k, \tag{3.14}$$

$$\max_{x\in T} \min_j |x - a^i_{k,j}| < 10 \cdot 2^{-k}, \qquad i = 1, 2,\ k \geq 0. \tag{3.15}$$

(B1). We construct pseudoballs $\mathcal{B}^1_n$, $\mathcal{B}^2_n$, $n \geq 1$, such that

$$\left.\begin{aligned} (10 - 2^{-4n+3})E^- \subset \mathcal{B}^2_{2n-1}, \qquad (10 - 2^{-4n+2})E^- \subset \mathcal{B}^1_{2n-1}, \\ (10 - 2^{-4n+1})E^+ \subset \mathcal{B}^2_{2n}, \qquad (10 - 2^{-4n})E^+ \subset \mathcal{B}^1_{2n}, \end{aligned}\right\} \tag{3.16}$$

and the pseudoballs $\mathcal{B}^i_n$ and $\mathcal{B}^i_{n+1}$ are $2^{-2n-10i-1}$-joined at points $b^{i,1}_n$ and $b^{i,0}_{n+1}$, $n \geq 1$, $i = 1, 2$.

Next, we construct pseudoballs $\mathcal{B}^3_n$, $\mathcal{B}^4_n$, and points $b^{3,j}_n$, $b^{4,j}_n$, $j = 0, 1$, $n \geq 1$, by the formulas $\mathcal{B}^{i+2}_n = \Phi(\mathcal{B}^i_n)$, $b^{i+2,j}_n = \Phi(b^{i,j}_n)$, $i = 1, 2$, $j = 0, 1$, $n \geq 1$, where $\Phi(x) = 10x/|x|^2 + 5$. Two pseudoballs $\mathcal{B}^1_0$, $\mathcal{B}^2_0$ are constructed in such a way that $\mathcal{B}^i_0$ and $\mathcal{B}^i_1$ are 2^{-10i-1}-joined at points $b^{i,1}_0$ and $b^{i,0}_1$ and $\mathcal{B}^i_0$ and $\mathcal{B}^{i+2}_1$ are 1-joined at points $b^{i,0}_0$ and $b^{i+2,0}_1$, $i = 1, 2$.

We put

$$f_0(b^{i,1}_n) = 1 - 2^{-2n-10i}, \quad f_0(b^{i,0}_n) = 1 - 2^{-2n-10i+1}, \quad i = 1, 2,\ n \geq 0,$$
$$f_0(b^{i+2,j}_n) = -f_0(b^{i,j}_n), \qquad i = 1, 2,\ j = 0, 1,\ n \geq 1.$$

(B2). We construct pseudoballs $\mathcal{B}^5_{k,j}$, $\mathcal{B}^6_{k,j}$, $k \geq 0$, $1 \leq j \leq 2^k$, such that

$$a^i_{k,j} \in \mathcal{B}^{4+i}_{k,j}, \tag{3.17}$$

$$\operatorname{diam} \mathcal{B}^{4+i}_{k,j} < 100 \cdot 2^{-k}, \qquad i = 1, 2,\ k \geq 0,\ 1 \leq j \leq 2^k, \tag{3.18}$$

the pseudoballs $\mathcal{B}^i_{k,j}$ and $\mathcal{B}^i_{k+1,2j-1}$ are 2^{-2k-3}-joined at points $b^{i,1}_{k,j}$ and $b^{i,0}_{k+1,2j-1}$, and the pseudoballs $\mathcal{B}^i_{k,j}$ and $\mathcal{B}^i_{k+1,2j}$ are 2^{-2k-3}-joined at points $b^{i,2}_{k,j}$ and $b^{i,0}_{k+1,2j}$, $i = 5, 6$, $k \geq 0$, $1 \leq j \leq 2^k$. We also require that $\mathcal{B}^5_{0,1}$ and $\mathcal{B}^6_{0,1}$ be 1-joined at points $b^{5,0}_{0,1}$ and $b^{6,0}_{0,1}$.

For $k \geq 0$, $1 \leq j \leq 2^k$, put

$$f_0(b^{5,0}_{k,j}) = 2^{-2k-1}, \qquad f_0(b^{5,i}_{k,j}) = 2^{-2k-2}, \quad i = 1, 2, \tag{3.19}$$
$$f_0(b^{6,i}_{k,j}) = -f_0(b^{5,i}_{k,j}), \qquad i = 0, 1, 2.$$

(B3). It remains to show how to construct the pseudoballs $\mathcal{B}$ satisfying properties (3.16), (3.17), (3.18), and joined in the indicated way. First, star-shaped domains

$$\mathcal{B}_{x,\varphi} = \{x + ry : r < \varphi(y),\ y \in T\}$$

with $x \in \mathbb{R}^3$, $\varphi \in C^\infty(T)$, $\varphi(y) \neq 0$, $y \in T$, are pseudoballs, and we seek for $\mathcal{B}^i_{k,j}$, $i = 3, 4, 5, 6$, among such domains. Second, if $\mathcal{B}$ is a pseudoball, and Φ is a C^∞-smooth diffeomorphism acting on a neighborhood of $\mathcal{B}$, for example $\Phi_y(z) = (z-y)/|z-y|$, where $y \in \mathbb{R}^3 \setminus \mathrm{Clos}\,\mathcal{B}$, then $\Phi_y(\mathcal{B})$ is also a pseudoball. We seek for $\mathcal{B}^i_{k,j}$, $i = 1, 2$ among domains $\Phi_y(\mathcal{B}_{x,\varphi})$.

We restrict ourselves to describing the construction for $\mathcal{B}^3_{k,j}$, $k > 0$. We are given a family A of points $a^1_{k,j}$, $k > 0$. Each pseudoball $\mathcal{B} \in \{\mathcal{B}^3_{k,j}\}$ should contain the point $a_{0,k,j} = a^1_{k,j}$ and should be joined with three other pseudoballs, containing each by a point in A; we denote these points by $a_{1,k,j}$, $a_{2,k,j}$, $a_{3,k,j}$. By induction, for every (k, j) we choose a point $x_{k,j}$ such that the set

$$S_{k,j} = \bigcup_{0 \le i \le 3} [x_{k,j}, a_{i,k,j})$$

consists of 4 intervals intersecting only by the point $x_{k,j}$, all the sets $S_{k,j}$ are disjoint, and

$$1 + \frac{2^{-k}}{10} < \inf\{|y| : y \in S_{k,j}\} < \sup\{|y| : y \in S_{k,j}\} < 1 + 30 \cdot 2^{-k}.$$

This is just the place where we use the fact that the dimension of the space is at least 3.

Finally, we put $\mathcal{B}^3_{k,j} = \mathcal{B}(x_{k,j}, \varphi_{k,j})$ with suitable $\varphi_{k,j}$.

Step C. On this step we are going to verify that $T \subset \mathfrak{M}(f_2, \mathcal{O} \cup T)$.

Fix an arbitrary point $x \in T$, and define $A^m_x = \{(k, j) : k \ge m, |x - a^1_{k,j}| < 100 \cdot 2^{-k}\}$. As a consequence of (3.15), for every $k \ge 0$ there exists j such that $(k, j) \in A^k_x$. Furthermore, by (3.14), for every pair $(k, j) \in A^k_x$, $k > 0$, the "preceding" pair $p(k, j) = (k - 1, [(j + 1)/2])$ is in A^{k-1}_x, where $[y]$ is the entire part of $y \in \mathbb{R}$. Therefore, $(0, 1) \in A'_x = \cap_{m<\infty} p^m(A^m_x) \subset A^0_x$. Put $j_0 = 1$. In an inductive process, on the step $k \ge 0$ put $j_{k+1} = 2j_k - 1$ if $(k + 1, 2j_k - 1) \in A'_x$, otherwise $(k+1, 2j_k) \in A'_x$ and we put $j_{k+1} = 2j_k$. As a result, we get a sequence of points a^1_{k,j_k} such that $j_{k+1} = 2j_k - 1$ or $j_{k+1} = 2j_k$, $k \ge 0$, and $|x - a^1_{k,j_k}| < 100 \cdot 2^{-k}$.

The properties of the pseudoballs $\mathcal{B}^5_{k,j}$ formulated on Step B and Lemma 4 imply that there exist (f_2) paths $\gamma_k : [0, 1] \to \partial\mathcal{B}^5_{k,j_k}$ with $\gamma_k(0) = b^{5,i_k}_{k,j_k}$, $\gamma_k(1) = b^{5,0}_{k,j_k}$, where i_k are defined by the relations $j_{k+1} = 2j_k + i_k - 2$. Condition (3.18) implies that diam $\gamma_k([0, 1]) < 100 \cdot 2^{-k}$.

Since the pseudoballs $\mathcal{B}^5_{k,j_k}$ and $\mathcal{B}^5_{k+1,j_{k+1}}$ are 2^{-2k-3}-joined (by cylinders C_k) at the points b^{5,i_k}_{k,j_k} and $b^{5,0}_{k+1,j_{k+1}}$, and the values of f_2 ($= f_0$) at these points are given by (3.19), Lemmas 4 and 3 imply that the linear maps γ'_k with $\gamma'_k(0) = b^{5,0}_{k+1,j_{k+1}}$, $\gamma'_k(1) = b^{5,i_k}_{k,j_k}$ are (f_2) paths. The lengths of the intervals $\gamma'_k([0,1])$ do not exceed 2^{-k} because these intervals are contained in the cylinders C_k.

Now we can define an (f_2) path $\gamma_x : [0,1] \to \mathcal{O} \cup T$ as follows: $\gamma_x(0) = x$,

$$\gamma_x(2^{-k}s) = \begin{cases} \gamma'_k(4s-2), & 1/2 < s \le 3/4, \\ \gamma_k(4s-3), & 3/4 < s \le 1, \quad k \ge 0. \end{cases}$$

Analogously, using the pseudoballs $\mathcal{B}^6_{k,j}$, we construct an (f_2) path $\gamma_x : [0,1] \to \mathcal{O} \cup T$ with $\gamma_x(1) = x$.

Step D. Now we prove that there are no continuous maps $\gamma : [0,1] \to \operatorname{Clos} \mathcal{O}$ connecting points in $\mathcal{O}$ with points in $\partial\mathcal{O} \setminus T$ such that $f_2 \circ \gamma$ is monotonic.

Indeed, suppose that $\gamma([0,1)) \subset \mathcal{O}$, $\gamma(1) \in 10T$ (the case $\gamma(1) \in \partial B(5e,1)$ is analogous). Since $\operatorname{dist}(T \setminus E^+, T \setminus E^-) > 0$, we obtain that for all t close to 1, $\gamma(t)/|\gamma(t)|$ belongs to one of the sets $E^\pm$, say E^+. Then $\gamma(t) \in |\gamma(t)|E^+$. For sufficiently big n, $\gamma([0,1))$ intersects $(10-2^{-n})T$. Define $t_n = \min\{t \in [0,1) : |\gamma(t)| = 10 - 2^{-n}\}$. Then t_n increase for big n, and $\lim_{n\to\infty} t_n = 1$. Therefore, for big n,

$$\begin{gathered} \gamma(t_{4n}) \in |\gamma(t_{4n})|E^+ = (10 - 2^{-4n})E^+ \subset \mathcal{B}^1_{2n}, \\ 1 - 2^{-4n-9} \le f_2(\gamma(t_{4n})) \le 1 - 2^{-4n-10}, \\ \gamma(t_{4n-1}) \in |\gamma(t_{4n-1})|E^+ = (10 - 2^{-4n+1})E^+ \subset \mathcal{B}^2_{2n}, \\ 1 - 2^{-4n-19} \le f_2(\gamma(t_{4n-1})) \le 1 - 2^{-4n-20}. \end{gathered}$$

These inequalities contradict to the assumption that $f_2 \circ \gamma$ is monotonic.

Step E. Now we are going to extend to $\mathbb{R}^3$ the function f given in $\operatorname{Clos} \mathcal{O}$ by formula (3.13).

First we define a function f_4 on the union of the sets $10^n(\mathcal{O} \cup T)$, $n \in \mathbb{Z}$, by the formula

$$f_4(x) = f(10^{-n}x) + n, \qquad x \in 10^n(\mathcal{O} \cup T), \;\; n \in \mathbb{Z}.$$

Then f_4 is C^∞-smooth on $\widetilde{\mathcal{O}} = \cup_{n\in\mathbb{Z}} 10^n(\mathcal{O} \cup T)$, $\widetilde{\mathcal{O}} = \mathfrak{M}(f_4, \widetilde{\mathcal{O}})$, and no (f_4) path $\gamma : [0,1] \to \widetilde{\mathcal{O}}$ connects points in $10^n(\mathcal{O} \cup T)$ for different n. If a function φ is C^∞-smooth and increasing on $\mathbb{R}$, $\varphi(x) = x$ for $x \in [0,1]$, and all the derivatives of φ vanish sufficiently rapidly at $-\infty$, then $f = \varphi \circ f_4$ possesses the same properties as f_4, and f extends C^∞-smoothly to $\widetilde{\mathcal{O}} \cup \{0\}$.

Our function f is now defined outside a countable union of disjoint closed balls Clos B_α, with centers on the line $\mathbb{R}e$. The function f extends by continuity to ∂B_α, say $f\big|\partial B_\alpha \equiv c_\alpha$. For every ball B_α we apply the following procedure. Consider a linear map Φ_α on $\mathbb{R}^3$, $\Phi_\alpha(x) = u_\alpha x + v_\alpha$, $u_\alpha \in (0,\infty)$, $v_\alpha \in \mathbb{R}e$, with $\Phi_\alpha(B_\alpha) = B(0,1)$, define $\Psi_\alpha(x) = \Phi_\alpha(x)/|\Phi_\alpha(x)|^2$, and for $x \in \mathcal{O}_\alpha = \{x \in B_\alpha : \Psi_\alpha(x) \in \mathcal{O} \cup T\}$ put

$$f(x) = d_\alpha f(\Psi_\alpha(x)) + c_\alpha,$$

where $d_\alpha > 0$. Now, f is defined in Clos $B_\alpha \setminus$ (Clos $B_{\alpha'} \cup$ Clos $B_{\alpha''}$) where Clos $B_{\alpha'}$, Clos $B_{\alpha''}$ are two new (smaller) disjoint closed balls contained in B_α, with centers on the line $\mathbb{R}e$. We repeat this procedure for every ball B_α, and as a result, define f on an open connected set Ω with $\mathbb{R}^3 \setminus \Omega = Ee$, where E is a Cantor type set on $\mathbb{R}$. Then $\Omega = \mathfrak{M}(f,\Omega)$, and no (f) path $\gamma : [-1,1] \to \Omega$ connects points in $10^n(\mathcal{O} \cup T)$, $\mathcal{O}_\alpha$ for different n, α. Finally, if the numbers d_α tend to 0 sufficiently rapidly, then f extends C^∞-smoothly to $\mathbb{R}^3$. □

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